# ON STRONG RIESZ SUMMABILITY FACTORS

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1. Suppose that $a$, $k$ are positive numbers, and that $p$ is the integer such that $k-1 \leqslant p < k$. Suppose that $\phi(w)$ is a positive unboundedly increasing function, as many times differentiable as may be required. Let $\lambda = \{\lambda_n\}$ be an unboundedly increasing sequence with $\lambda_1 > 0$.

Given a series, $\sum_{n=1}^{\infty} a_n$, and a number $m > -1$, we write

$$A_m(w) = \begin{cases} \sum_{\lambda_n < w} (w-\lambda_n)^m a_n & \text{if } w > \lambda_1, \\ 0 & \text{otherwise,} \end{cases}$$

and $A(w) = A_0(w)$.

If $w^{-m} A_m(w)$ tends to a finite limit as $w$ tends to infinity, the series, $\sum_{n=1}^{\infty} a_n$ is said to be summable $(R, \lambda, m)$; it is said to be strongly summable $(R, \lambda, m)$ with index $q > 0$, or summable $[R, \lambda; m, q]$, if there is a number $s$ such that

$$\int_0^w |t^{-(m-1)} A_{m-1}(t) - s|^q \, dt = o(w);$$

and it is said to be absolutely summable $(R, \lambda, m)$, or summable $|R, \lambda, m|$, if $w^{-m} A_m(w)$ is of bounded variation in the range $w \geqslant 0$.

We write summability $[R, \lambda, m]$ for summability $[R, \lambda; m, 1]$.

In this note, we shall be dealing with logarithmico-exponential functions (abbreviated to $L$-functions) for whose definition see [4].

We shall prove the following theorems:

Theorem 1. *For all $k > 0$, if*

(i) *$\phi(w)$ is an L-function,*

(ii) $\dfrac{1}{w} \prec \dfrac{\phi'(w)}{\phi(w)}$

(iii) $\psi(w) = \left\{ \dfrac{\phi(w)}{w\phi'(w)} \right\}^k,$

*then $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $[R, \phi(\lambda), k]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$.*

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THEOREM 2. *For all $k > 0$, if $\psi(w)$ is an L-function tending to a non-zero finite limit as $w$ tends to infinity, then $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $[R, \lambda, k]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$.*

THEOREM 3. *For all $k \geqslant 1$, if*

(i) *$\phi(w)$ is an L-function,*

(ii) $\dfrac{1}{w} \leqslant \dfrac{\phi'(w)}{\phi(w)}$,

(iii) $\psi(w) = \left\{\dfrac{\phi(w)}{w\phi'(w)}\right\}^k$,

*then $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $[R, \phi(\lambda), k]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$.*

Theorems analogous to Theorem 1 and to Theorem 3 for all values of $k > 0$, for ordinary Riesz summability, and for integer values of $k$ for absolute Riesz summability are due to Guha [**3**]. In a recent paper, [**1**], we have deduced the proof for non-integral values of $k$ for absolute Riesz summability. The theorems analogous to Theorem 2 are both due to Guha [**3**].

We wish to thank Dr. Kuttner for valuable suggestions including a draft of the proof of Theorem 2.

2. The following theorems are known:

THEOREM A. *$\sum_{n=1}^{\infty} a_n$ is summable $(R, \lambda, k)$ to sum $s$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$ to sum $s$.*

THEOREM B. *$\sum_{n=1}^{\infty} a_n (\lambda_n)^{-k+(1/q')}$ is summable $[R, e^{\lambda}; k, q]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda; k, q]$ where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.*

THEOREM C. (i) *Suppose that $k$ is a positive integer. If*

$$\int_a^w t^k |\phi^{(k+1)}(t)|\, dt = O\{\phi(w)\} \text{ for } w \geqslant a,$$

*then $\sum_{n=1}^{\infty} a_n$ is summable $[R, \phi(\lambda), k]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$.*

(ii) *Suppose that $k$ is any positive non-integral number greater than 1. If*

$$\int_a^w t^{p+1} |\phi^{(p+2)}(t)|\, dt = O\{\phi(w)\} \text{ for } w \geqslant a$$

*and either*

(a) $\phi'(w)$ *is a monotonic non-decreasing function for* $w \geqslant a$

*or*

(b) $\phi'(w)$ *is a monotonic non-increasing function for* $w \geqslant a$ *and* $w\phi''(w) = O\{\phi'(w)\}$ *for* $w \geqslant a$,

*then* $\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \phi(\lambda), k]$ *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \lambda, k]$.

These theorems are all due to Srivastava [**6, 7**]. She gives a counter-example to show that, for $0 < k < 1$, there is a series which is summable $[R, \lambda, k]$, but not summable $[R, \log\lambda, k]$.

From Theorem C we can immediately deduce

COROLLARY C. *For all* $k \geqslant 1$, *if*

(i) $\phi(w)$ *is an L-function*,

(ii) $\phi(w) = O(w^{\delta})$ *where* $\delta > 0$ *and for* $w \geqslant a$,

*then* $\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \phi(\lambda), k]$ *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \lambda, k]$.

3. The following lemmas are required.

LEMMA 1. (i) *Any L-function is continuous, of constant sign and monotonic from a certain value of the variable onwards.* (We suppose $a$ chosen so that all those $L$-functions which occur in the argument satisfy these conditions from $a$ onwards.)

(ii) *The derivative of an L-function is an L-function, and the ratio of two L-functions is an L-function.*

(iii) *If* $\phi_1(w)$, $\phi_2(w)$ *are L-functions not tending to finite limits, and* $\phi_1(w) \preccurlyeq \phi_2(w)$, *then* $\phi_1'(w) \preccurlyeq \phi_2'(w)$.

(iv) *If* $\phi(w)$ *is an L-function such that* $\phi(w) \succ e^w$, *then there exists a positive integer*, $N$, *such that*

$$e_N(w) \prec \phi(w) \preccurlyeq e_{N+1}(w)$$

*where* $e_N(w) = \exp\{e_{N-1}(w)\}$ *and* $e_0(w) = w$.

(v) *If* $\phi(w)$ *is a non-decreasing L-function such that* $\dfrac{1}{w} \prec \dfrac{\phi'(w)}{\phi(w)}$, *then* $\phi(w) \succ w^{\Delta}$ *for every* $\Delta$, *and*

$$\phi^{(n)}(w) \preccurlyeq \left\{\frac{\phi'(w)}{\phi(w)}\right\}^n \phi(w).$$

For proofs, see [**4**].

LEMMA 2. *The n-th derivative of* $\{f(t)\}^m$ *is a sum of constant multiples of terms like*

$$\{f(t)\}^{m-\mu}\prod_{\nu=1}^{n}\{f^{(\nu)}(t)\}^{\alpha_\nu}$$

*where* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are non-negative integers such that*

$$1 \leqslant \sum_{\nu=1}^{n}\alpha_\nu = \mu \leqslant \sum_{\nu=1}^{n}\nu\alpha_\nu = n.$$

*Further, if* $m$ *is a positive integer, then* $\mu \leqslant m$.

This simple result is a particular case of a theorem due to Faa di Bruno. See [**9**; pp. 88–89.]

LEMMA 3. *If* $\theta(t) \geqslant 0$, $m > 0$ *and* $m-n > 0$, *then the two assertions*

$$\int_0^w \theta(t)\,dt = o(w^m)$$

*and*

$$\int_0^w t^{-n}\,\theta(t)\,dt = o(w^{m-n})$$

*are equivalent, it being assumed that both integrals converge at the origin.*

Compare Lemma 2 in [**2**].

LEMMA 4. $\sum_{n=1}^{\infty} a_n = s[R, \lambda, k]$ *if and only if* $\sum_{n=1}^{\infty} a_n = s(R, \lambda, k)$ *and*

$$\int_0^w dx\left|x^{-k}\int_0^x (x-t)^{k-1}\,t\,dA(t)\right| = o(w).$$

*Proof.* Define

$$B_k(x) = \int_0^x (x-t)^k\,t\,dA(t)$$

and $C_k(x) = x^{-k}A_k(x)$. Now

$$\int_0^w \left|x^{-k}B_{k-1}(x)\right|dx = \int_0^w dx\left|x^{-k}\int_0^x (x-t)^{k-1}\,t\,dA(t)\right|$$

$$= \int_0^w dx\left|x^{-k}\int_0^x (x-t)^{k-1}\{x-(x-t)\}\,dA(t)\right|$$

$$= \int_0^w \left|C_{k-1}(x)-C_k(x)\right|dx.$$

Also,

$$\int_0^w \left|C_{k-1}(x)-s\right|dx \leqslant \int_0^w |C_{k-1}(x)-C_k(x)|\,dx + \int_0^w |C_k(x)-s|\,dx \qquad (3.1)$$

and

$$\int_0^w |C_{k-1}(x)-C_k(x)|\,dx \leqslant \int_0^w |C_{k-1}(x)-s|\,dx + \int_0^w |C_k(x)-s|\,dx. \quad (3.2)$$

Now, if $\sum_{n=1}^{\infty} a_n = s(R, \lambda, k)$, $C_k(x)$ tends to $s$ as $x$ tends to infinity, and hence

$$\int_0^w |C_k(x)-s|\,dx = o(w).$$

If also

$$\int_0^w |x^{-k} B_{k-1}(x)|\,dx = o(w),$$

we can deduce from (3.1) that

$$\int_0^w |C_{k-1}(x)-s|\,dx = o(w),$$

that is, that $\sum_{n=1}^{\infty} a_n = s\,[R, \lambda, k]$.

Conversely, if $\sum_{n=1}^{\infty} a_n = s[R, \lambda, k]$, by Theorem A, we have that $\sum_{n=1}^{\infty} a_n = s(R, \lambda, k)$, and hence, in view of (3.2), that

$$\int_0^w |x^{-k} B_{k-1}(x)|\,dx = o(w).$$

For a similar result on strong Cesaro summability, see [**5**].

LEMMA 5. *Assume that the expressions below have a meaning.*

(i) *If* $G_1(w) = \int_a^w f_1(w, t)\, g_1(t)\,dt$, *then*

$$\int_a^w |dG_1(w)| \leqslant \overline{\text{bd}}_{a<t<w} \left\{ |f_1(t, t)| + \int_t^w |d_w f_1(w, t)| \right\} . \int_a^w |g_1(t)|\,dt.$$

(ii) *If* $G_2(w) = \int_0^1 f_2(w, t)\, g_2(t)\,dt$, *then*

$$\int_a^w |dG_2(w)| \leqslant \overline{\text{bd}}_{0<t<1} \int_a^w |d_w f_2(w, t)| . \int_0^1 |g_2(t)|\,dt.$$

This is similar to Lemma 1 in [**8**], and is proved similarly. See also Lemma 5 in [**6**].

LEMMA 6. (i) *If* $\dfrac{t\phi''(t)}{\phi'(t)}$ *is non-negative non-decreasing for* $t \geqslant a$, *then* $\dfrac{y(1-v)\,\phi'(u+vy)}{\phi(u+y)-\phi(u+vy)}$ *is a non-negative monotonic non-increasing function of* $y$ *in the range* $[0, \infty)$ *for* $0 < v < 1$ *and* $u \geqslant a$.

(ii) *If* $\dfrac{t\phi'(t)}{\phi(t)}$ *is non-negative monotonic non-decreasing for* $t \leqslant a$, *then* $\dfrac{\phi(u+vy)}{\phi(u+y)}$ *is a non-negative monotonic non-increasing function of* $y$ *in the range* $[0, \infty)$ *for* $0 < v < 1$ *and* $u \geqslant a$.

See Lemma 6 in [1].

LEMMA 7. *Under the hypotheses of Theorem* 1,

$$w^n \psi^{(n)}(w) \prec \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^{k-n},$$

*for* $n = 0, 1, \ldots, p+1$ *and* $w \geqslant a$.

*Proof.* The result is trivially true for $n = 0$. Using Lemma 2, for $n = 1, 2, \ldots, p+1$ and $w \geqslant a$, $\psi^{(n)}(w)$ can be expressed as a sum of constant multiples of terms like

$$\left\{\frac{\phi(w)}{w\phi'(w)}\right\}^{k-\sigma} \prod_{\nu=1}^{n} \left\{\left(\frac{\partial}{\partial w}\right)^{\nu} \left(\frac{\phi(w)}{w\phi'(w)}\right)\right\}^{\alpha_\nu}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers such that

$$1 \leqslant \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leqslant \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

Also, by Leibnitz's theorem on the differentiation of a product,

$$\left(\frac{\partial}{\partial w}\right)^{\nu} \left(\frac{\phi(w)}{w\phi'(w)}\right)$$

can be expressed as a sum of constant multiples of terms like

$$\left(\frac{\partial}{\partial w}\right)^{i} \left(\frac{1}{w}\right) \left(\frac{\partial}{\partial w}\right)^{j} \left(\frac{1}{\phi'(w)}\right) . \phi^{(\nu-i-j)}(w)$$

where $i$ and $j$ are integers such that $0 \leqslant i \leqslant \nu$ and $0 \leqslant j \leqslant \nu - i$. This expression, in turn, can be expressed as a sum of constant multiples of terms like

$$\theta(w) = w^{-1-i} \phi^{(\nu-i-j)}(w) \{\phi'(w)\}^{-1-\mu} \prod_{m=1}^{j} \{\phi^{(m+1)}(w)\}^{\beta_m},$$

where $\beta_1, \beta_2, \ldots, \beta_j$ are non-negative integers such that

$$0 \leqslant \sum_{m=1}^{j} \beta_m = \mu \leqslant \sum_{m=1}^{j} m\beta_m = j.$$

Hence, in view of Lemma 1 (v) and condition (ii) of Theorem 1,

$$\theta(w) \leqslant w^{-1-i} \frac{\{\phi'(w)\}^{\nu-i-j}}{\{\phi(w)\}^{\nu-i-j-1}} \{\phi'(w)\}^{-1-\mu} \frac{\{\phi'(w)\}^{j+\mu}}{\{\phi(w)\}^{j}}$$

$$= w^{-1-i} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-i-1}$$

$$= w^{-1} \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^{i} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-1}$$

$$\prec w^{-1} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-1}.$$

Hence

$$\psi^{(n)}(w) \prec \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^{k-\sigma} \prod_{\nu=1}^{n} \left(w^{-1} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-1}\right)^{\alpha_\nu}$$

$$= \left\{\frac{\phi(w)}{w\phi'(w)}\right\}^{k-\sigma} w^{-\sigma} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{n-\sigma}$$

$$= w^{-k} \left\{\frac{\phi(w)}{\phi'(w)}\right\}^{k-n},$$

and hence the required result is immediately obtained.

4. *Proof of Theorem* 1.

We assume, without loss of generality, that the sum of the series is zero, and that

$$A(w) = 0 \text{ for } 0 \leqslant w \leqslant a.$$

Since $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$, in view of Theorem A, we have that $\sum_{n=1}^{\infty} a_n$ is summable $(R, \lambda, k)$, and so, since the conditions of the theorem are sufficient to prove that $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $\big(R, \phi(\lambda), k\big)$ [**3**; Theorem 2], in view of Lemmas 3 and 4, it is sufficient for the proof of Theorem 1, to show that, for $w \geqslant a$,

$$\int_a^w \phi'(x) \,|\, F_{k-1}(x) \,|\, dx = o\Big(\{\phi(w)\}^{k+1}\Big) \tag{4.1}$$

where

$$F_k(x) = \int_a^x \{\phi(x) - \phi(t)\}^k \phi(t) \psi(t) \, dA(t).$$

In view of Lemma 3, since the sum of the series is zero, we have that

$$\int_a^w |\, A_{k-1}(t) \,|\, dt = o(w^k). \tag{4.2}$$

(i) Suppose that $k$ is a positive integer. Integrating by parts $k$ times, we find that $F_{k-1}(x)$ can be expressed as a sum of constant multiples of

$$A_{k-1}(x)\psi(x)\phi(x)\{\phi'(x)\}^{k-1}$$

and

$$\int_a^x A_{k-1}(t)\left(\frac{\partial}{\partial t}\right)^k\left(\{\phi(x)-\phi(t)\}^{k-1}\phi(t)\psi(t)\right)dt. \tag{4.3}$$

Now, in view of (4.2) and Lemma 1 (v), we have that

$$\begin{aligned}\int_a^w \phi'(x)|A_{k-1}(x)\psi(x)\phi(x)\{\phi'(x)\}^{k-1}|\,dx &= \int_a^w |A_{k-1}(x)|\,x^{-k}\{\phi(x)\}^{k+1}\,dx\\ &\leqslant w^{-k}\{\phi(w)\}^{k+1}\int_a^w |A_{k-1}(x)|\,dx\\ &= o\left(\{\phi(w)\}^{k+1}\right).\end{aligned} \tag{4.4}$$

Also, in view of Lemma 2 and Leibnitz's theorem on the differentiation of a product,

$$\int_a^x A_{k-1}(t)\left(\frac{\partial}{\partial t}\right)^k\left(\{\phi(x)-\phi(t)\}^{k}\phi(t)\psi(t)\right)dt$$

can be expressed as a sum of constant multiples of integrals of the types

$$I_1(x) = \int_a^x A_{k-1}(t)\{\phi(x)-\phi(t)\}^{k-1-\mu}\phi^{(k-r-m)}(t)\,\psi^{(m)}(t)\,.\prod_{\nu=1}^{r}\{\phi^{(\nu)}(t)\}^{\alpha_\nu}\,dt,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are non-negative integers such that

$$\sum_{\nu=1}^{r}\alpha_\nu = \mu; \quad \sum_{\nu=1}^{r}\nu\alpha_\nu = r,$$

and

$$0 \leqslant \mu \leqslant k-1; \quad 0 \leqslant \mu \leqslant r \leqslant k; \quad 0 \leqslant m \leqslant k-r$$

Now, in view of (4.2) and Lemmas 1 (v) and 7,

$$\begin{aligned}&\int_a^w \phi'(x)|I_1(x)|\,dx\\ &= O\left\{\int_a^w \phi'(x)\,dx\int_a^x |A_{k-1}(t)|\,\{\phi(x)-\phi(t)\}^{k-1-\mu}.\right.\\ &\qquad\left.\times\frac{\{\phi'(t)\}^{k-r-m}}{\{\phi(t)\}^{k-r-m-1}}\,\frac{t^{-k}\{\phi(t)\}^{k-m}}{\{\phi'(t)\}^{k-m}}\,\frac{\{\phi'(t)\}^{r}}{\{\phi(t)\}^{r-\mu}}\,dt\right\}\\ &= O\left\{\int_a^w \phi'(x)\{\phi(x)\}^{k-1-\mu}\,dx\int_a^x |A_{k-1}(t)|\,t^{-k}\{\phi(t)\}^{\mu+1}\,dt\right\}\\ &= O\left\{\int_a^w \phi'(x)\{\phi(x)\}^{k}x^{-k}\,dx\int_a^x |A_{k-1}(t)|\,dt\right\}\\ &= o\left\{\int_a^w \phi'(x)\{\phi(x)\}^{k}\,dx\right\}\\ &= o\left(\{\phi(w)\}^{k+1}\right).\end{aligned} \tag{4.5}$$

Hence, in view of (4.4) and (4.5), we can deduce that (4.1) is true. This completes the proof of Theorem 1 for integer values of $k$.

(ii) Suppose that $k$ is any positive non-integral number. The relation, (4.1), that we must prove, can be written as

$$\int_a^w |d_x F_k(x)| = o\left(\{\phi(w)\}^{k+1}\right). \tag{4.6}$$

Integrating by parts $p+1$ times, we obtain that

$$F_k(x) = \frac{(-1)^{p+1}}{p!} \int_a^x A_p(t) \left(\frac{\partial}{\partial t}\right)^{p+1} \left(\{\phi(x)-\phi(t)\}^k \phi(t)\psi(t)\right) dt.$$

Using Lemma 2 and Leibnitz's theorem on the differentiation of a product, it follows that $F_k(x)$ can be expressed as a sum of constant multiples of integrals of the forms

$$I_2(x) = \int_a^x A_p(t)\, Q(x, t)\, dt,$$

where

$$\begin{aligned} Q(x, t) &= Q_{\mu, r, m}(x, t) \\ &= \{\phi(x)-\phi(t)\}^{k-\mu} \phi^{(p+1-r-m)}(t)\, \psi^{(m)}(t) \prod_{\nu=1}^{r} \{\phi^{(\nu)}(t)\}^{\alpha_\nu} \end{aligned}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are non-negative integers such that

$$0 \leqslant \sum_{\nu=1}^{r} \alpha_\nu = \mu \leqslant \sum_{\nu=1}^{r} \nu\alpha_\nu = r \leqslant p+1$$

and $0 \leqslant m \leqslant p+1-r$.

Now, we have that

$$\begin{aligned} I_2(x) &= \int_a^x A_p(t)\, Q(x, t)\, dt \\ &= \frac{\Gamma(p+1)}{\Gamma(k)\,\Gamma(p+1-k)} \int_a^x Q(x, t)\, dt \int_a^t (t-u)^{p-k} A_{k-1}(u)\, du \\ &= \frac{\Gamma(p+1)}{\Gamma(k)\,\Gamma(p+1-k)} \int_a^x A_{k-1}(u)\, du \int_u^x Q(x, t)(t-u)^{p-k}\, dt \\ &= \frac{\Gamma(p+1)}{\Gamma(k)\,\Gamma(p+1-k)} \int_a^x A_{k-1}(u)\, I_3(x, u)\, du \end{aligned}$$

say, where

$$I_3(x, u) = \int_u^x Q(x, t)(t-u)^{p-k}\, dt.$$

For $\mu = 0, 1, \ldots, p$, $I_3(u, u) = 0$, and for $\mu = p+1$, define

$$I_3(u, u) = \lim_{x \to u+} I_3(x, u).$$

Now, for $x > t$

$$Q(x, t) = (x-t)^{k-p-1}[\{\phi'(x)\}^k \psi(x)\phi(x) + \delta(x, t)],$$

where $\delta(x, t) \to 0$ as $x \to t+$, uniformly in $t$ for $t$ in some right-hand neighbourhood of $u$.

Hence,

$$\begin{aligned} I_3(u, u) &= \Gamma(k-p)\Gamma(p+1-k)\phi(u)\psi(u)\{\phi'(u)\}^k \\ &= \Gamma(k-p)\Gamma(p+1-k)u^{-k}\{\phi(u)\}^{k+1}, \end{aligned}$$

and so, in view of Lemma 1 (v),

$$\underset{a<u<w}{\overline{\text{bd}}} |I_3(u, u)| = O\left(w^{-k}\{\phi(w)\}^{k+1}\right).$$

Hence, in view of Lemma 5 (i) and (4.2), in order to prove that

$$\int_a^w |dI_2(x)| = o\left(\{\phi(w)\}^{k+1}\right),$$

it remains to prove that

$$\begin{aligned} I_4(w) &= \underset{a<u<w}{\overline{\text{bd}}} \int_u^w \left| d_x \int_u^x Q(x, t)(t-u)^{p-k}\, dt \right| \\ &= O\left(w^{-k}\{\phi(w)\}^{k+1}\right). \end{aligned} \tag{4.7}$$

To the "inner" integral in the expression defining $I_4(w)$, apply the transformation:

$$\begin{cases} x = u + y \\ t = u + vy. \end{cases}$$

Hence, in view of Lemma 5 (ii), and since $\int_0^1 v^{p-k}(1-v)^{k-p-1}\, dv$ is finite,

$$\begin{aligned} I_4(w) &= O\left\{ \underset{a<u<w}{\overline{\text{bd}}} \int_0^{w-u} \left| d_y \int_0^1 \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} v^{p-k}(1-v)^{k-p-1}\, dv \right| \right\} \\ &= O\left\{ \underset{a<u<w}{\overline{\text{bd}}}\ \underset{0<v<1}{\overline{\text{bd}}} \int_0^{w-u} \left| d_y \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} \right| . \int_0^1 v^{p-k}(1-v)^{k-p-1}\, dv \right\} \\ &= O\left\{ \underset{a<u<w}{\overline{\text{bd}}}\ \underset{0<v<1}{\overline{\text{bd}}} \int_0^{w-u} \left| d_y \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} \right| \right\} \\ &= O\left\{ \underset{a<u<w}{\overline{\text{bd}}}\ \underset{0<v<1}{\overline{\text{bd}}} \int_0^{w-u} \left| d_y P(y, u, v) S_0(y, u, v) \right| \right\}, \end{aligned}$$

say, where

$$S_0(y, u, v) = (u+vy)^{-k}\{\phi(u+vy)\}^{k-p}\{\phi(u+y)\}^{p+1},$$

and

$$P(y, u, v) = P_{\mu, r, m}(y, u, v)$$

is the product of

$$S_1(y, u, v) = \left\{\frac{y(1-v)\phi'(u+vy)}{\phi(u+y)-\phi(u+vy)}\right\}^{p+1-k}$$

$$S_2(y, u, v) = \left\{1-\frac{\phi(u+vy)}{\phi(u+y)}\right\}^{p+1-\mu}\left\{\frac{\phi(u+vy)}{\phi(u+y)}\right\}^{\mu}$$

$$S_3(y, u, v) = \frac{\phi^{(p+1-r-m)}(u+vy)}{\phi'(u+vy)}\left\{\frac{\phi(u+vy)}{\phi'(u+vy)}\right\}^{p-r-m}$$

$$S_4(y, u, v) = \prod_{\nu=1}^{r}\left(\frac{\phi^{(\nu)}(u+vy)}{\phi'(u+vy)}\left\{\frac{\phi(u+vy)}{\phi'(u+vy)}\right\}^{\nu-1}\right)^{\alpha_\nu}$$

and

$$S_5(y, u, v) = \psi^{(m)}(u+vy)\left\{\frac{\phi'(u+vy)}{\phi(u+vy)}\right\}^{k-m}(u+vy)^k$$

where

$$0 \leqslant \sum_{\nu=1}^{r} \alpha_\nu = \mu \leqslant \sum_{\nu=1}^{r} \nu\alpha_\nu = r \leqslant p+1,$$

and

$$0 \leqslant m \leqslant p+1-r.$$

Now, in view of Lemma 1 (v), it is clear that $S_0(y, u, v)$ is a monotonic non-decreasing function of $y$ in the range $[0, w-u]$ for $0 < v < 1$ and $u \geqslant a$, and its total variation with respect to $y$ in that range is, at most, $\{\phi(w)\}^{k+1}w^{-k}$. From condition (ii) of Theorem 1, we can deduce that both $\frac{t\phi''(t)}{\phi'(t)}$ and $\frac{t\phi'(t)}{\phi(t)}$ are non-negative monotonic non-decreasing functions of $t$ for $t \geqslant a$, and hence, that the results of Lemma 6 hold under the hypotheses of Theorem 1. Thus $P_{\mu,r,m}(y, u, v)$ is of uniformly bounded variation with respect to $y$ in the range $[0, w-u]$ for $0 < v < 1$ and $u \geqslant a$, since each function $S_i(y, u, v)$ $(i = 1, 2, 3, 4, 5)$ is uniformly bounded and of uniformly bounded variation with respect to $y$ in the range $[0, w-u]$ for $0 < v < 1$ and $u \geqslant a$; $S_1(y, u, v)$, because of Lemma 6 (i) since $p+1-k > 0$; $S_2(y, u, v)$, because of Lemma 6 (ii) since $p+1-\mu \geqslant 0$ and $\mu \geqslant 0$; $S_3(y, u, v)$ and $S_4(y, u, v)$, because of Lemma 1 (v); and $S_5(y, u, v)$, because of Lemma 7. Hence, we can deduce that

$$\begin{aligned}
I_4(w) &= O\left\{\underset{a<u<w}{\overline{\mathrm{bd}}}\ \underset{0<v<1}{\overline{\mathrm{bd}}}\int_0^{w-u}|d_y\,P(y, u, v)\,S_0(y, u, v)|\right\} \\
&= O\left\{\underset{a<u<w}{\overline{\mathrm{bd}}}\ \underset{0<v<1}{\overline{\mathrm{bd}}}\left[\underset{0<y<w-u}{\overline{\mathrm{bd}}}|P(y, u, v)|\int_0^{w-u}|d_y\,S_0(y, u, v)|\right.\right. \\
&\qquad\left.\left. +\underset{0<y<w-u}{\overline{\mathrm{bd}}}|S_0(y, u, v)|\int_0^{w-u}|d_y\,P(y, u, v)|\right]\right\} \\
&= O\left(\{\phi(w)\}^{k+1}w^{-k}\right).
\end{aligned}$$

That is, we have deduced that (4.7) is true, and so, that (4.6) is true. This completes the proof of Theorem 1.

5. *Proof of Theorem* 2.

Again, we assume without loss of generality, that the sum of the series is zero, and that $A(w) = 0$ for $0 \leqslant w \leqslant a$.

Since $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$, in view of Theorem A, we have that $\sum_{n=1}^{\infty} a_n$ is summable $(R, \lambda, k)$, and so, since the conditions of the theorem are sufficient to prove that $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $(R, \lambda, k)$ [**3**; remark following Theorem 2], in view of Lemma 4, it is sufficient for the proof of Theorem 2, to show that, for $w \geqslant a$,

$$\int_a^w |G_{k-1}(x)|\, dx = o(w^{k+1}), \tag{5.1}$$

where

$$G_k(x) = \int_a^x (x-t)^k \eta(t)\, dA(t)$$

where we define $\eta(t) = t\psi(t)$, and hence

$$\eta^{(n)}(t) = O(t^{1-n}) \text{ for } n = 0, 1, \ldots, p+1. \tag{5.2}$$

Also, in view of Lemma 3, since the sum of the series is zero, we again have that

$$\int_a^w |A_{k-1}(t)|\, dt = o(w^k). \tag{5.3}$$

(i) First, suppose that $k$ is a positive integer. Now $G_{k-1}(x)$ can be expressed as a sum of constant multiples of

$$A_{k-1}(x)\eta(x)$$

and

$$\int_a^x A_{k-1}(t)\left(\frac{\partial}{\partial t}\right)^k \{(x-t)^{k-1}\eta(t)\}\, dt.$$

Now, in view of (5.3) and (5.2),

$$\int_a^w |A_{k-1}(x)\eta(x)|\, dx = O\left\{\int_a^w x|A_{k-1}(x)|\, dx\right\}$$

$$= o(w^{k+1}).$$

Also, in view of Leibnitz's theorem on the differentiation of a product,

$$\int_a^x A_{k-1}(t)\left(\frac{\partial}{\partial t}\right)^k \{(x-t)^{k-1}\eta(t)\}\, dt$$

can be expressed as a sum of constant multiples of integrals of the types

$$J_m(x) = \int_a^x A_{k-1}(t)\,\eta^{(k-m)}(t)(x-t)^{k-1-m}\,dt$$

where $m = 0, 1, \ldots, k-1$.

Hence, in view of (5.2) and (5.3), for $m = 0, 1, \ldots, k-2$,

$$J_m(x) = O\left\{\int_a^x |A_{k-1}(t)|\, t^{1-k+m}(x-t)^{k-1-m}\,dt\right\}$$

$$= O\left\{\int_a^x \left[\int_a^t |A_{k-1}(u)|\,du\right] \frac{\partial}{\partial t}\{t^{1-k+m}(x-t)^{k-1-m}\}\,dt\right\}$$

$$= o\left\{\int_a^x t^k\,\frac{x(x-t)^{k-m-2}}{t^{k-m}}\,dt\right\}$$

$$= o\left\{x\int_a^x t^m(x-t)^{k-m-2}\,dt\right\}$$

$= o(x^k)$, the result being trivially true for $m = k-1$.

Hence $\int_a^w |J_m(x)|\,dx = o(w^{k+1})$, and so we can deduce that (5.1) is true. This completes the proof for integer values of $k$.

(ii) Suppose, now, that $k$ is any positive non-integral number. The relation, (5.1), that we must prove, can be written

$$\int_a^w |d_x G_k(x)| = o(w^{k+1}). \tag{5.4}$$

Integrating by parts $p+1$ times, and in view of Leibnitz's theorem on the differentiation of a product, $G_k(x)$ can be expressed as a sum of constant multiples of integrals of the types

$$K_m(x) = \int_a^x A_p(t)(x-t)^{k-p-1+m}\,\eta^{(m)}(t)\,dt$$

where $m = 0, 1, \ldots, p+1$.

Hence

$$\frac{\Gamma(k)\,\Gamma(p+1-k)}{\Gamma(p+1)} K_m(x) = \int_a^x (x-t)^{k-p-1+m}\,\eta^{(m)}(t)\,dt \int_a^t (t-u)^{p-k} A_{k-1}(u)\,du$$

$$= \int_a^x A_{k-1}(u)\,du \int_u^x (x-t)^{k-p-1+m}(t-u)^{p-k}\,\eta^{(m)}(t)\,dt$$

$$= \int_a^x A_{k-1}(u)\,q_m(x, u)\,du,$$

say, where

$$q_m(x, u) = \int_u^x (x-t)^{k-p-1+m}(t-u)^{p-k}\eta^{(m)}(t)\,dt.$$

Now

$$\frac{\partial K_m(x)}{\partial x} = \frac{\Gamma(p+1)}{\Gamma(k)\,\Gamma(p+1-k)}\{A_{k-1}(x)\,q_m(x, x)+H_m(x)\},$$

where

$$H_m(x) = \int_a^x A_{k-1}(u)\,\frac{\partial q_m(x, u)}{\partial x}\,du.$$

Now, for $m = 1, 2, \ldots, p+1$, $q_m(x, x) = 0$, but

$$q_0(x, x) = \Gamma(p+1-k)\,\Gamma(k-p)\,\eta(x) = O(x),$$

in view of (5.2). Hence, in view of (5.3),

$$\int_a^w |A_{k-1}(x)\,q_m(x, x)|\,dx = O\left\{\int_a^w x|A_{k-1}(x)|\,dx\right\}$$

$$= o(w^{k+1}). \tag{5.5}$$

To establish the truth of (5.4), it now remains to show that

$$\int_a^w |H_m(x)|\,dx = o(w^{k+1}). \tag{5.6}$$

Consider, first, $H_0(x)$. Set $t = u+(x-u)v$. Hence

$$q_0(x, u) = \int_0^1 v^{p-k}(1-v)^{k-p-1}\,\eta(u+\overline{x-u}\,v)\,dv$$

and so, in view of (5.2), that

$$\frac{\partial q_0(x, u)}{\partial x} = \int_0^1 v^{p-k+1}(1-v)^{k-p-1}\,\eta'(u+\overline{x-u}\,v)\,dv$$

$$= O(1),$$

since $k-p-1 > -1$ and $p-k+1 > 0$. Hence

$$H_0(x) = O\left\{\int_a^x |A_{k-1}(u)|\,du\right\} = o(x^k),$$

and so

$$\int_a^w |H_0(x)|\,dx = o(w^{k+1}). \tag{5.7}$$

Consider next $H_1(x)$. We have that

$$q_1(x, u) = \int_u^x (x-t)^{k-p}(t-u)^{p-k}\,\eta'(t)\,dt,$$

and so, in view of (5.2),

$$\frac{\partial q_1(x,u)}{\partial x} = (k-p)\int_u^x (x-t)^{k-p-1}(t-u)^{p-k}\eta'(t)\,dt$$

$$= O(1),$$

since $k-p-1 > -1$ and $p-k > -1$. Hence

$$H_1(x) = O\left\{\int_a^x |A_{k-1}(u)|\,du\right\} = o(x^k),$$

and so

$$\int_a^w |H_1(x)|\,dx = o(w^{k+1}). \tag{5.8}$$

Consider, finally, $H_m(x)$ for $m = 2, 3, \ldots, p+1$. Now

$$q_m(x, u) = \int_u^x (x-t)^{k-p-1+m}(t-u)^{p-k}\eta^{(m)}(t)\,dt,$$

and so, in view of (5.2),

$$\frac{\partial q_m(x, u)}{\partial x} = (k-p-1+m)\int_u^x (x-t)^{k-p-2+m}(t-u)^{p-k}\eta^{(m)}(t)\,dt$$

$$= O\left\{\int_u^x (x-t)^{k-p-2+m}(t-u)^{p-k}t^{1-m}\,dt\right\}$$

$$= O\left\{\left(\frac{x}{u}\right)^{m-1}\int_u^x (x-t)^{k-p-1}(t-u)^{p-k}\,dt\right\}$$

$$= O\left\{\left(\frac{x}{u}\right)^{m-1}\right\},$$

since $k-p-1 > -1$ and $p-k > -1$. Consequently, in view of Lemma 3 and (5.3), since $k+1-m > 0$,

$$H_m(x) = O\left\{\int_a^x \left(\frac{x}{u}\right)^{m-1}|A_{k-1}(u)|\,du\right\}$$

$$= O\left\{x^{m-1}\int_a^x u^{1-m}|A_{k-1}(u)|\,du\right\}$$

$$= o\{x^{m-1}.x^{k+1-m}\} = o(x^k),$$

and hence

$$\int_a^w |H_m(x)|\,dx = o(w^{k+1}). \tag{5.9}$$

Thus, in view of (5.7), (5.8) and (5.9), we can deduce that (5.6) is true, and so, in conjunction with (5.5), that (5.4) is true. This completes the proof of Theorem 2.

6. *Proof of Theorem* 3.

In view of Theorem 2 and Corollary C, the proof of Theorem 3 is immediate.

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