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1. *Introduction.* Let $\{\lambda_n\}$ be a strictly increasing unbounded sequence with $\lambda_0 \geqslant 0$, let

$$\mu = p + \delta \quad (p = 0,\ 1, \ldots;\ 0 \leqslant \delta < 1)$$

and let $\sum_{n=0}^{\infty} a_n$ be an arbitrary series. We write

$$A^{\mu}(w) = \sum_{\lambda_\nu < w} (w - \lambda_\nu)^{\mu}\, a_\nu,$$

$$\pi_n^{\mu}(t) = \begin{cases} (\lambda_{n+1} - t)^{\delta} & (p = 0), \\ (\lambda_{n+1+p} - t)^{\delta} \prod_{i=1}^{p} (\lambda_{n+i} - t) & (p \geqslant 1), \end{cases}$$

$$C_n^{\mu} = \sum_{\nu=0}^{n} \pi_n^{\mu}(\lambda_\nu)\, a_\nu,$$

and say that the series $\Sigma a_n$ is summable to $s$ by

(i) the Riesz method $(R, \lambda, \mu)$, if $w^{-\mu} A^{\mu}(w) \to s$ as $w \to \infty$,

(ii) the discrete Riesz method $(R^*, \lambda, \mu)$, if $\lambda_n^{-\mu} A^{\mu}(\lambda_n) \to s$,

(iii) the generalised Cesàro method $(C, \lambda, \mu)$, if $C_n^{\mu} / \pi_n^{\mu}(0) \to s$.

The summability methods $(R^*, \lambda, \mu)$ and $(C, \lambda, \mu)$ are identical when $0 \leqslant \mu \leqslant 1$. For integral values of the parameter $\mu$, the method $(C, \lambda, \mu)$, which is essentially the same as methods defined independently by Jurkat[1] and Burkill[2], reduces to the standard Cesàro method $(C, \mu)$ when $\lambda_n = n$. Burkill considered only the integral case but Jurkat extended his definition to non-integral values of the parameter in a manner different from the above.

1. Jurkat (7).
2. Burkill (4).

The inclusions

$$(1)\qquad (C, \lambda, \mu) \subseteq (R, \lambda, \mu),$$

$$(2)\qquad (R, \lambda, \mu) \subseteq (C, \lambda, \mu)$$

are known to be valid under various hypotheses on the sequence $\{\lambda_n\}$ and the parameter $\mu$. The most general results to date are as follows:

Russell[1] has proved that (1) holds (i.e. that every series summable $(C, \lambda, \mu)$ to $s$ is summable $(R, \lambda, \mu)$ to $s$) without any restriction on $\{\lambda_n\}$ when $\mu$ is an integer, and that (2) holds provided

$$(3)\qquad \Lambda_{n-1} = O(\Lambda_n), \quad \Lambda_n = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}$$

when $\mu = 3, 4, \ldots$ and unrestrictedly when $\mu = 0, 1, 2$. I have shown[2] that (2) holds if $\mu$ is an integer and

$$(4)\qquad \lambda_{n+1} = O(\lambda_n).$$

In the special case $\lambda_n = n$, I have proved[3] both (1) and (2) to hold for all $\mu \geqslant 0$ by showing $(C, \lambda, \mu)$ to be equivalent to $(C, \mu)$, well-known to be equivalent to $(R, \lambda, \mu)$.

Since the submission of this paper for publication, it has been shown that (2) holds without restriction on $\{\lambda_n\}$, for $\mu$ an integer by Meir[4] and, subsequently, for all $\mu \geqslant 0$ by Russell and myself[5]. (Added May 10, 1967).

For non-integral values of $\mu$, inclusion (1) is more difficult to deal with than (2). In the range $0 \leqslant \mu \leqslant 1$, (1) is the same as the inclusion

$$(5)\qquad (R^*, \lambda, \mu) \subseteq (R, \lambda, \mu)$$

and Jurkat[6] has shown (5) to hold without restriction on $\{\lambda_n\}$ in the said range. Peyerimhoff[7] has established the validity of (5) in the range $1 < \mu < \log 3/\log 2 = 1{\cdot}58\ldots$ subject to the conditions

$$\frac{\lambda_{n+1}}{\lambda_n} \downarrow \qquad \left(\text{i.e. } \frac{\lambda_{n+1}}{\lambda_n} \geqslant \frac{\lambda_{n+2}}{\lambda_{n+1}}\right)$$

and

$$\frac{\Delta\lambda_{n+1}}{\Delta\lambda_n} \to 1, \qquad \Delta\lambda_n = \lambda_n - \lambda_{n+1};$$

while Kuttner[8] has shown that, when $\lambda_n = n$, (5) holds for $1 < \mu < 2$ but fails

1. Russell (12).
2. Borwein (1).
3. Borwein (1).
4. Meir (10).
5. Borwein and Russell (3).
6. Jurkat (6).
7. Peyerimhoff (11).
8. Kuttner (9).



for every $\mu \geqslant 2$. The question whether or not (5) continues to hold for more general sequences $\{\lambda_n\}$ when $\log 3/\log 2 \leqslant \mu < 2$, remains open.

The main object of this paper is to prove that inclusion (1) holds whenever $1 < \mu < 2$, provided the sequence $\{\lambda_n\}$ satisfies

$$(6)\qquad \frac{\lambda_{n+1}}{\lambda_n} \downarrow$$

and

$$(7)\qquad \frac{\Delta\lambda_{n+1}}{\Delta\lambda_n} \downarrow .$$

Note that (6) implies both (3) and (4).

It is hoped that, by extending the arguments employed below, it will be possible to establish (1) for non-integral $\mu > 2$ under reasonably light restrictions on $\{\lambda_n\}$.

2. *Auxiliary results.* Let $(c_{n,\nu})$ be a normal matrix (i.e. $c_{n,\nu} = 0$ for $\nu > n$ and $c_{n,n} \neq 0$) and let

$$\sigma_n = \sum_{\nu=0}^{n} c_{n,\nu}\, s_\nu \quad (n = 0, 1, \ldots).$$

The following three lemmas, incorporating key results required in the rest of the paper, are due essentially to Jurkat and Peyerimhoff[1]. The proof of Lemma 2 is straightforward[2] and Lemma 3 is an immediate consequence of Lemmas 1 and 2.

LEMMA 1. *If*

$$(8)\qquad c_{n,\nu} > 0 \quad (0 \leqslant \nu \leqslant n),$$

$$(9)\qquad \frac{c_{n,\nu}}{c_{n-1,\nu}} \leqslant \frac{c_{n,\nu-1}}{c_{n-1,\nu-1}} \quad (1 \leqslant \nu \leqslant n-1),$$

*then the matrix* $(c_{n,\nu})$ *satisfies the "mean value" condition*

$$(10)\qquad \left| \sum_{\nu=0}^{m} c_{n,\nu}\, s_\nu \right| \leqslant \max_{0 \leqslant r \leqslant m} \frac{c_{n,0}}{c_{r,0}} \,|\sigma_r| \quad (0 \leqslant m \leqslant n).$$

LEMMA 2. *If* $\xi_n > 0$, $\sigma_n = o(\xi_n)$ *and*

$$(11)\qquad c_{n,0} = o(\xi_n),$$

$$(12)\qquad 0 < \frac{c_{n,0}}{\xi_n} \leqslant M \frac{c_{r,0}}{\xi_r} \quad (0 \leqslant r \leqslant n,\ M \text{ a positive constant}),$$

1. Jurkat and Peyerimhoff (8, Satz 1 and Satz 3; see also Peyerimhoff, 11, p. 71).
2. Jurkat and Peyerimhoff (8), p. 98.

then $$\max_{0\leqslant r\leqslant n} \frac{c_{n,0}}{c_{r,0}}\,|\sigma_r| = o(\xi_n).$$

LEMMA 3. *If* $\xi_n>0$, $\sigma_n=o(\xi_n)$ *and the matrix* $(c_{n,\nu})$ *satisfies conditions* (10), (11) *and* (12), *then*
$$s_n = o\left(\frac{\xi_n}{c_{n,n}}\right).$$

We require one additional lemma.

LEMMA 4. *The method* $(C, \lambda, \mu)$ *is regular for every* $\mu\geqslant 0$.

The case $0\leqslant\mu\leqslant 1$ of this lemma is well-known, and Russell[1] has proved it for $\mu$ an integer. Let
$$s_n=\sum_{\nu=0}^{n} a_\nu;$$
then
$$\frac{C_n^\mu}{\pi_n^\mu(0)} = \sum_{\nu=0}^{n} \gamma_{n,\nu}\, s_\nu,$$
where
$$\gamma_{n,\nu} = \frac{1}{\pi_n^\mu(0)}\{\pi_n^\mu(\lambda_\nu)-\pi_n^\mu(\lambda_{\nu+1})\}\geqslant 0 \qquad (0\leqslant\nu\leqslant n).$$
Now, for any fixed $\nu\geqslant 0$,
$$\frac{\pi_n^\mu(\lambda_\nu)}{\pi_n^\mu(0)}\to 1 \text{ as } n\to\infty,$$
so that
$$\gamma_{n,\nu}\to 0 \text{ as } n\to\infty;$$
and
$$\sum_{\nu=0}^{n} |\gamma_{n,\nu}| = \sum_{\nu=0}^{n} \gamma_{n,\nu} = \frac{\pi_n^\mu(\lambda_0)}{\pi_n^\mu(0)} \to 1.$$

It follows, by a standard result[2], that $C_n^\mu/\pi_n^\mu(0)\to s$ whenever $s_n\to s$, i.e. that $(C, \lambda, \mu)$ is regular.

3. *The main results.* Suppose throughout this section that
$$0<\delta<1.$$
In addition to the notations introduced in §1, we shall also use the following:
$$s_n=\sum_{\nu=0}^{n} a_\nu,$$
$$A(w)=\sum_{\lambda_\nu<w} a_\nu,$$
$$k_n=\lambda_{n+1}-\lambda_n=-\Delta\lambda_n.$$

1. Russell (12).
2. Hardy (5), p. 43.



$$c_n(t) = -\frac{d}{dt}\{(\lambda_{n+1}-t)(\lambda_{n+2}-t)^\delta-(\lambda_n-t)(\lambda_{n+1}-t)^\delta\}$$
$$= -\frac{d}{dt}\{\pi_n^{1+\delta}(t)-\pi_{n-1}^{1+\delta}(t)\} \qquad (0\leqslant t<\lambda_{n+1},\ n\geqslant 0),$$
$$c_{n,\nu}=\int_{\lambda_\nu}^{\lambda_{\nu+1}} c_n(t)\,dt \ (0\leqslant\nu\leqslant n), \quad c_{n,\nu}=0 \ (\nu>n).$$

Then

(13) $$C_n^{1+\delta}-C_{n-1}^{1+\delta}=\int_0^{\lambda_{n+1}}\{\pi_n^{1+\delta}(t)-\pi_{n-1}^{1+\delta}(t)\}\,dA(t)=\int_0^{\lambda_{n+1}} c_n(t)\,A(t)\,dt$$
$$= \sum_{\nu=0}^{n} c_{n,\nu}\, s_\nu \qquad (n=0, 1, \ldots).$$

Note that

(14) $$c_{n,n}=(\lambda_{n+1}-\lambda_n)(\lambda_{n+2}-\lambda_n)^\delta > k_n^{1+\delta}$$

and that

(15) $$-\frac{d}{dt}\pi_{n-1}^{1+\delta}(t) = (1+\delta)(\lambda_{n+1}-t)^\delta - \delta k_n(\lambda_{n+1}-t)^{\delta-1} \quad (0\leqslant t<\lambda_{n+1},\ n\geqslant 0).$$

THEOREM 1. *If* (6) *and* (7) *hold and if*

(16) $$\xi_n>0, \quad \frac{\lambda_{n+1}^{\ \delta}}{\xi_n}\downarrow 0,$$

*then the normal matrix* $(c_{n,\nu})$ *satisfies conditions* (8), (9), (11) *and* (12).

*Proof.* Let
$$k(u)=u-\lambda_{n+1}+\frac{k_n}{k_{n+1}}(\lambda_{n+2}-u) \qquad (\lambda_{n+1}\leqslant u<\lambda_{n+2},\ n\geqslant 0);$$
$$\phi=\phi(u,t)=(1+\delta)(u-t)^\delta-\delta k(u)(u-t)^{\delta-1}$$
$$\psi=\psi(u,t)=\frac{\partial\phi}{\partial t} \qquad (0\leqslant t<u,\ u\geqslant\lambda_1).$$

Then $k(\lambda_{n+1})=k_n$ $(n=0, 1, \ldots)$, and hence, in view of (15),

(17) $$c_n(t)=\phi(\lambda_{n+2}, t)-\phi(\lambda_{n+1}, t) \qquad (0\leqslant t<\lambda_{n+1},\ n\geqslant 0).$$

Further,

(18) $$\frac{\partial\phi}{\partial u}=\delta\left(\delta+\frac{k_n}{k_{n+1}}\right)(u-t)^{\delta-1}+\delta(1-\delta)k(u)(u-t)^{\delta-2}>0$$
$$(0\leqslant t<\lambda_{n+1}<u<\lambda_{n+2}).$$

It follows from (17) and (18) that $c_n(t)>0$ $(0\leqslant t<\lambda_{n+1})$ and hence that $c_{n,\nu}>0$ $(0\leqslant\nu\leqslant n)$, i.e. that the matrix satisfies condition (8).

Differentiating (18) with respect to $t$, we get

$$\text{(19)}\qquad \frac{\partial\psi}{\partial u}=\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial u}\right)=\frac{1-\delta}{u-t}\left\{\frac{\partial\phi}{\partial u}+\delta k(u)(u-t)^{\delta-2}\right\}\ (0\leqslant t<\lambda_{n+1}<u<\lambda_{n+2}).$$

Since $$\frac{c_{n,\nu}}{c_{n-1,\nu}}=\frac{c_n(t_\nu)}{c_{n-1}(t_\nu)}\quad(\lambda_\nu<t_\nu<\lambda_{\nu+1},\ 0\leqslant\nu\leqslant n-1),$$

inequality (9) will be established if we can show that

$$\frac{c_n(t)}{c_{n-1}(t)}$$

decreases as $t$ increases from 0 to $\lambda_n$, and to do this it suffices to prove

$$\text{(20)}\qquad \frac{c_n'(t)}{c_n(t)}\leqslant\frac{c_{n-1}'(t)}{c_{n-1}(t)}\quad(0\leqslant t<\lambda_n,\ n\geqslant 1).$$

In view of (17), (18) and (19), we have

$$\text{(21)}\qquad \frac{c_n'(t)}{c_n(t)}=\frac{\psi(\lambda_{n+2},t)-\psi(\lambda_{n+1},t)}{\phi(\lambda_{n+2},t)-\phi(\lambda_{n+1},t)}=\left[\frac{\partial\psi}{\partial u}\Big/\frac{\partial\phi}{\partial u}\right]_{u=u_n}$$

$$=\frac{1-\delta}{u_n-t}\left[1+\frac{1}{1-\delta+\left(\delta+\dfrac{k_n}{k_{n+1}}\right)\dfrac{u_n-t}{k(u_n)}}\right]$$

$$(0\leqslant t<\lambda_{n+1}<u_n<\lambda_{n+2}).$$

Now, for fixed $t\geqslant 0$,

$$\text{(22)}\qquad \frac{k(u)}{u-t}\ \text{decreases with increasing}\ u>\min(t,\lambda_1),$$

since, when $$0\leqslant t<u,\ \lambda_{n+1}\leqslant u<\lambda_{n+2},\ n\geqslant 0,$$

$$\frac{k(u)}{u-t}=1-\frac{k_n}{k_{n+1}}+\frac{1}{u-t}\left\{\frac{k_n}{k_{n+1}}(\lambda_{n+2}-t)-(\lambda_{n+1}-t)\right\}$$

and, in virtue of (6),

$$\frac{k_n}{k_{n+1}}\geqslant\frac{\lambda_{n+1}}{\lambda_{n+2}}\geqslant\frac{\lambda_{n+1}-t}{\lambda_{n+2}-t}.$$

Inequality (20) follows from (7), (21) and (22); and it remains to establish that the matrix satisfies (11) and (12).

In view of (17) and (18), we have

$$\text{(23)}\qquad c_{n,0}=k_0\,c_n(v_n)=k_0\,k_{n+1}\left.\frac{\partial\phi}{\partial u}\right]_{u=w_n,\ t=v_n}$$

$$=\delta k_0(w_n-v_n)^{\delta-1}k_n\left\{1+\frac{k_{n+1}}{k_n}\left(\delta+(1-\delta)\frac{k(w_n)}{w_n-v_n}\right)\right\}$$

$$(\lambda_0<v_n<\lambda_1<\lambda_{n+1}<w_n<\lambda_{n+2}\ n\geqslant 1).$$



Since $$\lambda_{n+2}>\lambda_{n+2}-v_n>\lambda_{n+1}-v_n>\lambda_{n+1}-\lambda_1\geqslant\lambda_{n+1}\frac{k_1}{\lambda_2}\ (n\geqslant 1),$$

it follows from (22) and (23) that

$$\text{(24)}\qquad \delta k_0\lambda_{n+2}{}^{\delta-1}k_n\left\{1+\frac{k_{n+1}}{k_n}\left(\delta+(1-\delta)\frac{k_{n+1}}{\lambda_{n+2}}\right)\right\}$$

$$\leqslant c_{n,0}$$

$$\leqslant\delta k_0\left(\frac{\lambda_2}{k_1}\right)^{2-\delta}\lambda_{n+1}{}^{\delta-1}k_n\left\{1+\frac{k_{n+1}}{k_n}\left(\delta+(1-\delta)\frac{k_n}{\lambda_{n+1}}\right)\right\}(n\geqslant 1).$$

Hence, by (7), (16) and (24),

$$c_{n,0}=O\left\{\xi_n\frac{\lambda_{n+1}{}^\delta}{\xi_n}\left(1+\frac{k_{n+1}}{k_n}\right)\right\}=o(\xi_n),$$

i.e. the matrix $(c_{n,\nu})$ satisfies (11).

Finally, by (6), (7) and (16),

$$\frac{\lambda_{n+1}}{\lambda_n}\downarrow,\ \frac{k_n}{\lambda_{n+1}}\downarrow,\ \frac{k_{n+1}}{k_n}\downarrow,\ \frac{\lambda_{n+1}{}^\delta}{\xi_n}\downarrow,$$

and so it follows from (24) that, for $0\leqslant r\leqslant n-1$,

$$\frac{c_{n,0}}{\xi_n}\leqslant\delta k_0\left(\frac{\lambda_2}{k_1}\right)^{2-\delta}\frac{\lambda_{r+1}{}^\delta}{\xi_r}\frac{k_r}{\lambda_{r+1}}\left\{1+\frac{k_{r+1}}{k_r}\left(\delta+(1-\delta)\frac{k_{r+1}}{\lambda_{r+2}}\right)\right\}$$

$$\leqslant\left(\frac{\lambda_2}{k_1}\right)^{2-\delta}\left(\frac{\lambda_{r+2}}{\lambda_{r+1}}\right)^{1-\delta}\frac{c_{r,0}}{\xi_r}\leqslant\left(\frac{\lambda_2}{k_1}\right)^{2-\delta}\left(\frac{\lambda_2}{\lambda_1}\right)^{1-\delta}\frac{c_{r,0}}{\xi_r},$$

showing that the matrix $(c_{n,\nu})$ satisfies (12).

In what follows, we suppose that

$$x=x(w)\quad(w>\lambda_0)$$

is the integer such that

$$\lambda_x<w\leqslant\lambda_{x+1}.$$

THEOREM 2. *If (6), (7) and (16) hold, if $\xi_n/k_n\uparrow$, and if*

$$\text{(25)}\qquad C_n{}^{1+\delta}=o(\xi_n),$$

*then*

$$\text{(26)}\qquad s_n=o\left(\frac{\xi_n}{k_n{}^{1+\delta}}\right),$$

$$\text{(27)}\qquad A^\delta(w)=o\left(\frac{\xi_x}{k_x}\right)\quad(w\to\infty),$$

$$\text{(28)}\qquad A^{1+\delta}(w)=o(\xi_x)\quad(w\to\infty).$$

*Proof.* Note first that, by (16), $\xi_n\uparrow$.

In virtue now of (13) and (25), we have

$$\sigma_n=\sum_{\nu=0}^{n} c_{n,\nu}\, s_\nu = C_n^{1+\delta} C_{n-1}^{1+\delta}=o(\xi_n), \tag{29}$$

from which it follows, by Theorem 1, Lemmas 1 and 3, and (14), that

$$s_n=o\left(\frac{\xi_n}{c_{n,n}}\right)=o\left(\frac{\xi_n}{k_n^{1+\delta}}\right),$$

i.e. that (26) holds.

Suppose now that $w>\lambda_3$, so that $x=x(w)\geqslant 3$. Then

$$A^\delta(w)-\delta\int_{\lambda_{x-1}}^{w}(w-t)^{\delta-1}A(t)\,dt=\delta\int_0^{\lambda_{x-1}}(w-t)^{\delta-1}A(t)\,dt \tag{30}$$

$$=\sum_{\nu=0}^{x-2} b_{x,\nu}\, s_\nu,$$

where
$$b_{x,\nu}=\delta\int_{\lambda_\nu}^{\lambda_{\nu+1}}(w-t)^{\delta-1}dt \quad (0\leqslant\nu\leqslant x-2).$$

In view of (26), we have

$$\delta\left|\int_{\lambda_{x-1}}^{w}(w-t)^{\delta-1}A(t)\,dt\right|\leqslant\delta|s_{x-1}|\int_{\lambda_{x-1}}^{\lambda_x}(w-t)^{\delta-1}dt+\delta|s_x|\int_{\lambda_x}^{w}(w-t)^{\delta-1}dt \tag{31}$$

$$\leqslant|s_{x-1}|k_{x-1}^{\delta}+|s_x|k_x^{\delta}=o\left(\frac{\xi_x}{k_x}\right)\ (w\to\infty).$$

Next we observe that

$$\frac{b_{x,\nu}}{c_{x-2,\nu}}=\frac{\delta(w-t_\nu)^{\delta-1}}{c_{x-2}(t_\nu)}\ (\lambda_\nu<t_\nu<\lambda_{\nu+1},\ 0\leqslant\nu\leqslant x-2). \tag{32}$$

As $t$ increases from 0 to $\lambda_{x-1}$,

$$\chi(t)=\frac{\delta(w-t)^{\delta-1}}{c_{x-2}(t)}$$

decreases, since, by (21),

$$\frac{\chi'(t)}{\chi(t)}=\frac{1-\delta}{w-t}-\frac{c_{x-2}'(t)}{c_{x-2}(t)}<\frac{1-\delta}{w-t}-\frac{1-\delta}{\lambda_x-t}<0;$$

and, therefore, it follows from (32) that

$$\frac{b_{x,\nu}}{c_{x-2,\nu}}\leqslant\frac{b_{x,\nu-1}}{c_{x-2,\nu-1}}\ (1\leqslant\nu\leqslant x-2).$$



Consequently, by (30) and partial summation,

$$\delta\left|\int_0^{\lambda_{x-1}}(w-t)^{\delta-1}A(t)\,dt\right|=\left|\sum_{\nu=0}^{x-2}\frac{b_{x,\nu}}{c_{x-2,\nu}}\,c_{x-2,\nu}\,s_\nu\right| \tag{33}$$

$$\leqslant\frac{b_{x,0}}{c_{x-2,0}}\max_{0\leqslant m\leqslant x-2}\left|\sum_{\nu=0}^{m}c_{x-2,\nu}\,s_\nu\right|.$$

Now, by (24) and (32),

$$\frac{b_{x,0}}{c_{x-2,0}}\leqslant\frac{\delta(\lambda_x-\lambda_1)^{\delta-1}}{\delta^2 k_0\lambda_x^{\delta-1}k_{x-1}}\leqslant\frac{1}{\delta k_0}\left(\frac{\lambda_2}{k_1}\right)^{1-\delta}\frac{1}{k_{x-1}}. \tag{34}$$

Also, by Theorem 1 and Lemmas 1 and 2, it follows from (29) that

$$\max_{0\leqslant m\leqslant x-2}\left|\sum_{\nu=0}^{m}c_{x-2,\nu}\,s_\nu\right|\leqslant\max_{0\leqslant r\leqslant x-2}\frac{c_{x-2,0}}{c_{r,0}}|\sigma_r|=o(\xi_{x-2}). \tag{35}$$

Hence, by (33), (34) and (35),

$$\int_0^{\lambda_{x-1}}(w-t)^{\delta-1}A(t)\,dt=o\left(\frac{\xi_{x-1}}{k_{x-1}}\right)\ (w\to\infty); \tag{36}$$

and conclusion (27) follows from (30), (31), and (36).

We are now in a position to establish (28), the one outstanding conclusion. By (35) and (27),

$$A^{1+\delta}(\lambda_n)=C_{n-2}^{1+\delta}+k_{n-1}A^\delta(\lambda_n)=o(\xi_{n-2})+o\left(k_{n-1}\frac{\xi_{n-1}}{k_{n-1}}\right)=o(\xi_{n-1}),$$

and consequently, by (27) again,

$$A^{1+\delta}(w)=A^{1+\delta}(\lambda_x)+(1+\delta)\int_{\lambda_x}^{w}A^\delta(t)\,dt$$

$$=o(\xi_{x-1})+o\left(k_x\frac{\xi_x}{k_x}\right)=o(\xi_x)\ (w\to\infty).$$

THEOREM 3. *If* (6) *and* (7) *hold, and if* $C_n^{1+\delta}=o(\lambda_{n+1}\lambda_{n+2}^{\delta})$, *then*

$$A^{1+\delta}(w)=o(w^{1+\delta})\ (w\to\infty).$$

*Proof.* Take $\xi_n=\lambda_{n+1}\lambda_{n+2}^{\delta}$; then

$$\frac{\xi_n}{\lambda_{n+1}^{\delta}}\to\infty \text{ and } \frac{\xi_{n+1}}{\lambda_{n+2}^{\delta}}\,\frac{\lambda_{n+1}^{\delta}}{\xi_n}=\left(\frac{\lambda_{n+3}}{\lambda_{n+2}}\right)^{\delta}\left(\frac{\lambda_{n+2}}{\lambda_{n+1}}\right)^{1-\delta}>1,$$

so that $\xi_n$ satisfies (16). Also, by (6), $\xi_n/k_n\uparrow$. Consequently, by Theorem 2,

$$A^{1+\delta}(w)=o(\lambda_{x+1}\lambda_{x+2}^{\delta})=o(w^{1+\delta})\,(w\to\infty),$$

since, in view of (6),

$$\frac{\lambda_{x+1}\lambda_{x+2}{}^{\delta}}{w^{1+\delta}} < \frac{\lambda_{x+1}}{\lambda_x}\left(\frac{\lambda_{x+2}}{\lambda_x}\right)^{\delta} = O(1).$$

Since $(R, \lambda, \mu)$ is known to be regular and $(C, \lambda, \mu)$ is regular, by Lemma 4, the following theorem is an immediate consequence of Theorem 3.

THEOREM 4. *If* $\frac{\lambda_{n+1}}{\lambda_n}\downarrow$, $\frac{\Delta\lambda_{n+1}}{\Delta\lambda_n}\downarrow$ *and* $1<\mu<2$, *then*

$$(C, \lambda, \mu) \subseteq (R, \lambda, \mu).$$

## REFERENCES


1. D. BORWEIN, "On a generalised Cesàro summability method of integral order", *Tôhoku Math. Jour.*, **18** (1966), 71-73.
2. D. BORWEIN, "On a method of summability equivalent to the Cesàro method", *Jour. London Math. Soc.*, **42** (1967), 339-343.
3. D. BORWEIN AND D.C. RUSSELL, "On Riesz and generalized Cesàro summability of arbitrary positive order," *Math. Zeit.*, 99 (1967), 171-177.
4. H. BURKILL, "On Riesz and Riemann summability", *Proc. Cambridge Phil. Soc.*, **57** (1961), 55-60.
5. G. H. HARDY, *Divergent Series* (Oxford, 1949).
6. W. B. JURKAT, "Über Rieszsche Mittel mit unstetigem Parameter", *Math. Zeit.*, **55** (1951), 8-12.
7. W. B. JURKAT, "Über Rieszsche Mittel und verwandte Klassen von Matrixtransformationen", *Math. Zeit.*, **57** (1953), 353-394.
8. W. B. JURKAT AND A. PEYERIMHOFF, "Mittelwertsätze bei Matrix-und Integraltransformationen", *Math. Zeit.*, **55** (1951), 92-108.
9. B. KUTTNER, "On discontinuous Riesz means of type *n*", *Jour. London Math. Soc.*, **37** (1962), 354-364.
10. A. MEIR, "An inclusion theorm for generalized Cesàro and Riesz means", *Candian Jour. Math.* (to appear).
11. A. PEYERIMHOFF, "On discontinuous Riesz means", *Indian Jour. Math.*, **6** (1964), 69-91.
12. D. C. RUSSELL, "On generalised Cesàro means of integral order", *Tôhoku Math. Jour.*, **17** (1965), 410-442.


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