ON A CLASS OF CONVERGENT SERIES OF POSITIVE TERMS

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Let $0 < u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n$, $s_n = \sum_1^n u_r$ $(n = 1, 2, \ldots)$.

In a recent issue of the *American Mathematical Monthly* [1964, p. 99] Hayman and Barry posed the problem of showing that:

$$\textit{if } u_n \leqslant n \textit{ and } \alpha > 1 \textit{ then } \sum_{n=1}^{\infty} \left(\frac{u_n}{s_n}\right)^{\alpha} < \infty.$$

The object of this note is to prove the following twofold generalisation of the above result.

Theorem. *If* (i) $u_n \leqslant n k_n$ *where* $k_n \geqslant 1$, (ii) $x f(x)$ *is positive and monotonic non-increasing for* $x \geqslant a > 0$ *and*

(iii)
$$\int_a^\infty f(x)\,dx < \infty,$$

then
$$\sum_{n=n_0}^{\infty} k_n f\left(\frac{k_n s_n}{u_n}\right) < \infty. \tag{1}$$

Here and subsequently n_0 denotes a suitably large positive integer. We first prove a simple lemma of independent interest.

Lemma. *If* $d_n \geqslant 0$, $D_n = \sum_1^n d_r \to \infty$ *and* $f(x)$ *is a positive monotonic non-increasing function for* $x \geqslant a$ *which satisfies* (iii), *then*

$$\sum_{n=n_0}^{\infty} d_n f(D_n) < \infty.$$

This is an immediate consequence of the inequality

$$d_n f(D_n) \leqslant \int_{D_{n-1}}^{D_n} f(x)\,dx \quad (D_{n-1} \geqslant a).$$

Proof of the theorem.

Case 1. $\lim u_n < \infty$. In this case $s_n/u_n > \tfrac{1}{2}n$ for all large n and hence, by (ii) and the lemma with $d_n = \tfrac{1}{2}$,

$$\sum_{n=n_0}^{\infty} k_n f\left(\frac{k_n s_n}{u_n}\right) \leqslant \sum_{n=n_0}^{\infty} f\left(\frac{s_n}{u_n}\right) \leqslant \sum_{n=n_0}^{\infty} f\left(\frac{n}{2}\right) < \infty.$$

Case 2. $\lim u_n = \infty$. Let

$$d_1 = u_1, \quad d_n = \frac{s_n}{n} - \frac{s_{n-1}}{n-1} = \frac{n u_n - s_n}{n(n-1)} \quad (n \geqslant 2).$$

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Then $d_n \geqslant 0$ and $D_n = \sum_1^n d_r = \frac{s_n}{n} \to \infty$. Let N_1 and N_2 be the sets of positive integers $n \geqslant n_0$ such that

$$\frac{s_n}{u_n} > \frac{n}{2} \text{ when } n \in N_1,$$

$$\frac{s_n}{u_n} \leqslant \frac{n}{2} \text{ when } n \in N_2.$$

Then

$$\sum_{n \in N_1} k_n f\left(\frac{k_n s_n}{u_n}\right) \leqslant \sum_{n \in N_1} f\left(\frac{s_n}{u_n}\right) \leqslant \sum_{n \in N_1} f\left(\frac{n}{2}\right) < \infty. \tag{2}$$

Further, for $n \in N_2$,

$$d_n f(D_n) \geqslant \frac{u_n}{2n} f\left(\frac{s_n}{n}\right) \geqslant \tfrac{1}{2} k_n f\left(\frac{k_n s_n}{u_n}\right)$$

and so, by the lemma,

$$\sum_{n \in N_2} k_n f\left(\frac{k_n s_n}{u_n}\right) < \infty. \tag{3}$$

The required conclusion (1) follows from (2) and (3).

Remark. If the sequence $\{k_n\}$ of the theorem is bounded then conclusion (1) can evidently be replaced by

$$\sum_{n=n_0}^{\infty} f\left(\frac{s_n}{u_n}\right) < \infty. \tag{1$'$}$$

On the other hand, given an unbounded sequence $\{k_n\}$ such that

$$1 \leqslant k_1 \leqslant 2k_2 \leqslant \ldots \leqslant nk_n,$$

we can construct a sequence $\{u_n\}$ satisfying the hypotheses of the theorem for which (1)′ is false. Choose a strictly increasing sequence of integers $\{n_\nu\}$ such that

$$n_1 = 1, \quad k_{n_{\nu+1}} \geqslant n_\nu k_{n_\nu} \quad (\nu = 1, 2, \ldots),$$

and put $\qquad u_n = n_\nu k_{n_\nu}$ for $n_\nu \leqslant n < n_{\nu+1}, \quad \nu = 1, 2, \ldots$.

Then $0 < u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n \leqslant nk_n$ and, for $m = 2, 3, \ldots$,

$$s_{n_m} - u_{n_m} = \sum_{\nu=1}^{m-1} (n_{\nu+1} - n_\nu) n_\nu k_{n_\nu} \leqslant n_m k_{n_m} = u_{n_m}.$$

Hence, for $n = n_m$, $m = 2, 3, \ldots$,

$$\frac{s_n}{u_n} \leqslant 2$$

and so (1)′ must be false.

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