ON MOMENT CONSTANT METHODS OF SUMMABILITY

D. BORWEIN†.

1. *Introduction.*

Suppose that $\chi(x)$ is a real non-decreasing and bounded function in the range $0 \leqslant x < X \leqslant \infty$, that $\chi(y) < \lim_{x \to X-} \chi(x)$ when $0 \leqslant y < X$, and that

$$0 < \mu_n = \lim_{x \to X-} \int_0^x t^n \, d\chi(t) < \infty \qquad (n = 0, 1, \ldots).$$

Let l, a_n $(n = 0, 1, \ldots)$ be arbitrary complex numbers, let

$$s_n = \sum_{r=0}^{n} a_r, \quad M(x) = \sum_{n=0}^{\infty} x^n/\mu_n,$$

and denote the radius of convergence of the power series by R.

Associated with the sequence of moment constants $\{\mu_n\}$ are methods of summability $(J_\chi)^*$, (J_χ), $(J_\chi')^*$, (J_χ') defined as follows:

If $\sum_{n=0}^{\infty} s_n z^n/\mu_n = S(z)$ is an analytic function in a neighbourhood of the origin admitting, in a region containing the real open interval $(0, R)$, an analytic continuation $S^*(z)$ such that

$$\lim_{x \to R-} S^*(x)/M(x) = l,$$

we write $\sum_0^{\infty} a_n = l(J_\chi)^*$ or $s_n \to l(J_\chi)^*$. We omit the stars when $S(z)$ is analytic in the region $|z| < R$. (See [5], §4.12.)

If $\sum_0^{\infty} a_n z^n/\mu_n = A(z)$ is an analytic function in a neighbourhood of the origin admitting, in a region containing the interval $(0, X)$, an analytic continuation $A^*(z)$ such that

$$\lim_{x \to X-} \int_0^x A^*(t)\, d\chi(t) = l,$$

we write $\sum_0^{\infty} a_n = l(J_\chi')^*$ or $s_n \to l(J_\chi')^*$. We omit the stars when $A(z)$ is analytic in the region $|z| < X$.

It is known ([5], §4.13) that the method (J_χ') is regular, *i.e.* $s_n \to l(J_\chi')$ whenever $s_n \to l$; and it follows that $(J_\chi')^*$ is regular. It is also known (see [2], Theorem 1) that $(J_\chi)^*$ and (J_χ) are regular if and only if $\lim_{x \to R-} M(x) = \infty$.

This note is concerned with relations linking certain of the "unprimed" and "primed" J-methods.

---

† Received 18 February, 1959; read 19 February, 1959.

Given summability methods P, Q we say that P includes Q and write $P \supseteq Q$ if $s_n \to l(P)$ whenever $s_n \to l(Q)$. If $P \supseteq Q$ and $Q \supseteq P$ we say that P and Q are equivalent and write $P \simeq Q$.

Borel-type methods. Denote by $(B, \alpha, \beta)^*$, (B, α, β), $(B', \alpha, \beta)^*$, (B', α, β) the J-methods based on the function

$$\chi(x) = \alpha^{-1} \int_0^x t^{(\beta-\alpha)/\alpha} e^{-t^{1/\alpha}} dt \qquad (\alpha > 0,\ \beta > 0;\ 0 \leqslant x < X = \infty).$$

In this case $\mu_n = \Gamma(\alpha n + \beta)$, $R = \infty$ and it is known (see [3]) that $\alpha M(x) \sim x^{(1-\beta)/\alpha} e^{x^{1/\alpha}}$ as $x \to \infty$. Note that $(B, 1, 1)$, $(B', 1, 1)$ are respectively the Borel exponential and the Borel integral methods, and that $(B, 1, 1)^*$ is the method (B^*) defined in [5], p. 192.

The above definitions of Borel-type methods are equivalent to ones appearing in [4]† where the following results are proved:

(a)*. $\sum_0^\infty a_n = l(B, \alpha, \beta)^*$ *if and only if* $\sum_0^\infty a_n = l(B', \alpha, \beta)^*$ *and* $a_n \to 0(B, \alpha, \beta)^*$.

(b)*. $(B, \alpha, \beta+1)^* \simeq (B', \alpha, \beta)^*$.

Also proved there are the versions of these results obtained by deleting the stars.

Our primary object is to obtain similar results for the following methods.

Abel-type methods. Denote by $(A_\alpha)^*$, (A_α), $(A_\alpha')^*$, (A_α') the J-methods based on the function

$$\chi(x) = -(1-x)^\alpha \qquad (\alpha > 0,\ 0 \leqslant x < X = 1).$$

Then $$\mu_n = 1 \Big/ \binom{n+\alpha}{n}, \quad R = 1, \quad M(x) = (1-x)^{-\alpha-1}.$$

We extend the definitions of $(A_\alpha)^*$, (A_α) to the range $-1 < \alpha \leqslant 0$ by taking μ_n to be $1 \Big/ \binom{n+\alpha}{n}$ in this range. These two methods are then regular for all $\alpha > -1$ (see [1], Theorem 1). The method (A_0) is the ordinary Abel method and it is easily verified that it is equivalent to (A_1').

The theorems established in this note are:

THEOREM 1*. *For* $\alpha > 0$, $\sum_0^\infty a_n = l(A_\alpha)^*$ *if and only if* $\sum_0^\infty a_n = l(A_\alpha')^*$ *and* $na_n \to 0(A_{\alpha-1})^*$;

THEOREM 2*. *For* $\alpha > 0$, $(A_\alpha')^* \simeq (A_{\alpha-1})^*$;

and Theorems 1 and 2 which denote these results with the stars omitted.

† See also Włodarski [6].

2. *Preliminary results.*

Let

$$\epsilon_n^\alpha = \binom{n+\alpha}{n}, \quad a(t, \alpha) = (1+t)^{-\alpha-1} \sum_{n=0}^\infty a_n \epsilon_n^\alpha \left(\frac{t}{1+t}\right)^n \quad (\alpha > -1),$$

let $\rho_a(\alpha)$ be the upper bound of real values of t for which the series is convergent, and define $s(t, \alpha)$, $\rho_s(\alpha)$ similarly. It is evident that, if $\alpha > -1$, $\beta > -1$, then $\rho_a(\alpha) = \rho_s(\beta) = \rho$ say.

We shall say that a function $f(z)$ is in class $\mathscr{A}$ if it is analytic in a region containing the real open interval $(0, \infty)$.

The following results are easily verified.

(A). *For* $\alpha > -1$, $\sum_0^\infty a_n = l(A_\alpha)$ *if and only if* $\rho = \infty$ *and* $\lim_{y\to\infty} s(y, \alpha) = l$.

(A)*. *For* $\alpha > -1$, $\sum_0^\infty a_n = l(A_\alpha)^*$ *if and only if* (i) $\rho > 0$, (ii) *there is a function* $s^*(z, \alpha)$ *in* $\mathscr{A}$ *such that* $s^*(t, \alpha) = s(t, \alpha)$ *for* $0 < t < \rho$, *and* (iii) $\lim_{t\to\infty} s^*(t, \alpha) = l$.

(A′). *For* $\alpha > 0$, $\sum_0^\infty a_n = l(A_\alpha')$ *if and only if* $\rho = \infty$ *and*

$$\lim_{y\to\infty} \alpha \int_0^y a(t, \alpha)\, dt = l.$$

(A′)*. *For* $\alpha > 0$, $\sum_0^\infty a_n = l(A_\alpha')^*$ *if and only if* (i) $\rho > 0$, (ii) *there is a function* $a^*(z, \alpha)$ *in* $\mathscr{A}$ *such that* $a^*(t, \alpha) = a(t, \alpha)$ *for* $0 < t < \rho$, *and* (iii) $\lim_{y\to\infty} \alpha \int_0^y a^*(t, \alpha)\, dt = l$.

We now prove some useful identities. Suppose that $0 < y < \rho$. Then, for $\alpha > -1$,

$$\begin{aligned} a(y, \alpha) &= (1+y)^{-\alpha-1} \left\{ \sum_{n=0}^\infty s_n \epsilon_n^\alpha \left(\frac{y}{1+y}\right)^n - \sum_{n=0}^\infty s_n \epsilon_{n+1}^\alpha \left(\frac{y}{1+y}\right)^{n+1} \right\} \\ &= s(y, \alpha) - (1+y)^{-\alpha-1} \sum_{n=0}^\infty s_n \epsilon_n^\alpha \frac{n+1+\alpha}{n+1} \left(\frac{y}{1+y}\right)^{n+1} \\ &= s(y, \alpha)\left(1 - \frac{y}{1+y}\right) - \alpha(1+y)^{-\alpha-1} \sum_{n=0}^\infty s_n \epsilon_n^\alpha \int_0^y \left(\frac{t}{1+t}\right)^n \frac{dt}{(1+t)^2} \\ &= (1+y)^{-1} s(y, \alpha) - \alpha(1+y)^{-\alpha-1} \int_0^y (1+t)^{\alpha-1} s(t, \alpha)\, dt; \qquad (1) \end{aligned}$$

the inversion being legitimate since $\Sigma |s_n| \epsilon_n^\alpha \left(\frac{y}{1+y}\right)^{n+1} < \infty$.

It follows that, for $\alpha > -1$,

$$\int_0^y a(u,\alpha)\,du = \int_0^y (1+u)^{-1}s(u,\alpha)\,du - \alpha\int_0^y (1+u)^{-\alpha-1}\,du\int_0^u (1+t)^{\alpha-1}s(t,\alpha)\,dt$$

$$= \int_0^y (1+u)^{-1}s(u,\alpha)\,du - \alpha\int_0^y (1+t)^{\alpha-1}s(t,\alpha)\,dt\int_t^y (1+u)^{-\alpha-1}\,du$$

$$= (1+y)^{-\alpha}\int_0^y (1+t)^{\alpha-1}s(t,\alpha)\,dt. \qquad (2)$$

Combining (1) and (2) we get

$$s(y,\alpha) = \alpha\int_0^y a(t,\alpha)\,dt + (1+y)\,a(y,\alpha) \qquad (\alpha > -1). \qquad (3)$$

Another useful identity in essence has been established elsewhere ([1], Lemma 2); it is:

$$s(y,\beta) = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta)}\,y^{-\alpha}\int_0^y (y-t)^{\alpha-\beta-1}t^{\beta}s(t,\alpha)\,dt \qquad (\alpha > \beta > -1), \qquad (4)$$

and the same identity holds with a in place of s.

In view of (3) and (4) we have, for $\alpha > 0$,

$$s(y,\alpha-1) = \alpha y^{-\alpha}\int_0^y t^{\alpha-1}s(t,\alpha)\,dt$$

$$= \alpha^2 y^{-\alpha}\int_0^y t^{\alpha-1}\,dt\int_0^t a(u,\alpha)\,du + \alpha y^{-\alpha}\int_0^y u^{\alpha-1}(1+u)\,a(u,\alpha)\,du$$

$$= \alpha\int_0^y a(u,\alpha)\,du + \alpha y^{-\alpha}\int_0^y u^{\alpha-1}a(u,\alpha)\,du, \qquad (5)$$

$$= \alpha\int_0^y a(u,\alpha)\,du + a(y,\alpha-1). \qquad (6)$$

We require the following results:

(I)* *If* $\rho > 0$, $\alpha > -1$, *then conditions* (A)*(ii) *and* (A′)*(ii) *are equivalent.*

This is a simple consequence of (1) and (3).

(II)*. *If* $\rho > 0$, $\alpha > \beta > -1$ *and condition* (A)*(ii) *holds, then there is a function* $s^*(z,\beta)$ *in* $\mathscr{A}$ *such that* $s^*(t,\beta) = s(t,\beta)$ *for* $0 < t < \rho$.

This is a consequence of (4) (see the proof of Lemma 5 in [4]).

(III)*. *If* $\alpha > \beta > -1$ *then* $(A_\beta)^* \supseteq (A_\alpha)^*$.

In virtue of (II)*, this can be deduced from (4) as in [1] where the "unstarred" version of the result is proved.

(IV)*. *If* $\alpha > -1$, p *is real, and if* $s_n \to l(A_\alpha)^*$ *and*

$$(n+p)\,u_n = s_n \qquad (n = 0, 1, \ldots),$$

then $u_n \to 0(A_\alpha)^*$.

This can be established by slightly modifying the proof given in [1] of the unstarred version of the result.

3. *Proof of Theorem 1*.*

From now on it is to be assumed that $\alpha > 0$.

(i) *Suppose that*

$$\sum_0^\infty a_n = l(A_\alpha)^*. \qquad (7)$$

Then, in view of (I)*, there are functions $s^*(z,\alpha)$, $a^*(z,\alpha)$ in $\mathscr{A}$ such that

$$s^*(t,\alpha) = s(t,\alpha), \quad a^*(t,\alpha) = a(t,\alpha) \qquad (0 < t < \rho) \qquad (8)$$

and

$$\lim_{t\to\infty} s^*(t,\alpha) = l.$$

Hence, by (2),

$$\alpha\int_0^y a^*(u,\alpha)\,du = \alpha(1+y)^{-\alpha}\int_0^y (1+t)^{\alpha-1}s^*(t,\alpha)\,dt \to l \text{ as } y\to\infty;$$

so that

$$\sum_0^\infty a_n = l(A_\alpha')^*. \qquad (9)$$

Further, by (3),

$$(1+y)\,a^*(y,\alpha) = s^*(y,\alpha) - \alpha\int_0^y a^*(t,\alpha)\,dt \to 0 \text{ as } y\to\infty.$$

But, for $0 < y < \rho$,

$$(1+y)\,a^*(y,\alpha) = (1+y)^{-\alpha}\sum_{n=1}^\infty \frac{\alpha+n}{\alpha n}\,na_n\,\epsilon_n^{\alpha-1}\left(\frac{y}{1+y}\right)^n + a_0(1+y)^{-\alpha};$$

and so

$$\frac{\alpha+n}{\alpha n}\,na_n \to 0 \quad (A_{\alpha-1})^*.$$

Consequently, by (IV)*,

$$na_n \to 0 \quad (A_{\alpha-1})^*. \qquad (10)$$

(ii) It remains to prove that (7) is a consequence of (9) and (10). *Suppose therefore that* (9) *and* (10) *hold.* In view of (I)*, it follows from (9) that there are functions $a^*(z,\alpha)$, $s^*(z,\alpha)$ in $\mathscr{A}$ satisfying (8), and that

$$\lim_{y\to\infty} \alpha\int_0^y a^*(u,\alpha)\,du = l.$$

Further, reversing the last part of the argument in (i), we find that a consequence of (10) is that

$$\lim_{y\to\infty} (1+y)\,a^*(y, \alpha) = 0.$$

Hence, by (3),

$$\lim_{y\to\infty} s^*(y, \alpha) = l;$$

and (7) follows.

4. *Proof of Theorem 2**.

(i) *Suppose that*

$$\sum_0^\infty a_n = l(A_\alpha')^*. \tag{11}$$

Then, in view of (I)* and (II)*, there are functions $a^*(z, \alpha)$, $s^*(z, \alpha-1)$ in $\mathscr{A}$ such that

$$a^*(t, \alpha) = a(t, \alpha), \quad s^*(t, \alpha-1) = s(t, \alpha-1) \qquad (0 < t < \rho),$$

and

$$\lim_{y\to\infty} \alpha \int_0^y a^*(u, \alpha)\, du = l.$$

Consequently, by (5),

$$s^*(y, \alpha-1) = \alpha \int_0^y a^*(u, \alpha)\, du + \alpha y^{-\alpha} \int_0^y u^{\alpha-1} a^*(u, \alpha)\, du$$

$$= \alpha \int_0^y a^*(u, \alpha)\, du + O(y^{-\alpha}) + O(y^{-1}) \to l \text{ as } y \to \infty;$$

and so

$$\sum_0^\infty a_n = l(A_{\alpha-1})^*. \tag{12}$$

(ii) It remains to prove that (11) is a consequence of (12). *Suppose that* (12) *holds*. Then, in view of (I)* and (6), there are functions $s^*(z, \alpha-1)$, $a^*(z, \alpha-1)$, $a^*(z, \alpha)$ in $\mathscr{A}$ such that

$$s^*(t, \alpha-1) = s(t, \alpha-1), \quad a^*(t, \alpha-1) = a(t, \alpha-1),$$

$$a^*(t, \alpha) = a(t, \alpha) \qquad (0 < t < \rho).$$

Further

$$\lim_{y\to\infty} s^*(y, \alpha-1) = l;$$

and so, by (1),

$$a^*(y, \alpha-1) = (1+y)^{-1} s^*(y, \alpha-1)$$

$$+ (\alpha-1)(1+y)^{-\alpha} \int_0^y (1+t)^{\alpha-2} s^*(t, \alpha-1)\, dt \to 0 \text{ as } y \to \infty.$$

Hence, by (6),

$$\alpha \int_0^y a^*(u, \alpha)\, du = s^*(y, \alpha-1) - a^*(y, \alpha-1) \to l \text{ as } y \to \infty;$$

and (11) follows.

We have now proved Theorems 1* and 2*. Since the radii of convergence of the series

$$\Sigma a_n \epsilon_n^{\alpha} x^n, \quad \Sigma s_n \epsilon_n^{\beta} x^n, \quad \Sigma n a_n \epsilon_n^{\gamma} x^n \quad (\alpha > -1, \beta > -1, \gamma > -1)$$

are all equal, we see that Theorem 1 is a consequence of Theorem 1* and Theorem 2 of Theorem 2*.

References.

1. D. Borwein, "On a scale of Abel-type methods of summability", *Proc. Cambridge Phil. Soc.*, 53 (1957), 318–322.
2. ———, "On methods of summability based on power series", *Proc. Royal Soc. Edinburgh*, 64 (1957), 342–349.
3. ———, "On methods of summability based on integral functions II", *Proc. Cambridge Phil. Soc.* (to appear).
4. ———, "Relations between Borel-type methods of summability", *Journal London Math. Soc.*, 35 (1960), 65–70.
5. G. H. Hardy, *Divergent Series* (Oxford, 1949).
6. L. Włodarski, "Propertiés des méthodes continues de limitation du type de Borel", *Bull. Acad. Polon. Sci., Cl. III*, 4 (1956), 173–175.

St. Salvator's College,
University of St. Andrews.

C. F. Hodgson & Son Ltd., Pakenham Street, London, W.C.1