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Strong Nörlund Summability

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1. Introduction

In this paper we give a definition of strong Nörlund summability, and show that in the case of Cesàro summability our definition is equivalent to the standard definition of strong Cesàro summability. We answer such questions as: "If one Nörlund method of summability includes another, is the same true of the associated strong methods?". We establish relations between strong Nörlund, absolute Nörlund and Nörlund summability. For a certain class of Nörlund methods of summability, we construct associated one parameter families of Nörlund methods, the methods in each family increasing in strength as the parameter increases. A special case is the family of Cesàro methods of summability (C, α) with $\alpha > -1$. Finally we consider a method of summability (C^*, α), known to be equivalent to (C, α) for $\alpha > 0$, and use our theorems to show that the associated strong methods of summability are equivalent.

2. Generalities and Definitions

Throughout this paper, H, H_1 etc. will denote positive constants, which will not necessarily take the same value at different occurences.

Suppose throughout that

$$s_n = \sum_{r=0}^{n} a_r, \qquad \varepsilon_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)\dots(\alpha+n)}{n!}.$$

Given an arbitrary sequence $\{w_n\}$, we define

$$\Delta w_n = w_n - w_{n-1}, \qquad w_{-1} = 0.$$

Let $\{p_n\}$ and $\{q_n\}$ be arbitrary sequences of complex numbers, let

$$P_n = \sum_{r=0}^{n} p_r, \qquad Q_n = \sum_{r=0}^{n} q_r,$$

and assume throughout that P_n and Q_n are non-zero for all values of n. Also let

$$P_n^* = \sum_{r=0}^{n} |p_r|, \qquad Q_n^* = \sum_{r=0}^{n} |q_r|.$$

The n-th (N, p_n) and n-th (N, q_n) transforms of the sequence $\{s_n\}$ are respectively

(2.1) $$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_{n-r} = \frac{1}{P_n} \sum_{r=0}^{n} P_r a_{n-r}$$

and

(2.2) $$u_n = \frac{1}{Q_n} \sum_{r=0}^{n} q_r s_{n-r} = \frac{1}{Q_n} \sum_{r=0}^{n} Q_r a_{n-r}.$$

The series

$$\sum_{r=0}^{\infty} a_r$$

is said to be summable (N, p_n) to s if $t_n \to s$ as $n \to \infty$. We denote this by

(2.3) $$\sum_{r=0}^{\infty} a_r = s(N, p_n) \quad \text{or} \quad s_n \to s(N, p_n).$$

All limits in the sequel will be taken as the variable tends to infinity, unless otherwise specified.

We shall also have occasion, in the case that $p_n \neq 0$ and $q_n \neq 0$ for all values of n, to use the notation

(2.4) $$t_n^\Delta = \frac{1}{p_n} \sum_{r=0}^{n} P_r a_{n-r}$$

and

(2.5) $$u_n^\Delta = \frac{1}{q_n} \sum_{r=0}^{n} Q_r a_{n-r}.$$

We note that (2.4) and (2.5) are respectively the n-th $(N, \Delta p_n)$ and n-th $(N, \Delta q_n)$ transforms of the sequence $\{s_n\}$.

A method of summability is *regular*, if it sums every convergent series to its ordinary sum.

If P and Q are methods of summability, Q is said to *include* P (written "$P \Rightarrow Q$") if every series summable by the method P is also summable by the method Q to the same sum. P and Q are said to be *equivalent* (written "$P \Leftrightarrow Q$") if each method includes the other.

Using standard results, [3, Theorem 2], we find that the Nörlund method (N, p_n) is regular if and only if

(2.6) $$P_n^* = O(|P_n|)$$

and

(2.7) $$p_n / P_n \to 0.$$

See also [4].

Thus for a regular Nörlund method (N, p_n), either,

(2.8) $$\sum_{r=0}^{\infty} |p_r| < \infty$$

or

(2.9) $$|P_n| \to \infty .$$

Since p_0 and q_0 are both non-zero, there exist sequences $\{k_n\}$, $\{l_n\}$ and $\{\gamma_n\}$ such that,

(2.10) $$k_0 p_n + \cdots + k_n p_0 = q_n, \qquad n=0,1,2,\ldots,$$

(2.11) $$l_0 q_n + \cdots + l_n q_0 = p_n, \qquad n=0,1,2,\ldots,$$

(2.12) $$\gamma_0 p_n + \cdots + \gamma_n p_0 = 0, \qquad n=1,2,3,\ldots,$$

(2.13) $$\gamma_0 = 1/p_0 .$$

Thus

$$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_{n-r},$$

if and only if

$$s_n = \sum_{r=0}^{n} \gamma_{n-r} P_r t_r .$$

Therefore every Nörlund transformation is invertible.

The following propositions give necessary and sufficient conditions for inclusion or equivalence relations to hold between two Nörlund methods. The proofs are closely modelled on proofs given by HARDY [3, Theorems 19 and 21], but are applicable to a larger class of Nörlund methods than are considered by HARDY. In connection with the second proposition see also [4, Corollary 1].

Proposition 1. *For Nörlund methods (N, p_n) and (N, q_n), (not necessarily regular), necessary and sufficient conditions that $(N, p_n) \Rightarrow (N, q_n)$ are*

(2.14) $$|k_0|\,|P_n| + \cdots + |k_n|\,|P_0| \leqq H\,|Q_n|$$

where H is independent of n, and

(2.15) $$k_{n-r}/Q_n \to 0, \qquad \textit{for each } r.$$

Proof. Referring to (2.2), we have

(2.16) $$u_n = \sum_{r=0}^{\infty} c_{n,r} t_r,$$

with $c_{n,r} = k_{n-r} P_r/Q_n$ for $r \leqq n$ and $c_{n,r}=0$ for $r>n$. cf. [3, Theorem 19].

Since a Nörlund method is invertible. $(N, p_n) \Rightarrow (N, q_n)$ if and only if the sequence to sequence transformation (2.16) is regular. A standard result now yields (2.14) and (2.15). See for example [3, Theorem 2].

Proposition 2. *For regular Nörlund methods (N, p_n) and (N, q_n), necessary and sufficient conditions that $(N, p_n) \Leftrightarrow (N, q_n)$ are*

(2.17) $$\sum_{r=0}^{\infty} |k_r| < \infty \quad \textit{and} \quad \sum_{r=0}^{\infty} |l_r| < \infty .$$

Proof. Necessity.

By Proposition 1 both $\{|P_n|/|Q_n|\}$ and $\{|Q_n|/|P_n|\}$ are bounded. Also by (2.14),

$$|k_0| + |k_1| \frac{|P_{n-1}|}{|P_n|} + \cdots + |k_r| \frac{|P_{n-r}|}{|P_n|} \leqq H\,|Q_n|/|P_n|$$

for $r \leqq n$. Now fixing r, letting n tend to infinity, and using the fact that for a regular Nörlund method (N, p_n), $P_{n-r} \sim P_n$ for each r, we have

$$|k_0| + \cdots + |k_r| \leqq H \limsup(|Q_n|/|P_n|) = H_1 < \infty ,$$

so that

$$\sum_{n=0}^{\infty} |k_n| < \infty .$$

Similarly

$$\sum_{n=0}^{\infty} |l_n| < \infty ,$$

and, hence we have (2.17).

Sufficiency.

We now have $k_{n-r} \to 0$ for each r, and, because of the regularity of, (N, q_n), $|Q_n| \geqq H Q_n^* \geqq H Q_0^* > 0$: it follows that (2.15) holds. Also, by (2.6) and its analogue for the method (N, q_n),

$$|P_n| \leqq |Q_0|\,|l_n| + \cdots + |Q_n|\,|l_0| \leqq H\,|Q_n| \sum_{r=0}^{\infty} |l_r| ,$$

and

$$|k_0|\,|P_n| + \cdots + |k_n|\,|P_0| \leqq H_1\,|Q_n| \sum_{r=0}^{\infty} |k_r| \sum_{r=0}^{\infty} |l_r| .$$

Thus (2.14) holds, and hence $(N, p_n) \Rightarrow (N, q_n)$. Similarly, $(N, p_n) \Rightarrow (N, q_n)$, and the proof is complete.

The n-th $(\overline{N}, p_n)$ transform of the sequence $\{s_n\}$ is

(2.18) $$\tau_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_r .$$

Associated with this transform is a method of summability, $(\overline{N}, p_n)$, defined in the same way as in the case of (N, p_n).

Using a standard result [3, Theorem 2], we find that $(\overline{N}, p_n)$ is regular if and only if (2.6) and (2.9) hold.

Proposition 3. *If $(\overline{N}, p_n)$ is regular, $p_n > 0$ for all n and either*

$$\{p_n\} \textit{ is non-decreasing and } \quad n \leqq H_1 \frac{P_n}{p_n}$$

or

$$\{p_n\} \textit{ is non-increasing and } \quad P_n/p_n \leqq H_2(n+1)$$

then

$$(\overline{N}, p_n) \Leftrightarrow (\overline{N}, 1).$$

Proof. This result is an immediate consequence of a theorem given by HARDY [3, Theorem 14].

Definitions. 1. *Strong summability* $[N, p_n]_\lambda$, $\lambda > 0$.

Let (N, p_n) be a Nörlund method with $p_n \neq 0$ for all values of n. We shall say that

$$\sum_{r=0}^{\infty} a_r$$

is strongly summable (N, p_n) *with index* λ *to* s, if

$$\frac{1}{|P_n|} \sum_{r=0}^{n} |p_r| \, |t_r^{\Delta} - s|^{\lambda} = o(1). \tag{2.19}$$

We shall denote this by

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda \quad \text{or} \quad s_n \to s[N, p_n]_\lambda .$$

Remark. Whenever (2.6) holds,

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda$$

if and only if

$$\frac{1}{P_n^*} \sum_{r=0}^{n} |p_r| \, |t_r^{\Delta} - s|^{\lambda} = o(1). \tag{2.20}$$

We shall take advantage of this result without further comment.

2. *Absolute summability* $|N, p_n|_\lambda$, $\lambda > 0$.

We shall say that

$$\sum_{r=0}^{\infty} a_r$$

is absolutely summable (N, p_n) *with index*, λ, or summable $|N, p_n|_\lambda$, if

$$\sum_{n=1}^{\infty} n^{\lambda-1} |t_n - t_{n-1}|^{\lambda} < \infty . \tag{2.21}$$

When $\lambda = 1$, this definition reduces to the customary definition of absolute Nörlund summability, as given by MEARS [5] for example. See also [1].

We recall now the standard definition of strong Cesàro summability $[C, \alpha+1]_\lambda$ and show that is it equivalent to our definition. For $\lambda > 0$, $\alpha > -1$, the series

$$\sum_{r=0}^{\infty} a_r$$

is said to be summable $[C, \alpha+1]_\lambda$ to s, if

$$\frac{1}{n+1} \sum_{r=0}^{n} |s_r^{\alpha} - s|^{\lambda} = o(1)$$

where

$$s_n^{\alpha} = \frac{1}{\varepsilon_n^{\alpha}} \sum_{r=0}^{n} \varepsilon_{n-r}^{\alpha-1} s_r .$$

Since the Cesàro method of summability $(C, \alpha+1)$ is the method (N, p_n) with $p_n = \varepsilon_n^{\alpha}$, in order to show that our definition of strong Cesàro summability as strong Nörlund summability is equivalent to the standard definition of strong Cesàro summability it suffices to observe that $(\overline{N}, 1) \Leftrightarrow (\overline{N}, \varepsilon_n^{\alpha})$ for $\alpha > -1$. This follows from Proposition 3, because $\{\varepsilon_n^{\alpha}\}$ is non-increasing when $-1 < \alpha \leqq 0$, non-decreasing when $\alpha > 0$, and

$$\varepsilon_n^{\alpha+1} / \varepsilon_n^{\alpha} \sim n/(\alpha+1) \qquad \text{as } n \to \infty .$$

3. Inclusion Theorems

In this section we shall prove certain theorems giving sufficient conditions for one strong Nörlund method of summability to include another. Before doing so however, we make the following simplifying remark.

Remark. If (N, p_n) is a Nörlund method with $p_n \neq 0$ for all values of n, and $\{s_n\}$ is a sequence, then $t_n^{\Delta} - s$ is the n-th $(N, \Delta p_n)$ transform of the sequence $\{s_n - s\}$. Thus we have,

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda$$

if and only if $s_n - s \to 0[N, p_n]_\lambda$. Hence in order to prove that $[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda$, it is sufficient to prove that $s_n \to 0[N, p_n]_\lambda$ implies $s_n \to 0[N, q_n]_\lambda$. For the remainder of this paper, we shall assume that if (N, p_n) is a Nörlund method, then $p_n \neq 0$ for all values of n, unless mention is made to the contrary.

Theorem 1. *If* $(N, p_n) \Rightarrow (N, q_n)$ *then* $[N, p_n]_1 \Rightarrow [N, q_n]_1$.

Proof. Referring to (2.4) and (2.5), we have, for a given sequence $\{s_n\}$,

$$q_n u_n^{\Delta} = \sum_{r=0}^{n} k_{n-r} p_r t_r^{\Delta}.$$

Thus

$$|q_r| \, |u_r^{\Delta}| \leqq \sum_{v=0}^{r} |k_{r-v}| \, |p_v| \, |t_v^{\Delta}|$$

and hence

$$\sum_{r=0}^{n} |q_r| \, |u_r^{\Delta}| \leqq \sum_{r=0}^{n} \sum_{v=0}^{r} |k_v| \, |p_{r-v}| \, |t_{r-v}^{\Delta}|$$

$$= \sum_{v=0}^{n} |k_v| \sum_{r=v}^{n} |p_{r-v}| \, |t_{r-v}^{\Delta}| .$$

Setting

$$\varphi_n = \sum_{v=0}^{n} |p_v| \, |t_v^{\Delta}| ,$$

we see that

$$\text{(a)} \qquad \sum_{r=0}^{n} |q_r|\,|u_r^{\Delta}| \leqq \sum_{\nu=0}^{n} |k_\nu|\,\varphi_{n-\nu} = \sum_{\nu=0}^{n} |k_{n-\nu}|\,|P_\nu|\,\frac{\varphi_\nu}{|P_\nu|}.$$

Supposing now, that $s_n \to 0\,[N, p_n]_1$, we have $\varphi_n/|P_n| = o(1)$. Thus

$$\text{(b)} \qquad \frac{1}{|Q_n|} \sum_{\nu=0}^{n} |k_{n-\nu}|\,|P_\nu|\,\frac{\varphi_\nu}{|P_\nu|} = o(1),$$

provided

$$|k_0|\,|P_n| + \cdots + |k_n|\,|P_0| = O(|Q_n|)$$

and

$$|k_{n-\nu}| = o(|Q_n|) \qquad \text{for each } \nu.$$

But, by Proposition 1 this is equivalent to our hypothesis $(N, p_n) \Rightarrow (N, q_n)$. It follows from (a) and (b) that $s_n \to 0\,[N, q_n]_1$.

Thus

$$[N, p_n]_1 \Rightarrow [N, q_n]_1,$$

and the proof is complete.

Corollary 1. *If $(N, p_n) \Leftrightarrow (N, q_n)$, then $[N, p_n]_1 \Leftrightarrow [N, q_n]_1$.*

Theorem 2. *If $(N, p_n) \Rightarrow (N, q_n)$ and*

$$\text{(3.1)} \qquad \sum_{r=0}^{n} |k_{n-r}|\,|p_r| = O(|q_n|),$$

then

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \text{for } \lambda > 1.$$

Proof.

$$|q_r|^\lambda\,|u_r^{\Delta}|^\lambda \leqq \left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|\right)^\lambda.$$

Using HÖLDER's inequality, we obtain

$$|q_r|^\lambda\,|u_r^{\Delta}|^\lambda \leqq \left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda\right)\left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\right)^{\lambda-1}$$
$$\leqq H\,|q_r|^{\lambda-1} \sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda.$$

Thus

$$\sum_{r=0}^{n} |q_r|\,|u_r^{\Delta}|^\lambda \leqq H \sum_{r=0}^{n} \sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda.$$

To complete the proof we now set

$$\varphi_n = \sum_{r=0}^{n} |p_r|\,|t_r^{\Delta}|^\lambda,$$

and proceed as in the proof of Theorem 1.

Corollary 2. *If $Q_n^* = O(|Q_n|)$,*

$$\text{(3.2)} \qquad k_{n-r}/Q_n \to 0 \qquad \text{for each } r$$

and (3.1) holds, then

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \text{for } \lambda > 1.$$

Proof. By summing both sides of (3.1) and using the fact that $Q_n^* = O(|Q_n|)$ we find that,

$$\sum_{r=0}^{n} |k_{n-r}|\,|P_r| = O(|Q_n|),$$

which together with (3.2) implies that $(N, p_n) \Rightarrow (N, q_n)$. The result now follows from Theorem 2.

Corollary 3. *If (N, p_n) and (N, q_n) are regular Nörlund methods with $\{|p_n|\}$ and $\{|q_n|\}$ non-decreasing, and if $(N, p_n) \Leftrightarrow (N, q_n)$, then*

$$[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda \qquad \text{for } \lambda > 1.$$

Proof. By Proposition 2, we know that

$$\sum_{n=0}^{\infty} |k_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |l_n| < \infty.$$

Thus

$$\sum_{r=0}^{n} |k_{n-r}|\,|p_r| \leqq H\,|p_n|$$

and

$$|p_n| \leqq \sum_{r=0}^{n} |l_{n-r}|\,|q_r| \leqq H_1\,|q_n|.$$

Hence

$$\sum_{r=0}^{n} |k_{n-r}|\,|p_r| = O(|q_n|).$$

Similarly

$$\sum_{r=0}^{n} |l_{n-r}|\,|q_r| = O(|p_n|).$$

The conclusion now follows from Theorem 2.

As (3.1) is in general a difficult condition to check, we proceed now to obtain some theorems which guarantee that (3.1) is satisfied. We shall suppose now that the sequences $\{p_n\}$ and $\{q_n\}$ are sequences of positive terms. We begin by quoting part of a result proved by HARDY [3, Theorem 22].

If (N, p_n) is a regular Nörlund method and

$$\text{(3.3)} \qquad p_0 = 1, \quad p_n > 0, \quad p_{n+1}/p_n \geqq p_n/p_{n-1} \qquad (n > 0)$$

then $\gamma_0 = 1$ and $\gamma_n \leqq 0$ for all $n > 0$, where the γ_n are defined by (2.12) and (2.13).

Theorem 3. *If (N, q_n) is a regular Nörlund method, $\{p_n\}$ satisfies (3.3), $q_n>0$, $q_n/q_{n-1} \leqq p_n/p_{n-1}$ for $n>0$, and $p_n=O(q_n)$, then (N, p_n) is regular, (3.1) holds, and $(N, p_n) \Rightarrow (N, q_n)$.*

Proof.

$$\frac{1}{q_0} Q_n \leqq P_n$$

and

$$p_n \leqq H q_n \quad \text{so that} \quad \frac{p_n}{P_n} \leqq q_0 \frac{H q_n}{Q_n} \quad \text{and hence} \quad \frac{p_n}{P_n} \to 0.$$

The condition $P_n^* = O(|P_n|)$ is satisfied because $p_n>0$. Thus (N, p_n) is regular.

We write $\gamma_n = -c_n$ for $n>0$, so that, in view of the above mentioned result of HARDY, $c_n \geqq 0$ for $n>0$.

Now

$$p_n - c_1 p_{n-1} - \cdots - c_n p_0 = 0 \qquad \text{for } n>0$$

and by (2.10) and (2.11)

$$q_n - c_1 q_{n-1} - \cdots - c_n q_0 = k_n \qquad \text{for } n \geqq 0.$$

Hence

$$\frac{k_n}{q_n} = 1 - c_1 \frac{q_{n-1}}{q_n} - \cdots - c_n \frac{q_0}{q_n}$$

$$\leqq 1 - c_1 \frac{p_{n-1}}{p_n} - \cdots - c_n \frac{p_0}{p_n} = 0 \qquad \text{for } n>0$$

and thus

$$k_n \leqq 0 \qquad \text{for } n>0.$$

So we have

$$|k_0| p_n + \cdots + |k_n| p_0 = k_0 p_n - k_1 p_{n-1} - \cdots - k_n p_0$$
$$= 2k_0 p_n - k_0 p_n - \cdots - k_n p_0$$
$$= 2k_0 p_n - q_n \leqq 2k_0 p_n + q_n \leqq H_1 q_n.$$

Since $p_n = O(q_n)$. This proves (3.1). It follows by summing both sides of (3.1) that

$$\sum_{r=0}^{n} |k_{n-r}| P_r = O(Q_n).$$

Also we have

$$|k_n| p_0 \leqq |k_0| p_n + \cdots + |k_n| p_0 = O(q_n) = o(Q_n)$$

since (N, q_n) is regular. Further $0 < Q_n \leqq Q_{n+r}$ for $r>0$ and so

$$k_{n-r}/Q_n \to 0 \qquad \text{as } n \to \infty.$$

It follows now from Proposition 1 that $(N, p_n) \Rightarrow (N, q_n)$.

Corollary 4. *If (N, p_n) and (N, q_n) satisfy the hypotheses of Theorem 3, then*

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \textit{for } \lambda > 1.$$

Proof. This result is an immediate consequence of Theorems 2 and 3.

Theorem 4. *If (N, p_n) and (N, q_n) are regular Nörlund methods, $\{p_n\}$ satisfies (3.3), and $q_n>0$, $p_n/p_{n-1} \leqq q_n/q_{n-1}$ for $n>n_0$, then*

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \textit{for } \lambda > 1.$$

Proof. In the case $n_0=0$ the result is an immediate consequence of Theorem 2 and HARDY's Theorem 23 in [3], which yields $(N, p_n) \Rightarrow (N, q_n)$ and $k_n \geqq 0$ for all n, so that

$$|k_0| p_n + \cdots + |k_n| p_0 = k_0 p_n + \cdots + k_n p_0 = q_n.$$

For the general case we modify the second part of HARDY's proof of Theorem 23 in [3].

We have

$$p_n/p_{n-1} \leqq q_n/q_{n-1} \qquad \text{for } n = n_0+1,\ n_0+2, \ldots.$$

Let

$$r_n = p_n \qquad \text{for } n = n_0,\ n_0+1, \ldots$$

and define r_n recursively for $n = n_0-1, n_0-2, \ldots, 0$, so that $r_n>0$ and

$$r_{n+1}/r_n \leqq \min(r_{n+2}/r_{n+1},\ q_{n+1}/q_n,\ p_{n+1}/p_n).$$

Let $\eta_n = r_n/r_0$, then

$$\eta_0 = 1, \quad \eta_n > 0, \quad \eta_{n+1}/\eta_n \geqq \eta_n/\eta_{n-1}, \quad \eta_n/\eta_{n-1} \leqq q_n/q_{n-1},$$

and

$$\eta_n/\eta_{n-1} \leqq p_n/p_{n-1}$$

for $n>0$. Also we have $p_n = O(\eta_n)$.

Thus we have $[N, p_n]_\lambda \Rightarrow [N, \eta_n]_\lambda$ by Corollary 4 and $[N, \eta_n]_\lambda \Rightarrow [N, q_n]_\lambda$ by the case $n_0=0$. Thus $[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda$ for $\lambda>1$, as required.

Theorem 5. *If (N, p_n) and (N, q_n) are regular Nörlund methods,*

$$p_0 = q_0 = 1, \quad p_n > 0, \quad q_n > 0, \quad p_{n+1}/p_n \geqq p_n/p_{n-1}, \quad q_{n+1}/q_n \geqq q_n/q_{n-1} \qquad \text{for } n>0$$

$$p_n/p_{n-1} \leqq q_n/q_{n-1} \qquad \textit{for } n > n_0$$

and $q_n = O(p_n)$, then

$$(N, p_n) \Leftrightarrow (N, q_n)$$

and

$$[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda \qquad \textit{for } \lambda \geqq 1.$$

Proof. That $(N, p_n) \Rightarrow (N, q_n)$ follows directly from HARDY's Theorem 23 in [3].

To prove that $(N, q_n) \Rightarrow (N, p_n)$ let $\{\eta_n\}$ be defined as in the proof of Theorem 4.

Now, $(N, q_n) \Rightarrow (N, \eta_n)$ by Theorem 3, $(N, \eta_n) \Rightarrow (N, p_n)$ by Hardy's Theorem 23 in [3], and so $(N, q_n) \Rightarrow (N, p_n)$.

Thus $(N, p_n) \Leftrightarrow (N, q_n)$ and consequently, by Corollary 1, $[N, p_n]_1 \Leftrightarrow [N, q_n]_1$.

To show that $[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda$ for $\lambda > 1$, we observe that $[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda$ by Theorem 4, $[N, \eta_n]_\lambda \Rightarrow [N, p_n]_\lambda$ by the case $n_0 = 0$ of Theorem 4, and $[N, q_n] \Rightarrow [N, \eta_n]_\lambda$ by Corollary 4 and hence $[N, q_n]_\lambda \Rightarrow [N, p_n]_\lambda$. This completes the proof.

4. Relations Between Strong Nörlund, Absolute Nörlund, and Nörlund Summability Methods

Theorem 6. $[N, p_n]_1 \Rightarrow (N, p_n)$.

Proof. Suppose $s_n \to s\,[N, p_n]_1$.

Now

$$\frac{1}{|P_n|}\left|\sum_{r=0}^{n} p_r (t_r^{\Delta} - s)\right| \leqq \frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - s|.$$

Thus

$$\frac{1}{P_n}\sum_{r=0}^{n} p_r t_r^{\Delta} \to s.$$

But

$$\frac{1}{P_n}\sum_{r=0}^{n} p_r t_r^{\Delta} = \frac{1}{P_n}\sum_{r=0}^{n} p_r s_{n-r}, \tag{4.1}$$

and so

$$s_n \to s\,(N, p_n).$$

Theorem 7. *If $P_n^* = O(|P_n|)$ then $[N, p_n]_\lambda \Rightarrow (N, p_n)$ for $\lambda > 1$.*

Proof. Using the fact that $P_n^* = O(|P_n|)$ in conjunction with Theorem 1 in [1], we find that $[N, p_n]_\lambda \Rightarrow [N, p_n]_1$ for $\lambda > 1$. The result now follows from Theorem 6.

Theorem 8. *If $(\overline{N}, p_n)$ is regular, and $\lambda \geqq 1$ then, $s_n \to s\,[N, p_n]_\lambda$ if and only if,*

$$s_n \to s\,(N, p_n)$$

and

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - t_r|^{\lambda} = o(1).$$

The proof of this theorem follows closely the proof of Borwein's Theorem 7 in [1].

Proof. We have to prove that

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - s|^{\lambda} = o(1) \tag{4.2}$$

if and only if

$$t_n \to s \tag{4.3}$$

and

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - t_r|^{\lambda} = o(1). \tag{4.4}$$

(i) Suppose that (4.2) holds. Then, by Theorem 7, (4.3) holds, and so

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r - s|^{\lambda} = o(1) \tag{4.5}$$

since $(\overline{N}, p_n)$ is regular. Hence, by Minkowski's inequality and (4.2),

$$\left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - t_r|^{\lambda}\right\}^{1/\lambda} \leqq \left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - s|^{\lambda}\right\}^{1/\lambda} + \left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r - s|^{\lambda}\right\}^{1/\lambda} = o(1)$$

and (4.4) follows.

(ii) Suppose that (4.3) and (4.4) hold. Again, (4.5) holds. Hence, by Minkowski's inequality and (4.4),

$$\left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - s|^{\lambda}\right\}^{1/\lambda} \leqq \left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^{\Delta} - t_r|^{\lambda}\right\}^{1/\lambda} + \left\{\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r - s|^{\lambda}\right\}^{1/\lambda} = o(1),$$

so that (4.2) holds. The proof is thus complete.

Remark.

If $\sum_{r=0}^{\infty} a_n$ is summable $|N, p_n|_1$, then $\sum_{n=0}^{\infty} a_n = s\,(N, p_n)$

where

$$s = \sum_{n=1}^{\infty} (t_n - t_{n-1}) + t_0. \tag{4.6}$$

Theorem 9. *If $(\overline{N}, p_n)$ is regular and*

$$\sum_{n=0}^{\infty} a_n \quad \text{is summable } |N, p_n|_1,$$

then

$$\sum_{n=0}^{\infty} a_n = s\,[N, p_n]_1,$$

where s is given by (4.6).

Proof. By Theorem 8 it suffices to prove that,

(4.7)
$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^\Delta - t_r| = o(1).$$

Now, for $r>0$,

$$t_r^\Delta - t_r = \frac{1}{p_r}\sum_{v=0}^{r}\Delta p_v s_{r-v} - \frac{1}{P_r}\sum_{v=0}^{r} p_v s_{r-v}$$
$$= \frac{P_r\sum_{v=0}^{r} p_v s_{r-v} - P_r\sum_{v=0}^{r-1} p_v s_{r-v-1} - p_r\sum_{v=0}^{r} p_v s_{r-v}}{p_r P_r}$$
$$= \frac{P_{r-1}\sum_{v=0}^{r} p_v s_{r-v} - P_r\sum_{v=0}^{r-1} p_v s_{r-v-1}}{p_r P_r}$$
$$= \frac{P_{r-1} t_r - P_{r-1} t_{r-1}}{p_r},$$

so that

(4.8)
$$p_r(t_r^\Delta - t_r) = P_{r-1}(t_r - t_{r-1}),$$

and hence

$$|t_r^\Delta - t_r| \leqq \frac{P_{r-1}^*}{|p_r|}\,|t_r - t_{r-1}|.$$

Consequently, since $t_0^\Delta = t_0$,

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^\Delta - t_r| \leqq \frac{1}{P_n^*}\sum_{r=1}^{n} P_{r-1}^*\,|t_r - t_{r-1}|.$$

Let

$$b_r = |t_r - t_{r-1}|, \qquad B_n = \sum_{r=1}^{n} b_r;$$

then

$$\frac{1}{P_n^*}\sum_{r=1}^{n} P_{r-1}^* b_r = B_n - \frac{1}{P_n^*}\sum_{r=1}^{n} B_r|p_r| = o(1)$$

by the regularity of $(\overline{N}, p_n)$. The required conclusion follows.

Theorem 10. *If* $\lambda>1$, $(\overline{N}, p_n)$ *is regular and*

(4.9)
$$P_{n-1}^* = O(n|p_n|);$$

and if the series

$$\sum_{n=0}^{\infty} a_n$$

is summable (N, p_n) *to* s *and is summable* $|N, p_n|_\lambda$ *then the series is summable* $[N, p_n]_\lambda$ *to* s.

Proof. Using (4.8), we find that

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^\Delta - t_r|^\lambda \leqq \frac{1}{P_n^*}\sum_{r=1}^{n}\frac{P_{r-1}^*}{|p_r|^{\lambda-1}}\,|t_r - t_{r-1}|^\lambda$$
$$\leqq \frac{1}{P_n^*}\sum_{r=1}^{n} P_{r-1}^*\, r^{\lambda-1}\,|t_r - t_{r-1}|^\lambda.$$

Using the same technique as in the proof of Theorem 9 we find that the final term is $o(1)$, so that

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^\Delta - t_r|^\lambda = o(1)$$

Finally, using Theorem 8, we obtain the desired conclusion.

Remark. Condition (4.9) is satisfied when $\{|p_n|\}$ is non-decreasing, and also if (N, p_n) is the (C, α) method of summability with $\alpha > -1$.

5. Construction of a Scale of Nörlund Methods

We shall restrict ourselves now to Nörlund methods (N, p_n) for which $p_0>0$ and $p_n \geqq 0$.

Given any sequence $\{v_n\}$ we use the notation

(5.1)
$$v_n^\alpha = \sum_{r=0}^{n}\varepsilon_r^{\alpha-1} v_{n-r},$$

so that

(5.2)
$$\Delta v_n = v_n^{-1}.$$

The following identities are immediate:

(5.3)
$$\sum_{r=0}^{n}\varepsilon_r^{\beta-1} v_{n-r}^\alpha = v_n^{\alpha+\beta},$$

(5.4)
$$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^\alpha.$$

We are going to consider the family of Nörlund methods (N, p_n^α) for $\alpha > -1$, and, when $p_n \neq 0$ for all values of n, we shall alow $\alpha = -1$. In the special case $p_0 = 1$, $p_n = 0$ for $n>0$ we have $p_n = \varepsilon_n^{\alpha-1}$, so that (N, p_n^α) is the Cesàro method (C, α).

Theorem 11. *If either* (i) $\beta>\alpha>-1$ *for* (ii) $\beta>\alpha=-1$, $p_n>0$ *and* $P_n\to\infty$, *then* $(N, p_n^\alpha) \Rightarrow (N, p_n^\beta)$.

Proof. We use Proposition 1 with p_n^α and p_n^β in place of p_n and q_n. Let $\beta - \alpha = \delta > 0$.

Now the appropriate $k_n=\varepsilon_n^{\delta-1}>0$ and, by (5.3),

$$\varepsilon_n^{\delta-1}P_0^\alpha+\cdots+\varepsilon_0^{\delta-1}P_n^\alpha=P_n^\beta.$$

So the appropriate form of (2.14) in Proposition 1 holds.

Since in this case $k_{n-r}\sim k_n$, in order to verify the appropriate form of (2.15) it suffices to show that

(5.5) $$\varepsilon_n^{\delta-1}/P_n^\beta=o(1).$$

Now, for $\alpha>-1$,

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta}=\varepsilon_n^{\delta-1}\Big/\sum_{r=0}^{n}\varepsilon_r^{\alpha+\delta}p_{n-r}\leqq\varepsilon_n^{\delta-1}/\varepsilon_n^{\alpha+\delta}p_0=O(1/n^{1+\alpha})=o(1).$$

Finally when $\alpha=-1$ and $P_n\to\infty$, we may suppose without loss in generality that $0<\delta\leqq 1$ and then obtain

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta}\leqq\varepsilon_n^{\delta-1}/\varepsilon_n^{\delta-1}P_n=1/P_n=o(1).$$

This completes the proof of the theorem.

Corollary 5. *If (N,p_n) is a regular Nörlund method, then so also is (N,p_n^α) for $\alpha>0$.*

Theorem 12. *If either* (i) $\alpha>0$ *or* (ii) $\alpha=0$, $p_n>0$ *and* $P_n\to\infty$, *then* $(N,p_n^{\alpha-1})\Rightarrow[N,p_n^\alpha]_\lambda$ *for* $\lambda>0$.

Proof. Let $s_n\to s(N,p_n^{\alpha-1})$, i.e. let

$$w_n=\frac{1}{P_n^{\alpha-1}}\sum_{r=0}^{n}p_{n-r}^{\alpha-1}s_r\to s.$$

Then

(5.6) $$\frac{1}{P_n^\alpha}\sum_{r=0}^{n}p_r^\alpha|w_r-s|^\lambda=o(1)$$

if $(\overline{N},p_n^\alpha)$ is regular, which is the case when either (i) or (ii) is satisfied. Since (5.6) is equivalent to $s_n\to s[N,p_n^\alpha]_\lambda$, this completes the proof.

Theorem 13. *For $\alpha\geqq 0$ and $\lambda\geqq 1$,*

$$[N,p_n^\alpha]_\lambda\Rightarrow(N,p_n^\alpha)$$

provided $p_n>0$ when $\alpha=0$.

Proof. This result follows immediately from Theorems 6 and 7, and p_n^α in place of p_n.

Theorem 14. *If $(N,p_n)\Rightarrow(N,q_n)$, then $(N,p_n^\alpha)\Rightarrow(N,q_n^\alpha)$ for $\alpha>0$.*

Proof. We have, by hypothesis and Proposition 1,

$$|k_0|P_n+\cdots+|k_n|P_0\leqq H\,Q_n.$$

Now

$$\begin{aligned}|k_0|P_n^\alpha+\cdots+|k_n|P_0^\alpha&=\sum_{r=0}^{n}|k_{n-r}|\sum_{\nu=0}^{r}\varepsilon_\nu^{\alpha-1}P_{r-\nu}\\&=\sum_{\nu=0}^{n}\varepsilon_\nu^{\alpha-1}\sum_{r=\nu}^{n}|k_{n-r}|P_{r-\nu}\\&\leqq H\sum_{\nu=0}^{n}\varepsilon_\nu^{\alpha-1}Q_{n-\nu}\\&=H\,Q_n^\alpha.\end{aligned}$$

Also $|k_{n-r}|/Q_n^\alpha\leqq|k_{n-r}|/Q_n=o(1)$ by hypothesis and Proposition 1. The conclusion now follows from Proposition 1.

Corollary 6. *If (N,p_n) is regular, then, for $\alpha>0$.*

$$(C,\alpha)\Rightarrow(N,p_n^\alpha)\quad\textit{and}\quad[C,\alpha]_\lambda\Rightarrow[N\,p_n^\alpha]_\lambda\qquad\textit{for }\lambda\geqq 1.$$

Theorem 15. *If $\beta>\alpha\geqq 0$ and $\lambda\geqq 1$ then $[N,p_n^\alpha]_\lambda\Rightarrow[N,p_n^\beta]_\lambda$ provided $p_n>0$ when $\alpha=0$.*

Proof. This follows immediately from Theorems 1 and 11 in the case $\lambda=1$, and from Theorems 2 and 11 in the case $\lambda>1$, because

$$\sum_{r=0}^{n}\varepsilon_r^{\beta-\alpha-1}p_{n-r}^\alpha=p_n^\beta\quad\text{and}\quad\varepsilon_r^{\beta-\alpha-1}>0.$$

6. An Application

The method (C^*,μ) is defined by BORWEIN [2] as follows: Let $\mu=m+\delta$, where m is a non-negative integer, and $0\leqq\delta<1$, and let

$$\pi_\mu(x)=m!\,\varepsilon_m^x(x+m+1)^\delta=(x+1)\dots(x+m)(x+m+1)^\delta.$$

A series

$$\sum_{r=0}^{\infty}a_r$$

is said to be summable (C^*,μ) to s if,

$$\sigma_n=\frac{1}{\pi_\mu(n)}\sum_{r=0}^{n}\pi_\mu(n-r)\,a_r\to s.$$

The method (C^*,μ) is the Nörlund method (N,p_n) with

(6.1) $$p_n=\pi_\mu(n)-\pi_\mu(n-1).$$

BORWEIN [2] has proved that

(6.2) $$(C^*,\mu)\Leftrightarrow(C,\mu)\qquad\text{for }\mu\geqq 0.$$

We now define the strong method $[C^*, \mu]_\lambda$ to be the method $[N, p_n]_\lambda$ with p_n given by (6.1), and prove the following theorem.

Theorem 16. *For* $\mu > 0$, $\lambda \geqq 1$, $[C^*, \mu]_\lambda \Leftrightarrow [C, \mu]_\lambda$.

Proof. The case $\lambda = 1$ follows immediately from (6.2) and Theorem 1. Suppose therefore that $\lambda > 1$, and let $q_n = \varepsilon_n^{\mu-1}$. We consider two cases, (i) $\mu \geqq 1$ and (ii) $\mu < 1$.

Case (i) $\mu > 1$. Now $\pi_\mu(n) \sim \Gamma(\mu+1)\, \varepsilon_n^\mu$.

Also

$$\frac{\pi_\mu(n) - \pi_\mu(n-1)}{\pi_{\mu-1}(n)} = (n+m)\left[(1+1/(n+m))^\delta - 1\right] + m$$

$$\rightarrow \delta + m = \mu \quad \text{as} \quad n \rightarrow \infty .$$

Thus

$$p_n = \pi_\mu(n) - \pi_\mu(n-1) \sim \mu\, \pi_{\mu-1}(n) \sim \Gamma(\mu+1)\, \varepsilon_n^{\mu-1} = \Gamma(\mu+1)\, q_n .$$

Now $q_n = \varepsilon_n^{\mu-1} \leqq q_{n+1}$ since $\mu \geqq 1$, and, by (6.2) and Proposition 2,

$$\sum_{n=0}^{\infty} |l_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |k_n| < \infty .$$

Hence

$$|k_0|\, p_n + \cdots + |k_n|\, p_0 = O(|k_0|\, q_n + \cdots + |k_n|\, q_0) = O(q_n)$$

and

$$|l_0|\, q_n + \cdots + |l_n|\, q_0 = O(q_n) = O(p_n) .$$

The desired result now follows from Theorem 2.

Case (ii) $\mu < 1$ i.e. $\mu = \delta$ with $0 < \delta < 1$.

Now

(6.3) $$q_{n+1}/q_n \geqq q_n/q_{n-1} \qquad \text{for } n > 0,$$

because

$$\frac{q_{n+1}\, q_{n-1}}{q_n^2} = \frac{n^2 + n\delta}{n^2 + n\delta + \delta - 1} \geqq 1 \qquad \text{for } n > 0 .$$

Also

(6.4) $$p_{n+1}/p_n \geqq p_n/p_{n-1} \qquad \text{for } n > 0,$$

because

$$p_{n+1}/p_n = \left(\frac{n+1+\theta_n}{n+\theta_n}\right)^{\delta-1} \qquad (0 < \theta_n < 1)$$

$$= \left(1 + \frac{1}{n+\theta_n}\right)^{\delta-1}$$

$$\geqq \left(1 + \frac{1}{n-1+\theta_{n-1}}\right)^{\delta-1}$$

$$= p_n/p_{n-1} .$$

We show next that there is an integer n_0 such that

(6.5) $$p_{n+1}/p_n \geqq q_{n+1}/q_n \qquad \text{for } n > n_0 .$$

Let

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

$$= \tfrac{1}{2}(1+2x)^{1+\delta} + \tfrac{1}{2}(1+2x)^\delta + 1 + \delta x - (1+\delta)(1+x)^{1+\delta} - (1-\delta)(1+x)^\delta .$$

An elementary computation shows that

$$f_0 = f_1 = f_2 = 0 \quad \text{and} \quad 2 f_3 = \delta(1-\delta)^2 > 0 .$$

Hence, for n sufficiently large,

$$n^{1+\delta} f(1/n) = (n+1)\{(n+2)^\delta - (n+1)^\delta\} - (n+\delta)\{(n+1)^\delta - n^\delta\} > 0$$

and (6.5) follows, because

$$\frac{p_{n+1}\, q_n}{p_n\, q_{n+1}} = \frac{n+1}{n+\delta} \cdot \frac{(n+2)^\delta - (n+1)^\delta}{(n+1)^\delta - n^\delta} .$$

Since $p_0 = q_0 = 1$, $p_n = O(q_n)$ and (6.3), (6.4) and (6.5) hold, we obtain the desired conclusion in Case (ii) by appealing to Theorem 5.

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