RELATIONS BETWEEN BOREL-TYPE METHODS OF SUMMABILITY

D. Borwein†.

1. *Introduction.*

Suppose throughout that l, a_n $(n = 0, 1, \ldots)$ are arbitrary complex numbers, that $\alpha > 0$ and β is real, and that N is a non-negative integer greater than $-\beta/\alpha$. Let

$$s_n = \sum_{\nu=0}^{n} a_\nu, \quad s_{-1} = 0; \quad a(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}, \quad s(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)},$$

and let $r_a = R_a^{1/\alpha}$, $r_s = R_s^{1/\alpha}$ where R_a, R_s are the radii of convergence of the series $\Sigma a_n z^n/\Gamma(\alpha n+\beta)$, $\Sigma s_n z^n/\Gamma(\alpha n+\beta)$.

A function $f(z)$ will be said to be in class $\mathscr{A}$ if it is analytic in a region containing the half-line $z = x > 0$.

We shall be concerned with regular methods of summability $(B, \alpha, \beta)^*$, (B, α, β), $(B', \alpha, \beta)^*$, (B', α, β) defined as follows (*cf.* [2] and [4]).

B. If (i) $r_s > 0$, (ii) there is a function $s^*(z)$ in $\mathscr{A}$ such that $s^*(x) = s(x)$ for $0 < x < r_s$, and (iii) $\lim_{x\to\infty} \alpha e^{-x} s^*(x) = l$, we write

$$\sum_0^\infty a_n = l(B, \alpha, \beta)^* \quad \text{or} \quad s_n \to l(B, \alpha, \beta)^*.$$

We omit the stars when $r_s = \infty$; in this case condition B(ii) is automatically satisfied.

B′. If (i) $r_a > 0$, (ii) there is a function $a^*(z)$ in $\mathscr{A}$ such that $a^*(x) = a(x)$ for $0 < x < r_a$, and (iii) $\lim_{x\to\infty} \int_0^x e^{-t} a^*(t)\, dt + s_{N-1} = l$, we write

$$\sum_0^\infty a_n = l(B', \alpha, \beta)^*.$$

We omit the star when $r_a = \infty$.

$(B, 1, 1)$ and $(B', 1, 1)$ are respectively the Borel exponential and the Borel integral methods; and $(B', 1, 1)^*$ is the method (B^*) defined in [5], p. 192.

The principal theorems established in this note are:

Theorem 1*. *$\sum_0^\infty a_n = l(B, \alpha, \beta)^*$ if and only if $\sum_0^\infty a_n = l(B', \alpha, \beta)^*$ and $a_n \to 0(B, \alpha, \beta)^*$;*

† Received 29 December, 1958; read 15 January, 1959.

[Journal London Math. Soc. 35 (1960), 65–70]

THEOREM 2*. $\sum_0^\infty a_n = l(B, \alpha, \beta+1)^*$ *if and only if* $\sum_0^\infty a_n = l(B', \alpha, \beta)^*$;

and Theorems 1 and 2 which denote these results with the stars omitted.

The cases $\alpha = \beta = 1$ of Theorems 1 and 2 are standard results ([5], Theorems 123 and 126). The case $\beta = 1$ of Theorem 1 is also known [2]. Other known results are:

(I) *If* $\lambda > \alpha$, $\sum_0^\infty a_n = l(B, \lambda, \mu)$ *and* $r_s > 0$, *then* $\sum_0^\infty a_n = l(B, \alpha, \beta)^*$;

(II)* *If* $\beta > \mu$ *and* $\sum_0^\infty a_n = l(B, \alpha, \mu)^*$, *then* $\sum_0^\infty a_n = l(B, \alpha, \beta)^*$;

and (II) which denotes result (II)* with the stars omitted. Proofs of these appear in [1]; the definitions there given of $(B, \alpha, \beta)^*$, (B, α, β) are equivalent to the above.

Replace r_s by r_a in (I), B by B' in (I), (II)*, (II), and label the propositions so obtained (I′), (II′)*, (II′). That these propositions are true emerges from Theorem 2*, Theorem 2, and Lemma 4 (below) which states that $r_a = r_s$. Result (II′) and the case $r_a = \infty$, $\mu = \beta$ of (I′) are due to Good [4].

Since it is clear that the actual choice of N in definitions B and B′ is immaterial, we shall assume in all that follows that

$$\alpha N + \beta \geqslant 1;$$

so that, whenever $r_a > 0$, $r_s > 0$, the functions $a(x)$ and $s(x)$ are continuous in an interval $[0, \epsilon]$. We shall also assume, evidently without loss in generality, that

$$a_0 = a_1 = \ldots = a_{N-1} = 0. \tag{1}$$

Given a function $f(x)$ which is continuous for $0 \leqslant x \leqslant c$, we write

$$f_0(x) = f(x), \quad f_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t)\, dt \quad (0 \leqslant x \leqslant c,\ \delta > 0).$$

This notation will also be used with other letters in place of f.

2. *Preliminary results.*

LEMMA 1. *Let* $f(x)$ *be continuous for* $x \geqslant 0$ *and let* $\delta > 0$.

(i) *If* $\lim_{x\to\infty} e^{-x} f(x) = l$, *then* $\lim_{x\to\infty} e^{-x} f_\delta(x) = l$.

(ii) *If* $\lim_{x\to\infty} \int_0^x e^{-t} f(t)\, dt = l$, *then* $\lim_{x\to\infty} \int_0^x e^{-t} f_\delta(t)\, dt = l$.

A proof of this lemma is given in [2].

LEMMA 2. *If* $\delta > 0$ *and* $r_s > 0$, *then*

$$s_\delta(x) = \sum_{n=N}^\infty \frac{s_n x^{\alpha n + \beta + \delta - 1}}{\Gamma(\alpha n + \beta + \delta)} \quad (0 < x < r_s).$$

The proof of this lemma is immediate.

LEMMA 3. *If* $\gamma > 0$, $\delta > 0$ *and* $f(z)$, $g(z)$ *are bounded analytic functions in the sector* $0 < |z| < c < \infty$, $|\arg z| < \Delta \leqslant \pi$, *then there is a function* $h(z)$, *analytic in the sector, such that*

$$h(x) = \int_0^x (x-t)^{\gamma-1} t^{\delta-1} f(t)\, g(x-t)\, dt \quad (0 < x < c).$$

Proof. Denote the sector in question by D and, for $z \in D$, let

$$\phi(z) = \int_0^1 (1-u)^{\gamma-1} u^{\delta-1} f(zu)\, g(z-zu)\, du,$$

$$\phi(z, n) = \int_{1/n}^{(n-1)/n} (1-u)^{\gamma-1} u^{\delta-1} f(zu)\, g(z-zu)\, du \quad (n = 3, 4, \ldots).$$

Then, by a standard result ([3], p. 108), $\phi(z, n)$ is an analytic function of z in D. Further, by hypothesis, there is a positive constant M such that, for $z \in D$,

$$|\phi(z) - \phi(z, n)| \leqslant M \int_0^{1/n} \{(1-u)^{\gamma-1} u^{\delta-1} + u^{\gamma-1}(1-u)^{\delta-1}\}\, du;$$

and so, when $n \to \infty$, $\phi(z, n) \to \phi(z)$ uniformly in D. Hence $\phi(z)$ is analytic in D. But, for $0 < x < c$,

$$x^{\gamma+\delta-1} \phi(x) = \int_0^x (x-t)^{\gamma-1} t^{\delta-1} f(t)\, g(x-t)\, dt;$$

so that $h(z) = z^{\gamma+\delta-1}\phi(z)$ has the required properties, provided $z^{\gamma+\delta-1}$ is taken to be $|z|^{\gamma+\delta-1} e^{i\theta(\gamma+\delta-1)}$ where θ is the principal value of $\arg z$.

LEMMA 4. $r_a = r_s$.

Proof. Having assumed that $s_{N-1} = 0$, we find that

$$\sum_{n=N}^\infty \frac{a_n z^n}{\Gamma(\alpha n + \beta)} = \sum_{n=N}^\infty \frac{s_n z^n}{\Gamma(\alpha n + \beta)} - \sum_{n=N}^\infty \frac{s_n z^{n+1}}{\Gamma(\alpha n + \alpha + \beta)} \tag{2}$$

whenever both series on the right-hand side are convergent. Since $\lim_{n\to\infty} \{\Gamma(\alpha n + \alpha + \beta)/\Gamma(\alpha n + \beta)\}^{1/n} = 1$, the two series on the right have the same radius of convergence, namely R_s; and so

$$R_a \geqslant R_s.$$

Suppose now that $R_a > 0$ and let γ, δ be any numbers such that $R_a > \gamma > \delta > 0$. Then there is a positive integer $m > N$ such that, for $n > m$,

$$|a_n| < \gamma^{-n} \Gamma(\alpha n + \beta) < \gamma^{-n-1} \Gamma(\alpha n + \alpha + \beta).$$

Hence, for $n > m$,

$$|s_n| < \sum_{\nu=0}^{m} |a_\nu| + n\gamma^{-n}\Gamma(\alpha n+\beta);$$

and so, for all sufficiently large n,

$$|s_n| < \delta^{-n}\Gamma(\alpha n+\beta).$$

It follows that $R_s \geqslant \delta$ and consequently that

$$R_s \geqslant R_a.$$

The required result is thus established.

From now on we shall denote the common value of r_a and r_s by r.

LEMMA 5. *If* $r > 0$, *then conditions* B(ii) *and* B′(ii) *are equivalent.*

Proof. In view of (1), (2) and Lemma 2,

$$a(x) = s(x) - s_\alpha(x) \quad (0 < x < r). \tag{3}$$

Further, it is easily verified that $\lim_{n\to\infty} s_{\alpha n}(x) = 0$ $(0 < x < r)$, and so (*cf.* [2]),

$$s(x) = \sum_{n=0}^{\infty} \{s_{\alpha n}(x) - s_{\alpha n+\alpha}(x)\} = a(x) + \sum_{n=1}^{\infty} a_{\alpha n}(x)$$

$$= a(x) + \sum_{n=1}^{\infty} \frac{1}{\Gamma(\alpha n)} \int_0^x t^{\alpha n-1} a(x-t)\,dt$$

$$= a(x) + \int_0^x a(x-t)\psi(t)\,dt \quad (0 < x < r), \tag{4}$$

where $\psi(t) = \sum_{n=1}^{\infty} \frac{t^{\alpha n-1}}{\Gamma(\alpha n)}$.

Assume first B′(ii). Then, given $c > 0$, there is a $\Delta > 0$ such that $a^*(z)$ and $z^{1-\alpha}\psi(z)$ are analytic and bounded in the sector $0 < |z| < c$, $|\arg z| < \Delta$. Consequently, by Lemma 3, there is a function $\phi(z)$ in $\mathscr{A}$ such that

$$\phi(x) = \int_0^x a^*(x-t)\psi(t)\,dt \quad (x > 0);$$

whence, by (4), condition B(ii) is satisfied by

$$s^*(z) = a^*(z) + \phi(z).$$

Now assume B(ii). Applying Lemma 3 as above, we find that there is a function $\theta(z)$ in $\mathscr{A}$ such that

$$\theta(x) = s^*_\alpha(x) \quad (x > 0).$$

Hence, by (3), condition B′(ii) is satisfied by

$$a^*(z) = s^*(z) - \theta(z).$$

This completes the proof of the lemma. It is evident that the above argument also yields a proof of:

LEMMA 6. *If* $r > 0$ *and conditions* (1), B(ii) *and* B′(ii) *hold, then, for all* $x > 0$, $a^*(x) = s^*(x) - s_\alpha(x)$.

3. *Proofs of the principal results.*

Theorem 1 can be deduced from (3) as in [2]; and, in virtue of Lemmas 5 and 6, the same basic argument can be used to establish Theorem 1*.

Proof of Theorem 2*. (i) *Suppose first that*

$$\sum_0^\infty a_n = l(B', \alpha, \beta)^*. \tag{5}$$

Then, in view of Lemma 5, there are functions $a^*(z)$, $s^*(z)$ in $\mathscr{A}$ such that $a^*(x) = a(x)$, $s^*(x) = s(x)$ $(0 < x < r)$; also, since (1) is assumed to hold,

$$\lim_{x\to\infty} \int_0^x e^{-t} a^*(t)\,dt = l. \tag{6}$$

Let

$$\delta = k\alpha > 4 \tag{7}$$

where k is an integer. Then, by Lemmas 5 and 6,

$$\sum_{n=0}^{k-1} a^*_{\alpha n}(x) = s^*(x) - s^*_\delta(x) \quad (x > 0). \tag{8}$$

Arguing as in [2], pp. 132-133, we can now deduce from (6), (7) and (8) that

$$\lim_{x\to\infty} \alpha e^{-x} s^*_\delta(x) = l;$$

whence, by Lemma 1(i),

$$\lim_{x\to\infty} \alpha e^{-x} s^*_{\delta+1}(x) = l. \tag{9}$$

Further,

$$\int_0^x e^{-t} a^*(t)\,dt = -e^{-x} a^*_1(x) + \int_0^x e^{-t} a^*_1(t)\,dt \quad (x > 0); \tag{10}$$

so that, by Lemma 1(ii) and (6),

$$\lim_{x\to\infty} e^{-x} a^*_1(x) = 0. \tag{11}$$

It follows from (8) and (11), by Lemma 1(i), that

$$\lim_{x\to\infty} \alpha e^{-x}\{s^*_1(x) - s^*_{\delta+1}(x)\} = 0;$$

so that, by (9),

$$\lim_{x\to\infty} \alpha e^{-x} s^*_1(x) = l. \tag{12}$$

In addition, by Lemma 3, there is a function $\sigma(z)$ in $\mathscr{A}$ such that $\sigma(x) = s_1^*(x)$ for all $x > 0$. Since

$$\sigma(x) = s_1(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta+1)} \qquad (0 < x < r),$$

it follows from (12) that

$$\sum_0^{\infty} a_n = l(B, \alpha, \beta+1)^*. \tag{13}$$

(ii) It remains to prove that (5) is a consequence of (13). *Suppose therefore that* (13) *holds*. Then, in view of Lemma 5, there is a function $\gamma(z)$ in $\mathscr{A}$ such that

$$\gamma(x) = a_1(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta+1)} \qquad (0 < x < r).$$

Let $a^*(z) = \gamma'(z)$. Then $a(x) = a^*(x)$ for $0 < x < r$ and $\gamma(x) = \alpha_1^*(x)$ for all $x > 0$. Applying now Theorem 1, we find that (11) holds and that

$$\lim_{x\to\infty} \int_0^x e^{-t} a_1^*(t)\, dt = l. \tag{14}$$

Hence, by (10), (11) and (14),

$$\lim_{x\to\infty} \int_0^x e^{-t} a^*(t)\, dt = l;$$

and (5) follows.

This completes the proof of Theorem 2*. Since the series

$$\sum_{n=N}^{\infty} \frac{a_n z^n}{\Gamma(\alpha n+\beta)}, \quad \sum_{n=N}^{\infty} \frac{s_n z^n}{\Gamma(\alpha n+\beta+1)}$$

have the same radius of convergence, namely $R_a = R_s$, Theorem 2 is an immediate consequence of Theorem 2*.

References.

1. D. Borwein, "On methods of summability based on integral functions II", *Proc. Cambridge Phil. Soc.* (to appear).
2. ———, "On Borel-type methods of summability", *Mathematika*, 5 (1958), 128–133.
3. E. T. Copson, *Functions of a Complex Variable* (Oxford, 1935).
4. I. J. Good, "Relations between methods of summation of series", *Proc. Cambridge Phil. Soc.*, 38 (1942), 144–165.
5. G. H. Hardy, *Divergent Series* (Oxford, 1949).

St. Salvator's College,
University of St. Andrews.

Printed by C. F. Hodgson & Son, Ltd., Pakenham Street London, W.C.1.