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# Multiplication Theorems for Strong Nörlund Summability

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## 1. Introduction

In [1] a definition of strong Nörlund summability was given, and some of its properties were investigated. In this paper, some theorems concerning the strong Nörlund summability of the Cauchy product of two given series are established, which generalise known theorems about strong Cesàro summability.

## 2. Preliminaries

Throughout this paper $H$, $H_1$ etc. will denote positive constants, which will not necessarily be the same at different occurrences. Let $\{p_n\}$ be a sequence of real numbers with $p_0>0$ and $p_n \geqq 0$ for all $n>0$, and let

$$P_n = \sum_{r=0}^{n} p_r .$$

**Definition 1.** Let $\sum_{r=0}^{\infty} a_r$ be a given series. Define

$$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r}\, s_r = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r}\, a_r \qquad (n=0,1,2,\ldots), \tag{2.1}$$

where

$$s_n = \sum_{r=0}^{n} a_r .$$

If $t_n \to s$ as $n \to \infty$, we shall write

$$\sum_{r=0}^{\infty} a_r = s(N, p_n)$$

or $s_n \to s(N, p_n)$.

This is the standard definition of *Nörlund summability* $(N, p_n)$. See Tamarkin [8], Nörlund [7] or Hardy [4, p. 64].

**Definition 2.** If $t_n = O(1)$ we say $\sum_{r=0}^{\infty} a_r$ is *bounded* $(N, p_n)$, where $t_n$ is defined by (2.1).

**Definition 3.** A method of summability is *regular* if it sums every convergent series to its ordinary sum.

The method $(N, p_n)$ is regular if and only if $p_n/P_n = o(1)$. See Hardy [4, Theorem 16, p. 64].

**Definition 4.** If $P$ and $Q$ are methods of summability, $Q$ is said to *include* $P$ (written $P \Rightarrow Q$) if every series summable by the method $P$ is also summable by the method $Q$ to the same sum. If, further, $P$ includes $Q$, $P$ and $Q$ are said to be *equivalent* (written $P \Leftrightarrow Q$).

If $p_n > 0$ for all $n \geqq 0$ and $\sum_{r=0}^{\infty} a_r$ is a given series, we define $t_n^A$ by

$$t_n^A = \frac{1}{p_n} \sum_{r=0}^{n} p_{n-r} a_r \qquad (n=0, 1, 2, \ldots). \tag{2.2}$$

**Definition 5. Strong Nörlund Summability** $[N, p_n]_\lambda$, $\lambda > 0$.

Let $(N, p_n)$ be a Nörlund method with $p_n > 0$ for all $n \geqq 0$, and let $\sum_{r=0}^{\infty} a_r$ be a given series. We shall say that $\sum_{r=0}^{\infty} a_r$ *is strongly summable* $(N, p_n)$ *with index* $\lambda$ *to* $s$, if

$$\frac{1}{P_n} \sum_{r=0}^{n} p_r |t_r^A - s|^\lambda = o(1). \tag{2.3}$$

We shall denote this by

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda \quad \text{or} \quad s_n \to s[N, p_n]_\lambda.$$

This is the definition of strong Nörlund summability given in [1], except that in [1], the sequence $\{p_n\}$ is allowed to be a sequence of complex numbers. It is proved in [1] that for $\alpha > 0$ and $\lambda > 0$, $[C, \alpha]_\lambda \Leftrightarrow [N, \varepsilon_n^{\alpha-1}]_\lambda$, where $[C, \alpha]_\lambda$ denotes the strong Cesàro method of summability with index $\lambda$, as defined in [2] for example. See also Winn [9]. For the remainder of this paper we shall use the notation "$[C, \alpha]_\lambda$" to denote the method of summability $[N, \varepsilon_n^{\alpha-1}]_\lambda$ $(\alpha > 0, \lambda > 0)$.

**Definition 6.** Let $(N, p_n)$ be a Nörlund method with $p_n > 0$ for all $n \geqq 0$, and let $\sum_{r=0}^{\infty} a_r$ be a given series. We shall say that $\sum_{r=0}^{\infty} a_r$ *is strongly bounded* $(N, p_n)$ *with index* $\lambda$ or $\sum_{r=0}^{\infty} a_r$ *is bounded* $[N, p_n]_\lambda$, if

$$\frac{1}{P_n} \sum_{r=0}^{n} p_r |t_r^A|^\lambda = O(1).$$

**Definition 7. Absolute Nörlund Summability** $|N, p_n|$.

Let $(N, p_n)$ be a Nörlund method with $p_n > 0$ for all $n \geqq 0$ and let $\sum_{r=0}^{\infty} a_r$ be a given series. We shall say that $\sum_{r=0}^{\infty} a_r$ *is absolutely summable* $(N, p_n)$ *to* $s$ if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$$

where $s = \lim_{n \to \infty} t_n$. We denote this by

$$\sum_{r=0}^{\infty} a_r = s\,|N, p_n|.$$

This is the definition of absolute Nörlund summability given by Mears in [5].

**Definition 8.** The method $(N, p_n)$ *is absolutely regular* if whenever $\sum_{r=0}^{\infty} a_r$ is absolutely convergent, it is also summable $|N, p_n|$.

See Mears [5] and Miesner [6].

## 3. Some Known Results

Let $\sum_{n=0}^{\infty} c_n$ denote the Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, i.e.

$$c_n = \sum_{v=0}^{n} a_v b_{n-v}. \tag{3.1}$$

The following propositions ($\alpha$), ($\beta$) and ($\gamma$) about Cesàro summability have been established; the first two by Winn in [9], and the third by Boyd in [3].

($\alpha$) *If*

$$\sum_{n=0}^{\infty} a_n = s[C, k]_1 \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n = t(C, l),$$

*where* $k > 0$ *and* $l \geqq 0$, *then*

$$\sum_{n=0}^{\infty} c_n = s\,t(C, k+l).$$

($\beta$) *If*

$$\sum_{n=0}^{\infty} a_n = s[C, k]_1 \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n = t[C, l]_1,$$

*where* $k > 0$ *and* $l > 0$, *then*

$$\sum_{n=0}^{\infty} c_n = s\,t[C, k+l]_1.$$

($\gamma$) *If*

$$\sum_{n=0}^{\infty} a_n = s[C, k]_1 \quad \textit{where } k > 0,$$

$\sum_{n=0}^{\infty} b_n$ *is absolutely convergent with sum* $t$, *then*

$$\sum_{n=0}^{\infty} c_n = s\,t[C, k]_1.$$

Propositions ($\alpha$), ($\beta$) and ($\gamma$) are special cases of our Theorems 2, 4 and 6 respectively.

## 4. Main Theorems

Suppose that $\{p_n\}$ and $\{q_n\}$ are sequences of real numbers with $p_0 > 0$, $q_0 > 0$, $p_n \geqq 0$ and $q_n \geqq 0$ for all $n > 0$. Let

$$P_n = \sum_{r=0}^{n} p_r, \quad Q_n = \sum_{r=0}^{n} q_r.$$

Also let,

$$r_n=\sum_{\nu=0}^{n} p_\nu q_{n-\nu} \quad \text{and} \quad R_n=\sum_{\nu=0}^{n} r_\nu,$$

then $r_0>0$ and $r_n\geqq 0$ for $n>0$.

Given any sequence $\{\zeta_n\}$ let $\zeta(x)$ denote the formal power series $\sum_{n=0}^{\infty}\zeta_n x^n$.

**Theorem 1.** *If $p_n>0$ for all $n$, $q_0>0$, $q_n\geqq 0$ for all $n>0$, $\lambda\geqq 1$, $(N, q_n)$ is regular,*

$$\sum_{n=0}^{\infty} a_n=0[N, p_n]_\lambda$$

*and $\sum_{n=0}^{\infty} b_n$ is bounded $(N, q_n)$, then*

$$\sum_{n=0}^{\infty} c_n=0(N, r_n).$$

*Proof.* Using Hölder's inequality, it is easy to show that for $\lambda>1$, $[N, p_n]_\lambda\Rightarrow[N, p_n]_1$ (see [2, Theorem 1]). Thus it is sufficient to prove Theorem 1 for the case $\lambda=1$. Let

$$w_n=\frac{1}{Q_n}\sum_{\nu=0}^{n} Q_{n-\nu} b_\nu$$

and

$$v_n=\frac{1}{R_n}\sum_{\nu=0}^{n} R_{n-\nu} c_\nu.$$

Now $R_n v_n$ is the coefficient of $x^n$ in the series

$$\sum_{n=0}^{\infty} p_n t_n^{\Delta} x^n \sum_{n=0}^{\infty} Q_n w_n x^n=p(x)\, a(x)(1-x)^{-1} q(x)\, b(x)$$

$$=p(x)\, q(x)(1-x)^{-1} c(x)=R(x)\, c(x).$$

Thus

$$R_n v_n=\sum_{\nu=0}^{n} p_\nu t_\nu^{\Delta} Q_{n-\nu} w_{n-\nu}$$

and so,

$$R_n |v_n|\leqq \sum_{\nu=0}^{n} p_\nu |t_\nu^{\Delta}|\, Q_{n-\nu} |w_{n-\nu}|.$$

Since by hypothesis,

$$\sum_{r=0}^{n} p_r |t_r^{\Delta}|=o(P_n)$$

and

$$|w_n|=O(1),$$

it follows that

$$R_n |v_n|\leqq H\sum_{\nu=0}^{n} p_\nu |t_\nu^{\Delta}|\, Q_{n-\nu}$$

$$=H\sum_{\nu=0}^{n} q_{n-\nu}\sum_{r=0}^{\nu} p_r |t_r^{\Delta}|$$

$$=H\sum_{\nu=0}^{n} q_{n-\nu}\, o(P_\nu).$$



Since $(N, q_n)$ is regular the final sum is $o(R_n)$. Thus $v_n=o(1)$, and so

$$\sum_{n=0}^{\infty} c_n=0(N, r_n)$$

as required.

**Theorem 2.** *If $p_n>0$ for all $n$, $q_0>0$, $q_n\geqq 0$ for all $n>0$, $\lambda\geqq 1$, $(N, p_n)$ and $(N, q_n)$ are regular,*

$$\sum_{n=0}^{\infty} a_n=s[N, p_n]_\lambda \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n=t(N, q_n), \quad \textit{then} \quad \sum_{n=0}^{\infty} c_n=s\,t(N, r_n).$$

*Proof.* If $s=0$ the result is an immediate consequence of Theorem 1. Suppose $s\neq 0$. Let

$$a_0'=a_0-s, \quad a_n'=a_n \quad \text{for } n>0$$

and

$$c_n'=\sum_{\nu=0}^{n} a_\nu' b_{n-\nu},$$

then

$$c_n'=c_n-s\, b_n.$$

Thus, since

$$\sum_{n=0}^{\infty} a_n'=0[N, p_n]_1$$

by hypothesis, we have

$$\sum_{n=0}^{\infty} c_n'=0(N, r_n)$$

by Theorem 1. Further, since $(N, p_n)$ is regular, $(N, q_n)\Rightarrow(N, r_n)$ (see Hardy [4, Theorem 17, p. 65]). Thus

$$\sum_{n=0}^{\infty} b_n=t(N, r_n).$$

Hence, since

$$\sum_{n=0}^{m} c_n=\sum_{n=0}^{m} c_n'+s\sum_{n=0}^{m} b_n \qquad \text{for } m=0, 1, 2, \ldots,$$

it follows that

$$\sum_{n=0}^{\infty} c_n=s\,t(N, r_n).$$

This completes the proof.

**Theorem 3.** *If $p_n>0$, $q_n>0$ for all $n$, $\lambda\geqq 1$, $(N, q_n)$ is regular*

$$\sum_{n=0}^{\infty} a_n=0[N, p_n]_\lambda$$

*and $\sum_{n=0}^{\infty} b_n$ is bounded $[N, q_n]_\lambda$, then*

$$\sum_{n=0}^{\infty} c_n=0[N, r_n]_\lambda.$$

*Proof.* Let

$$v_n^{\Delta}=\frac{1}{r_n}\sum_{\nu=0}^{n} r_{n-\nu}\, c_\nu$$

and

$$w_n^{\Delta}=\frac{1}{q_n}\sum_{\nu=0}^{n} q_{n-\nu}\, b_\nu .$$

Now

$$\sum_{n=0}^{\infty} r_n\, v_n^{\Delta}\, x^n=\sum_{n=0}^{\infty} p_n\, t_n^{\Delta}\, x^n \sum_{n=0}^{\infty} q_n\, w_n^{\Delta}\, x^n .$$

Thus

$$r_n\, v_n^{\Delta}=\sum_{\nu=0}^{n} p_\nu\, t_\nu^{\Delta}\, q_{n-\nu}\, w_{n-\nu}^{\Delta},$$

and so, we have

$$\{r_n\,|v_n^{\Delta}|\}^{\lambda}\leqq\left\{\sum_{\nu=0}^{n} p_\nu\,|t_\nu^{\Delta}|\, q_{n-\nu}\,|w_{n-\nu}^{\Delta}|\right\}^{\lambda}.$$

Using Hölder's inequality, we find that

$$\{r_n\,|v_n^{\Delta}|\}^{\lambda}\leqq\left\{\sum_{\nu=0}^{n} p_\nu\,|t_\nu^{\Delta}|^{\lambda}\, q_{n-\nu}\,|w_{n-\nu}^{\Delta}|^{\lambda}\right\}\left\{\sum_{\nu=0}^{n} p_\nu\, q_{n-\nu}\right\}^{\lambda-1}.$$

Thus

$$\sum_{n=0}^{m} r_n\,|v_n^{\Delta}|^{\lambda}\leqq\sum_{n=0}^{m}\sum_{\nu=0}^{n} p_\nu\,|t_\nu^{\Delta}|^{\lambda}\, q_{n-\nu}\,|w_{n-\nu}^{\Delta}|^{\lambda}=\sum_{\nu=0}^{m} p_\nu\,|t_\nu^{\Delta}|^{\lambda}\sum_{n=\nu}^{m} q_{n-\nu}\,|w_{n-\nu}^{\Delta}|^{\lambda}.$$

Now, by hypothesis

$$\sum_{\nu=0}^{m} p_\nu\,|t_\nu^{\Delta}|^{\lambda}=o(P_m)\quad\text{and}\quad\sum_{\nu=0}^{m} q_\nu\,|w_\nu^{\Delta}|^{\lambda}=O(Q_m).$$

Thus

$$\begin{aligned}\sum_{n=0}^{m} r_n\,|v_n^{\Delta}|^{\lambda}&\leqq H\sum_{\nu=0}^{m} p_\nu\,|t_\nu^{\Delta}|^{\lambda}\, Q_{m-\nu}\\&=H\sum_{\nu=0}^{m} q_{m-\nu}\sum_{r=0}^{\nu} p_r\,|t_r^{\Delta}|^{\lambda}\\&=H\sum_{\nu=0}^{m} q_{m-\nu}\, o(P_\nu)\\&=o(R_m)\end{aligned}$$

since $(N,q_n)$ is regular. Thus

$$\sum_{n=0}^{\infty} c_n=0\,[N,r_n]_\lambda$$

as required.

**Theorem 4.** *If $p_n>0$, $q_n>0$ for all $n$, $\lambda\geqq 1$, $(N,p_n)$ and $(N,q_n)$ are regular,*

$$\sum_{n=0}^{\infty} a_n=s\,[N,p_n]_\lambda \quad\textit{and}\quad \sum_{n=0}^{\infty} b_n=t\,[N,q_n]_\lambda,\quad\textit{then}\quad \sum_{n=0}^{\infty} c_n=s\,t\,[N,r_n]_\lambda .$$



*Proof.* When $s=t=0$, the result is an immediate consequence of Theorem 3. Suppose $s\neq 0$ and $t=0$. Let $a_0'=a_0-s$, $a_n'=a_n$ for $n>0$ and

$$c_n'=\sum_{\nu=0}^{n} a_\nu'\, b_{n-\nu},$$

then

$$\sum_{n=0}^{\infty} a_n'=0\,[N,p_n]_\lambda .$$

Thus by Theorem 3,

$$\sum_{n=0}^{\infty} c_n'=0\,[N,r_n]_\lambda .$$

Now

$$\sum_{n=0}^{\infty} b_n=0\,[N,r_n]_\lambda ,$$

since $[N,q_n]_\lambda\Rightarrow[N,r_n]_\lambda$ for $\lambda\geqq 1$ by Theorems 1 and 2 in [1]. Thus

$$\sum_{n=0}^{\infty} c_n=0\,[N,r_n]_\lambda$$

as required. The case $s\neq 0$, $t\neq 0$ is proved similarly by defining $b_0'=b_0-t$, $b_n'=b_n$ for $n>0$. The proof is now complete.

**Theorem 5.** *If $p_n>0$ for all $n$, $\{p_n\}$ is a non-decreasing sequence, $\lambda\geqq 1$,*

$$\sum_{n=0}^{\infty} a_n=0\,[N,p_n]_\lambda$$

*and $\sum_{n=0}^{\infty} b_n$ is absolutely convergent, then*

$$\sum_{n=0}^{\infty} c_n=0\,[N,p_n]_\lambda .$$

*When $\lambda=1$ the condition "$\{p_n\}$ is a non-decreasing sequence" may be omitted.*

*Proof.* Let

$$v_n^{\Delta}=\frac{1}{p_n}\sum_{\nu=0}^{n} p_{n-\nu}\, c_\nu .$$

Now $p_n\, v_n^{\Delta}$ is the coefficient of $x^n$ in the series

$$\sum_{n=0}^{\infty} p_n\, x^n \sum_{n=0}^{\infty} c_n\, x^n=p(x)\, c(x)=p(x)\, a(x)\, b(x).$$

Thus

$$p_n\, v_n^{\Delta}=\sum_{\nu=0}^{n} p_\nu\, t_\nu^{\Delta}\, b_{n-\nu},$$

and so

$$\{p_n\,|v_n^{\Delta}|\}^{\lambda}\leqq\left\{\sum_{\nu=0}^{n} p_\nu\,|t_\nu^{\Delta}|\,|b_{n-\nu}|\right\}^{\lambda}.$$

Using Hölder's inequality, we find that

$$\{p_n |v_n^A|\}^\lambda \leqq \left\{\sum_{v=0}^{n} p_n |t_v^A|^\lambda |b_{n-v}|\right\} \left\{\sum_{v=0}^{n} p_v |b_{n-v}|\right\}^{\lambda-1}. \tag{4.1}$$

Since by hypothesis $\sum_{n=0}^{\infty} b_n$ is absolutely convergent and $\{p_n\}$ is non-decreasing, we find

$$p_n |v_n^A|^\lambda \leqq H \left\{\sum_{v=0}^{n} p_v |t_v^A|^\lambda |b_{n-v}|\right\}.$$

We do not need to use the fact that $\{p_n\}$ is non-decreasing in the case $\lambda=1$, since the last sum in (4.1) does not appear when $\lambda=1$. Now, we have

$$\sum_{n=0}^{m} p_n |v_n^A|^\lambda \leqq H \sum_{n=0}^{m} \sum_{v=0}^{n} p_v |t_v^A|^\lambda |b_{n-v}|$$

$$= H \sum_{v=0}^{n} p_v |t_v^A|^\lambda \sum_{n=v}^{m} |b_{n-v}|$$

$$\leqq H_1 \sum_{v=0}^{m} p_v |t_v^A|^\lambda,$$

since $\sum_{n=0}^{\infty} b_n$ is absolutely convergent. The final term is $o(P_m)$ by hypothesis, hence

$$\sum_{n=0}^{m} p_n |v_n^A|^\lambda = o(P_m)$$

so

$$\sum_{m=0}^{\infty} c_n = 0[N, p_n]_\lambda \qquad \text{for } \lambda \geqq 1.$$

This completes the proof.

**Theorem 6.** *If* $p_n>0$ *for all* $n$, $P_n\to\infty$, $(N, p_n)$ *is absolutely regular,*

$$\sum_{n=0}^{\infty} a_n = s[N, p_n]_1$$

*and* $\sum_{n=0}^{\infty} b_n$ *is absolutely convergent with sum* $t$, *then*

$$\sum_{n=0}^{\infty} c_n = s\,t[N, p_n]_1.$$

*Proof.* If $s=0$, the result is an immediate consequence of Theorem 5. Suppose $s\neq 0$, and let $a_0'=a_0-s$, $a_n'=a_n$ for $n>0$ then

$$\sum_{n=0}^{\infty} a_n' = 0[N, p_n]_1$$

Let

$$c_n' = \sum_{v=0}^{n} a_v' b_{n-v},$$

then

$$c_n' = c_n - s\,b_n,$$



and

$$\sum_{n=0}^{\infty} c_n' = 0[N, p_n]_1,$$

by Theorem 5. Further $\sum_{n=0}^{\infty} b_n$ is summable $|N, p_n|_1$ by the absolute regularity of $(N, p_n)$. Thus

$$\sum_{n=0}^{\infty} b_n = t[N, p_n]_1$$

by Theorem 9 in [1]. Hence

$$\sum_{n=0}^{\infty} c_n = s\,t[N, p_n]_1.$$

This completes the proof.

## 5. Corollaries to Theorems 2, 4 and 6

Let $\alpha$ be real. Define

$$\varepsilon_0^\alpha = 1, \qquad \varepsilon_n^\alpha = \frac{(\alpha+1)\dots(\alpha+n)}{n!}, \qquad n=1,2,\dots.$$

Given any sequence $\{v_n\}$ we use the notation

$$v_n^\alpha = \sum_{r=0}^{n} \varepsilon_r^{\alpha-1} v_{n-r}$$

so that

$$\Delta v_n = v_n^{-1}.$$

The following identities are immediate:

$$\sum_{r=0}^{n} \varepsilon_r^{\beta-1} v_{n-r}^\alpha = v_n^{\alpha+\beta}, \tag{5.1}$$

$$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^\alpha. \tag{5.2}$$

We shall now suppose that $\alpha \geqq -1$, and consider the Nörlund methods $(N, p_n^\alpha)$ where $p_n>0$ for all $n$ when $\alpha=-1$. These methods of summability were studied in some detail in [1]. The corollaries stated below include as special cases propositions $(\alpha)$, $(\beta)$ and $(\gamma)$ about strong Cesàro summability.

**Corollary 2.1.** *If* $\alpha>0$ *or* $\alpha=0$ *and* $p_n>0$ *for all* $n$, $\beta>0$, $\lambda\geqq 1$, $(N, p_n)$ *is regular,*

$$\sum_{b=0}^{\infty} a_n = s[N, p_n^\alpha]_\lambda \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = t(C, \beta), \quad \text{then} \quad \sum_{n=0}^{\infty} c_n = s\,t(N, p_n^{\alpha+\beta}).$$

**Corollary 2.2.** *If* $\beta\geqq 0$, $\alpha>0$, $\lambda\geqq 1$, $(N, p_n)$ *is regular,*

$$\sum_{n=0}^{\infty} a_n = s[C, \alpha]_\lambda \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = t(N, p_n^\beta), \quad \text{then} \quad \sum_{n=0}^{\infty} c_n = s\,t(N, p_n^{\alpha+\beta}).$$

**Corollary 3.1.** *If* $\beta>0$ *or* $\beta=0$ *and* $p_n>0$ *for all* $n$, $\alpha>0$, $\lambda\geqq 1$

$$\sum_{n=0}^{\infty} a_n = s[N, p_n^{\beta}]_{\lambda} \quad and \quad \sum_{n=0}^{\infty} b_n = t[C,\alpha]_{\lambda}, \quad then \quad \sum_{n=0}^{\infty} c_n = s\,t[N, p_n^{\alpha+\beta}]_{\lambda}.$$

**Corollary 4.1.** *If* $(N, p_n)$ *is absolutely regular,* $P_n\to\infty$, $\alpha>0$,

$$\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha}]_1$$

*and* $\sum_{n=0}^{\infty} b_n$ *is absolutely convergent with sum* $t$, *then*

$$\sum_{n=0}^{\infty} c_n = s\,t[N, p_n^{\alpha}]_1.$$

To prove Corollary 4.1, one notes that if $(N, p_n)$ is absolutely regular, then $(N, p_n^{\alpha})$ is absolutely regular for each $\alpha>0$, then applies Theorem 4.

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