LINEAR FUNCTIONALS CONNECTED WITH STRONG CESÀRO SUMMABILITY

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1. *Introduction*

The object of this note is to characterise all linear functionals on the normed linear spaces w_p and W_p defined below. By a linear functional we mean one that is real-valued, additive, homogeneous [1; p. 27] and continuous. It is to be supposed throughout that $\infty > p \geqslant 1$ and that $1/p + 1/q = 1$.

Definitions. 1. w_p is the space of real sequences $x = \{x_n\}$ for which there is a number $l = l_x$ such that

$$\sum_{n=1}^{N} |x_n - l|^p = o(N),$$

with norm

$$\|x\| = \sup_{N \geqslant 1} \left(\frac{1}{N} \sum_{n=1}^{N} |x_n|^p \right)^{1/p}.$$

2. W_p is the space of real valued functions x, measurable (in the Lebesgue sense) in the interval $(1, \infty)$, for which there is a number $l = l_x$ such that

$$\int_1^T |x(t) - l|^p \, dt = o(T),$$

with norm

$$\|x\| = \sup_{T \geqslant 1} \left(\frac{1}{T} \int_1^T |x(t)|^p \, dt \right)^{1/p}.$$

3. Given any real sequence $\alpha = \{\alpha_n\}$ we define a sequence $\{m_n(\alpha, p)\}$ by:

$$m_n(\alpha, p) = \begin{cases} \sup\limits_{2^n \leqslant \nu < 2^{n+1}} |\nu \alpha_\nu| & \text{if } p = 1, \\ \left(\dfrac{1}{2^n} \sum\limits_{\nu = 2^n}^{2^{n+1}-1} |\nu \alpha_\nu|^q \right)^{1/q} & \text{if } p > 1. \end{cases}$$

4. Given any real valued function α measurable in $(1, \infty)$ we define a sequence $\{M_n(\alpha, p)\}$ by:

$$M_n(\alpha, p) = \begin{cases} \operatorname{ess.sup}\limits_{2^n < t < 2^{n+1}} |t\alpha(t)| & \text{if } p = 1, \\ \left(\dfrac{1}{2^n} \displaystyle\int_{2^n}^{2^{n+1}} |t\alpha(t)|^q \, dt \right)^{1/q} & \text{if } p > 1. \end{cases}$$

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The spaces w_p and W_p are intimately linked with the strong Cesàro summability method $[C, 1]_p$. In fact, a sequence $x = \{x_n\} \in w_p$ and $l = l_x$ if and only if $x_n \to l[C, 1]_p$; and any function in W_p can be similarly characterised.

We prove two theorems.

Theorem 1. (i) *If f is a linear functional on w_p, then there is a real number a and a real sequence sequence $\alpha = \{\alpha_n\}$ such that*

$$f(x) = al_x + \sum_{n=1}^{\infty} \alpha_n x_n \tag{1}$$

for every $x = \{x_n\} \in w_p$ and

$$\sum_{n=0}^{\infty} m_n(\alpha, p) < \infty. \tag{2}$$

(ii) *If a is a real number and $\alpha = \{\alpha_n\}$ is a real sequence satisfying* (2), *then* (1) *defines a linear functional f on w_p with*

$$\|f\| \leqslant |a| + 2^{1/p} \sum_{n=0}^{\infty} m_n(\alpha, p),$$

and the series in (1) *is absolutely convergent for every $x = \{x_n\} \in w_p$.*

Theorem 2. (i) *If f is a linear functional on W_p, then there is a real number a and a real-valued function α, measurable in $(1, \infty)$, such that*

$$f(x) = al_x + \int_1^{\infty} \alpha(t)\, x(t)\, dt \tag{3}$$

for every $x \in W_p$ and

$$\sum_{n=0}^{\infty} M_n(\alpha, p) < \infty. \tag{4}$$

(ii) *If a is a real number and α is a real valued function, measurable in $(1, \infty)$, satisfying* (4), *then* (3) *defines a linear functional f on W_p with*

$$\|f\| \leqslant |a| + 2^{1/p} \sum_{n=0}^{\infty} M_n(\alpha, p),$$

and the integral in (3) *is absolutely convergent for every $x \in W_p$.*

Theorem 2 will be proved first. I am indebted to the referee for pointing out that the first part of Theorem 1 can be deduced from Theorem 2.

2. *Proof of Theorem* 2

Part (i). Let L_p be the linear space of real-valued functions x measurable in $(1, \infty)$ for which

$$\int_1^{\infty} |x(t)|^p \, dt < \infty,$$

with norm

$$\|x\|_{L_p} = \left(\int_1^{\infty} |x(t)|^p \, dt \right)^{1/p}.$$

Clearly, if $x \in L_p$, then $x \in W_p$, $l_x = 0$ and

$$\|x\| = \|x\|_{W_p} \leqslant \|x\|_{L_p}.$$

Consequently the restriction to L_p of the given linear functional f on W_p is linear on L_p. It follows from standard results [1; pp. 64–65] that there is a real-valued function α, measurable in $(1, \infty)$, such that

$$f(x) = \int_1^\infty \alpha(t)\, x(t)\, dt \tag{5}$$

for all $x \in L_p$ and either

$$\operatorname*{ess.sup}_{1<t<\infty} |\alpha(t)| < \infty \quad \text{if} \quad p = 1,$$

or

$$\int_1^\infty |\alpha(t)|^q\, dt < \infty \quad \text{if} \quad p > 1.$$

To show that α must necessarily satisfy (4) we consider the cases $p = 1$ and $p > 1$ separately.

Case a. $p = 1$. Let $M_n = M_n(\alpha, 1)$. There is a measurable set e_n of positive measure $|e_n|$ in the interval $(2^n, 2^{n+1})$ such that

$$|t\alpha(t)| > M_n - \frac{1}{2^n}$$

for all $t \in e_n$. Let

$$x(t) = \begin{cases} \dfrac{2^n}{|e_n|} \operatorname{sign} \alpha(t) & \text{if} \quad t \in e_n,\ n \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in L_1$ and so, by (5),

$$\|f\|\|x\| \geqslant f(x) = \int_1^\infty x(t)\, \alpha(t)\, dt = \sum_{n=0}^r \int_{e_n} \frac{2^n}{|e_n|} |\alpha(t)|\, dt$$

$$\geqslant \tfrac{1}{2} \sum_{n=0}^r \frac{1}{|e_n|} \int_{e_n} |t\,\alpha(t)|\, dt \geqslant \tfrac{1}{2} \sum_{n=0}^r \left(M_n - \frac{1}{2^n}\right). \tag{6}$$

Further, for $2^s \leqslant T < 2^{s+1} \leqslant 2^{r+1}$,

$$\frac{1}{T} \int_1^T |x(t)|\, dt \leqslant \frac{1}{2^s} \int_1^{2^{s+1}} |x(t)|\, dt = \frac{1}{2^s} \sum_{n=0}^s \int_{e_n} |x(t)|\, dt \leqslant \frac{1}{2^s} \sum_{n=0}^s 2^n < 2,$$

and, for $T > 2^{r+1}$,

$$\frac{1}{T} \int_1^T |x(t)|\, dt \leqslant \frac{1}{2^{r+1}} \int_1^{2^{r+1}} |x(t)|\, dt < 1.$$

Hence $\|x\| < 2$ and so, by (6),

$$2\|f\| + \tfrac{1}{2} \sum_{n=0}^\infty \frac{1}{2^n} = 2\|f\| + 1 \geqslant \tfrac{1}{2} \sum_{n=0}^\infty M_n,$$

which establishes (4) in this case.

Case (b). $p > 1$. Let $M_n = M_n(\alpha, p)$ and let

$$x(t) = \begin{cases} \dfrac{t^q}{2^n} \left|\dfrac{\alpha(t)}{M_n}\right|^{q/p} \operatorname{sign} \alpha(t) & \text{if} \quad 2^n \leqslant t < 2^{n+1} \leqslant 2^{r+1} \quad \text{and} \quad M_n \neq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Then $x \in L_p$ and so, by (5),

$$f(x) = \int_1^{2^{r+1}} |\alpha(t)\, x(t)|\, dt = \sum_{n=0}^r \int_{2^n}^{2^{n+1}} |\alpha(t)\, x(t)|\, dt$$

$$= \sum_{n=0}^r M_n \leqslant \|f\|\|x\|. \tag{7}$$

Further, for $2^s \leqslant T < 2^{s+1} \leqslant 2^{r+1}$,

$$\frac{1}{T} \int_1^T |x(t)|^p\, dt \leqslant \frac{1}{2^s} \int_1^{2^{s+1}} |x(t)|^p\, dt = \frac{1}{2^s} \sum_{n=0}^s \int_{2^n}^{2^{n+1}} |x(t)|^p\, dt$$

$$\leqslant 2^{p-s} \sum_{n=0}^s 2^n < 2^{p+1},$$

and, for $T \geqslant 2^{r+1}$,

$$\frac{1}{T} \int_1^T |x(t)|^p\, dt \leqslant \frac{1}{2^{r+1}} \int_1^{2^{r+1}} |x(t)|^p\, dt < 2^p.$$

Hence $\|x\| < 2^{1+1/p}$, and so, by (7),

$$\sum_{n=0}^\infty M_n \leqslant 2^{1+1/p} \|f\|,$$

which established (4) in Case (b).

Suppose now that $p \geqslant 1$, $M_n = M_n(\alpha, p)$ and $x \in W_p$. Then, by Hölders inequality,

$$\int_1^\infty |\alpha(t)\, x(t)|\, dt = \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} |\alpha(t)\, x(t)|\, dt$$

$$\leqslant \sum_{n=0}^\infty M_n \left(2^{p(1-1/p)n} \int_{2^n}^{2^{n+1}} \left|\frac{x(t)}{t}\right|^p dt\right)^{1/p}$$

$$\leqslant \sum_{n=0}^\infty M_n \left(2^{-n} \int_{2^n}^{2^{n+1}} |x(t)|^p\, dt\right)^{1/p}$$

$$\leqslant 2^{1/p} \|x\| \sum_{n=0}^\infty M_n. \tag{8}$$

It follows that $\int_1^\infty |\alpha(t)\, x(t)|\, dt < \infty$ whenever $x \in W_p$, and in particular, since the characteristic function of $(1, \infty)$ is in W_p, that

$$\int_1^\infty |\alpha(t)|\, dt < \infty.$$

Suppose next that $x \in W_p$ and $l = l_x$. Let

$$y(t) = x(t) - l,$$

$$y_n(t) = \begin{cases} y(t) & \text{for } 1 \leqslant t \leqslant n, \\ 0 & \text{for } t > n. \end{cases}$$

Then $y \in W_p$, $y_n \in L_p$ and

$$\|y_n - y\| = \sup_{T \geqslant n} \left(\frac{1}{T} \int_n^T |x(t) - l|^p \right)^{1/p} = o(1) \text{ as } n \to \infty.$$

But

$$|f(y_n - y)| = |f(y_n) - f(y)| \leqslant \|y_n - y\| \, \|f\|,$$

and so, by (5),

$$f(y) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \int_1^n y(t)\, \alpha(t)\, dt = \int_1^\infty x(t)\, \alpha(t)\, dt - l \int_1^\infty \alpha(t)\, dt,$$

since both integrals on the right-hand side have been shown to be absolutely convergent. Taking δ to be characteristic function of $(1, \infty)$, we see that

$$f(x) = f(y + l\delta) = f(y) + lf(\delta) = \int_1^\infty x(t)\, \alpha(t)\, dt + al,$$

where

$$a = f(\delta) - \int_1^\infty \alpha(t)\, dt.$$

This completes the proof of Part (i) of the theorem.

Part (ii). It follows from (8) that if $x \in W_p$, $l = l_x$ and $M_n = M_n(\alpha, p)$, then

$$|f(x)| = \left| \int_1^\infty \alpha(t)\, x(t)\, dt + al \right| \leqslant 2^{1/p} \|x\| \sum_{n=0}^\infty M_n + |al| \tag{9}$$

Further, by Minkowski's inequality,

$$(1 - 1/T)^{1/p} |l| \leqslant \left(\frac{1}{T} \int_1^T |x(t) - l|^p\, dt \right)^{1/p} + \left(\frac{1}{T} \int_1^T |x(t)|^p\, dt \right)^{1/p}$$

and the first term on the right-hand side is $o(1)$. Hence

$$|l| \leqslant \|x\|,$$

and consequently, by (9),

$$|f(x)| \leqslant \|x\| \left(|a| + 2^{1/p} \sum_{n=0}^\infty M_n \right)$$

for every $x \in W_p$.

The additive and homogeneous functional f defined by (3) is therefore also continuous on W_p and

$$\|f\| \leqslant |a| + 2^{1/p} \sum_{n=0}^\infty M_n.$$

Finally, by (8), the integral in (3) is absolutely convergent, and the proof of Theorem 2 is thus complete.

3. *Proof of Theorem* 1

Part (i). Given any real sequence $x = \{x_n\}$ define a function x^* by

$$x^*(t) = x_n \text{ for } n < t \leqslant n+1,\ n = 1, 2, \ldots .$$

It is easily verified that this defines a one-one correspondence between w_p and a linear subspace $W_p{}^*$ of W_p such that

$$l_{x^*} = l_x \text{ and } \|x^*\| \leqslant \|x\| \leqslant 2^{1/p} \|x^*\|.$$

Hence, given a linear functional f on w_p, the functional f^* defined by

$$f^*(x^*) = f(x)$$

is linear on $W_p{}^*$. Consequently, by the Hahn-Banach theorem [**1**; pp. 27–28] and Theorem 2, there is a real number a and a real valued function α^*, integrable over $(1, \infty)$, such that

$$\sum_{n=0}^\infty M_n(\alpha^*, p) < \infty$$

and, for every $x \in w_p$,

$$f(x) = f^*(x^*) = al_{x^*} + \int_1^\infty \alpha^*(t)\, x^*(t)\, dt$$

$$= al_x + \sum_{n=1}^\infty \alpha_n x_n,$$

where

$$\alpha_n = \int_n^{n+1} \alpha^*(t)\, dt.$$

Further, for $\alpha = \{\alpha_n\}$,

$$\sum_{n=0}^\infty m_n(\alpha, p) \leqslant \sum_{n=0}^\infty M_n(\alpha^*, p);$$

and this completes the proof of Part (i).

Part (ii). If $x = \{x_n\} \in w_p$, $l = l_x$ and $m_n = m_n(\alpha, p)$, then, by Hölder's and Minkowski's inequalities, as in the proof of Part (ii) of Theorem 2,

$$f(x) = al + \sum_{n=1}^{\infty} \alpha_n x_n \leqslant |al| + \sum_{n=1}^{\infty} |\alpha_n x_n|$$

$$\leqslant |al| + 2^{1/p} \|x\| \sum_{n=0}^{\infty} m_n \leqslant \|x\| \left(|\alpha| + 2^{1/p} \sum_{n=0}^{\infty} m_n \right).$$

The functional f defined by (1) is therefore linear on W_p,

$$\|f\| \leqslant |a| + 2^{1/p} \sum_{n=0}^{\infty} m_n$$

and the series in (1) is absolutely convergent. This completes the proof of Theorem 1.

Reference

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).

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