# NOTE ON SUMMABILITY FACTORS

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1. *Introduction.* It is assumed throughout that $\lambda > 0$ and that all functions are real. The main object of this paper is to establish version (a) of the following theorem.

Theorem 1. (a) *In order that* $\int_1^\infty x(t)\, k(t)\, dt$ *be summable* $|C, \lambda|$ *whenever* $\int_1^\infty x(t)\, dt$ *is summable* $|C, \lambda|$ *it is necessary and sufficient that, for some constant* $c \geqslant 1$,

(i) $k(t)$ *be measurable and essentially bounded in* $(1, c)$,

(ii) $$\frac{k(t)}{t} = \frac{1}{\Gamma(\lambda)} \int_t^\infty (u-t)^{\lambda-1} h(u)\, du \quad p.p. \text{ in } (c, \infty),$$

*where* $u^{\lambda+1}\, h(u)$ *is measurable and essentially bounded in* $(c, \infty)$.

(b) *Replace* $|C, \lambda|$ *by* $(C, \lambda)$ *and "essentially bounded" in* (ii) *by "of bounded variation".*

Version (b) of the theorem has been proved by Sargent†. We shall, however, give a somewhat simpler proof of the necessity part of this result.

There are results‡ similar to the above which involve the additional

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† The result as stated here follows from Lemma 5, Theorem 2 and the proof of Theorem 1 in Sargent (4).

‡ For a $|C, \lambda|$ result see Borwein (2) where references are given to $(C, \lambda)$ results and to series analogues. Further references appear in Sargent (4).

hypothesis that the $\lambda$-th derivative of $k(t)$ exists and is absolutely continuous in $[1, w]$ for all $w \geqslant 1$.

2. *Notation and some preliminary results.* Let $x(u)$ be integrable $L$ in every finite interval in $(1, \infty)$. Then, for $w > 1$,

$$\int_1^w \left(1-\frac{u}{w}\right)^\lambda x(u)\,du = \lambda \int_1^w x(u)\,du \int_u^w \left(1-\frac{u}{t}\right)^{\lambda-1} \frac{u}{t^2}\,dt$$

$$= \lambda \int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} u x(u)\,du.$$

Hence $\int_1^\infty x(u)\,du$ is

(i) summable $(C, \lambda)$ if and only if $\int_1^\infty t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} u\,x(u)\,du$ is convergent;

(ii) summable $|C, \lambda|$ if and only if $\int_1^\infty t^{-\lambda-1}\,dt \left|\int_1^t (t-u)^{\lambda-1} u\,x(u)\,du\right| < \infty$;

(iii) bounded $(C)$ if and only if $\int_1^w t^{-\mu-1}\,dt \int_1^t (t-u)^{\mu-1} u x(u)\,du$ is bounded in $(1, \infty)$ for some $\mu > 0$.

We shall be concerned with the following function spaces (it is to be assumed that $1 \leqslant a \leqslant b \leqslant \infty$):

$M(a, b)$: the space of functions measurable and essentially bounded in $(a, b)$.

$L(a, b)$: the normed vector space of functions $x(t)$ integrable $L$ in $(a, b)$, the norm being defined by the equation

$$\|x\| = \int_a^b |x(t)|\,dt.$$

The general linear functional in this space is given by an equation of the form*

$$f(x) = \int_a^b x(t)\,\alpha(t)\,dt,$$

where $\alpha(t) \in M(a, b)$.

$BV(a, b)$: the space of functions having bounded variation in $[a, b)$.

$F$: the normed vector space of functions $x(t)$ continuous in $[1, \infty)$ and tending to finite limits as $t \to \infty$, the norm being defined by the equation

$$\|x\| = \overline{\text{bound}}_{t \geqslant 1}\, |x(t)|.$$

* Banach (1), 65.



The general linear functional in this space is given by an equation of the form*

$$f(x) = \int_1^\infty x(t)\,d\alpha(t) + \gamma \lim_{t\to\infty} x(t),$$

where $\alpha(t) \in BV(1, \infty)$ and $\gamma$ is a constant independent of $x$.

$B$: the space† of functions $x(t)$ such that $\int_1^\infty x(t)\,dt$ is bounded $(C)$.

$S_\lambda$: the normed vector space of functions $x(t)$ such that $\int_1^\infty x(t)\,dt$ is summable $(C, \lambda)$, the norm being defined by the equation

$$\|x\| = \overline{\text{bound}}_{w \geqslant 1} \left|\int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} u\,x(u)\,du\right|.$$

$S_\lambda{}^a$: the vector subspace of $S_\lambda$ which consists of all functions $x(t)$ such that $x(t) = 0$ for $t < a$ and $t x(t) \in L(a, \infty)$.

$V_\lambda$: the normed vector space of functions $x(t)$ such that $\int_1^\infty x(t)\,dt$ is summable $|C, \lambda|$, the norm being defined by the equation

$$\|x\| = \int_1^\infty t^{-\lambda-1}\,dt \left|\int_1^t (t-u)^{\lambda-1} u\,x(u)\,du\right|.$$

$V_\lambda{}^a$: the vector subspace of $V_\lambda$ which consists of all functions $x(t)$ such that $x(t) = 0$ for $t < a$ and $t x(t) \in L(a, \infty)$.

3. We shall require the following lemmas.

LEMMA‡ 1. (a) *For $c \geqslant 1$, the general linear functional in the space $V_\lambda{}^c$ is given by an equation of the form*

$$f(x) = \frac{1}{\Gamma(\lambda)} \int_1^\infty u\,x(u)\,du \int_u^\infty (t-u)^{\lambda-1} h(t)\,dt,$$

*where* $t^{\lambda+1} h(t) \in M(1, \infty)$.

(b) *Replace $V$ by $S$ and $M$ by $BV$.*

*Proof of* (a). It is easily seen that the equation

$$y(t) = t^{-\lambda-1} \int_1^t (t-u)^{\lambda-1} u\,x(u)\,du \quad (t \geqslant 1)$$

* Banach (1), 59–60; see also Sargent (4), Lemma 1.

† It is implicit in the definition of this space and of $S_\lambda$ and $V_\lambda$ that they are contained in $L(1, w)$ whenever $1 < w < \infty$.

‡ *Cf.* Sargent (4), Lemma 2.

defines a linear and isometric transformation between all functions $x$ of $V_\lambda{}^c$ and a vector subspace of functions $y$ of $L(1, \infty)$. Hence, by the Hahn-Banach extension theorem, the general linear functional in $V_\lambda{}^c$ is given by an equation of the form

$$f(x) = \int_1^\infty \alpha(t)\, t^{-\lambda-1}\, dt \int_1^t (t-u)^{\lambda-1} u\, x(u)\, du,$$

where $\alpha(t) \,\varepsilon\, M(1, \infty)$. Since $\int_1^\infty |x(u)|\, du < \infty$ when $x(u) \,\varepsilon\, V_\lambda{}^c$, we can change the order of integration and then obtain the required result by putting

$$h(t) = \Gamma(\lambda)\, t^{-\lambda-1} \alpha(t) \quad (t \geqslant 1).$$

*Proof of* (b). The equation

$$y(w) = \int_1^w t^{-\lambda-1}\, dt \int_1^t (t-u)^{\lambda-1} u\, x(u)\, du \quad (w \geqslant 1)$$

defines a linear and isometric transformation between all functions $x$ of $S_\lambda{}^c$ and a vector subspace of functions $y$ of $F$. Hence the general linear functional in $S_\lambda{}^c$ is given by an equation of the form

$$f(x) = \int_1^\infty d\alpha(w) \int_1^w t^{-\lambda-1}\, dt \int_1^t (t-u)^{\lambda-1} u\, x(u)\, du + \gamma \int_1^\infty t^{-\lambda-1}\, dt \int_1^t (t-u)^{\lambda-1} u\, x(u)\, du,$$

where $\alpha(w) \,\varepsilon\, BV(1, \infty)$ and $\gamma$ is a constant. Since $\int_1^\infty |x(u)|\, du < \infty$ when $x(u) \,\varepsilon\, S_\lambda{}^c$, we can change the order of integration and then obtain the required result by putting

$$h(t) = \Gamma(\lambda)\, t^{-\lambda-1} \left\{\gamma + \int_t^\infty d\alpha(w)\right\} \quad (t \geqslant 1).$$

LEMMA* 2. (a) *If* $x(t)\, k(t) \,\varepsilon\, B$ *whenever* $x(t) \,\varepsilon\, V_\lambda$, *then* (i) $k(t) \,\varepsilon\, M(1, \infty)$, (ii) *the functional*

$$f(x) = \int_1^\infty x(t)\, k(t)\, dt$$

*is linear in* $V_\lambda{}^c$ *for some* $c \geqslant 1$.

(b) *Replace* $V$ *by* $S$.

Since $S_\lambda \supset V_\lambda$, result b(i) follows from a(i) which has been established elsewhere†.

*Proof of* a(ii). Since $k(t) \,\varepsilon\, M(1, \infty)$, $f(x)$ is defined and additive in $V_\lambda{}^c$ for all $c \geqslant 1$. *Suppose there is no* $c \geqslant 1$ *for which* $f(x)$ *is linear in* $V_\lambda{}^c$.

* *Cf.* Sargent (4), Lemma 4.

† Borwein (2), Lemma 10.



Then we can define by induction a sequence of functions $\{x_n\}$ and an increasing unbounded sequence of real numbers $\{c_n\}$ as follows:

Let $c_0 = 1$ and suppose that $c_1, c_2, \ldots, c_{n-1}, x_1, x_2, \ldots, x_{n-1}$ have been defined and that $x_r \,\varepsilon\, V_\lambda^{c_{r-1}}$ for $r = 1, 2, \ldots, n-1$. Since $f(x)$ is not linear in $V_\lambda^{c_{n-1}}$, there is a function $x_n$ such that*

$$x_n \,\varepsilon\, V_\lambda^{c_{n-1}}, \quad \|x_n\| < 2^{-n} \quad \text{and} \quad f(x_n) > 1.$$

Let
$$c_n = 2c_{n-1} + \sum_{r=1}^n \int_1^\infty u\, |x_r(u)\, k(u)|\, du.$$

Now define a function $x(t)$ by putting

$$x(t) = x_1(t) + x_2(t) + \ldots + x_n(t)$$

when $1 \leqslant t < c_n$ and $n = 1, 2, \ldots$.

Then, for any $\mu \geqslant 1$ and $n = 1, 2, \ldots$,

$$\int_1^{c_n} t^{-\mu-1}\, dt \int_1^t (t-u)^{\mu-1} u\, x(u)\, k(u)\, du$$

$$= \sum_{r=1}^n \int_1^{c_n} t^{-\mu-1}\, dt \int_1^t (t-u)^{\mu-1} u\, x_r(u)\, k(u)\, du$$

$$= \sum_{r=1}^n \int_1^\infty t^{-\mu-1}\, dt \int_1^t (t-u)^{\mu-1} u\, x_r(u)\, k(u)\, du - \sum_{r=1}^n \int_{c_n}^\infty t^{-\mu-1}\, dt \int_1^t (t-u)^{\mu-1} u\, x_r(u)\, k(u)\, du$$

$$\geqslant \sum_{r=1}^n \int_1^\infty u\, x_r(u)\, k(u)\, du \int_u^\infty (t-u)^{\mu-1} t^{-\mu-1}\, dt - \sum_{r=1}^n \int_{c_n}^\infty t^{-2}\, dt \int_1^\infty u\, |x_r(u)\, k(u)|\, du$$

$$= \frac{1}{\mu} \sum_{r=1}^n f(x_r) - \frac{1}{c_n} \sum_{r=1}^n \int_1^\infty u\, |x_r(u)\, k(u)|\, du \geqslant \frac{n}{\mu} - 1.$$

Hence $\int_1^\infty x(u)\, k(u)\, du$ is not bounded $(C, \mu)$ for any $\mu \geqslant 1$ and so $x(u)\, k(u)$ is not in $B$.

On the other hand, for $n = 1, 2, \ldots$,

$$\int_1^{c_n} t^{-\lambda-1}\, dt \left| \int_1^t (t-u)^{\lambda-1} u\, x(u)\, du \right| \leqslant \sum_{r=1}^n \int_1^{c_n} t^{-\lambda-1}\, dt \left| \int_1^t (t-u)^{\lambda-1} u\, x_r(u)\, du \right|$$

$$\leqslant \sum_{r=1}^n \|x_r\| < \sum_{r=1}^n 2^{-r} < 1,$$

and hence $x(t) \,\varepsilon\, V_\lambda$.

Since this contradicts the hypothesis, the required result is established.

* *Cf.* Sargent (4), Lemma 3.

*Proof of* b(ii). We can proceed as above, with $S$ in place of $V$, up to and including the statement:

$$x(u)\,k(u) \text{ is not in } B;$$

and then, to complete the proof, obtain a contradiction as follows.

Let $s$ be an arbitrary positive integer. Then

$$\overline{\lim_{\substack{w\to\infty\\ v\to\infty}}}\left|\int_v^w t^{-\lambda-1}\,dt\int_1^t (t-u)^{\lambda-1}u\,x(u)\,du\right|$$

$$\leqslant \overline{\lim_{\substack{w\to\infty\\ v\to\infty}}}\left\{\sum_{r=1}^{s}\left|\int_v^w t^{-\lambda-1}\,dt\int_1^t (t-u)^{\lambda-1}ux_r(u)\,du\right| + \sum_{r=s+1}^{\infty}\left|\int_v^w t^{-\lambda-1}\,dt\int_1^t (t-u)^{\lambda-1}u\,x_r(u)\,du\right|\right\}$$

$$\leqslant 2\sum_{r=s+1}^{\infty}\|x_r\| < 2\sum_{r=s+1}^{\infty}2^{-r} = 2^{1-s}.$$

Hence $x(t)\,\varepsilon\,S_\lambda$.

LEMMA 3. *If $c \geqslant 1$, $r>0$ and $x(t)\,\varepsilon\,L(1, w)$ for all $w>1$, then a necessary and sufficient condition for $x(t)$ to be in $V_\lambda$ is that*

$$\int_c^\infty t^{-\lambda-r}\,dt\left|\int_c^t (t-u)^{\lambda-1}u^r x(u)\,du\right| < \infty.$$

This follows from a result established elsewhere*.

LEMMA 4. *If $\lambda>1$ and $x(t)\,\varepsilon\,V_\lambda$, then $t^{-3}\int_1^t u^2x(u)\,du\,\varepsilon\,V_{\lambda-1}$.*

This also follows from the above-mentioned result.

LEMMA 5. *If $x(t)\,\varepsilon\,V_\lambda$, then $tx(t)=o(1)\,|\,C, \lambda+1\,|$ as $t\to\infty$.*

This well-known result follows from the identity

$$t^{-\lambda-1}\int_1^t (t-u)^\lambda u\,x(u)\,du = t^{-\lambda}\int_1^t (t-u)^\lambda x(u)\,du - t^{-\lambda-1}\int_1^t (t-u)^{\lambda+1}x(u)\,du \qquad (t\geqslant 1).$$

4. We shall now prove two theorems; the first of these includes both necessity parts of Theorem 1 and the second is simply a restatement of the sufficiency part of Theorem 1(a).

* Borwein (3), Theorem 1 with $\rho=-r$, $\alpha=\lambda$.



THEOREM* 2. (a) *If $x(t)k(t)\,\varepsilon\,B$ whenever $x(t)\,\varepsilon\,V_\lambda$, then there is a number $c\geqslant 1$ such that*

(i) $k(t)\,\varepsilon\,M(1, c)$,

(ii) $$\frac{k(t)}{t} = \frac{1}{\Gamma(\lambda)}\int_t^\infty (u-t)^{\lambda-1}h(u)\,du \text{ p.p. in } (c, \infty),$$

*where $u^{\lambda+1}h(u)\,\varepsilon\,M(c, \infty)$.*

(b) *Replace $V$ by $S$, and $M$ in* (ii) *by $BV$.*

Parts a(i) and b(i) follow from the corresponding parts of Lemma 2.

*Proof of* a(ii). In view of Lemmas 1(a) and 2(a) there is a number $c\geqslant 1$ such that, for all $x(t)$ in $V_\lambda{}^c$,

$$\int_1^\infty x(t)\,k(t)\,dt = \frac{1}{\Gamma(\lambda)}\int_1^\infty t\,x(t)\,dt\int_t^\infty (u-t)^{\lambda-1}h(u)\,du,$$

where $u^{\lambda+1}h(u)\,\varepsilon\,M(1, \infty)$. The required result follows since, for arbitrary $w>c$, $V_\lambda{}^c$ contains the characteristic function of the interval $[c, w]$.

*Proof of* b(ii). Replace (a) by (b), $V$ by $S$ and $M$ by $BV$ in the above proof.

THEOREM 3. *If $x(t)\,\varepsilon\,V_\lambda$ and, for some number $c\geqslant 1$,*

(i) $k(t)\,\varepsilon\,M(1, c)$,

(ii) $$\frac{k(t)}{t} = \frac{1}{\Gamma(\lambda)}\int_t^\infty (u-t)^{\lambda-1}h(u)\,du \text{ p.p. in } (c, \infty),$$

*where $u^{\lambda+1}h(u)\,\varepsilon\,M(c, \infty)$, then $x(t)k(t)\,\varepsilon\,V_\lambda$.*

Write, for $t\geqslant c$, $\mu>0$,

$$g(t) = t^2x(t),$$

$$g_\mu(t) = \frac{1}{\Gamma(\mu)}\int_c^t (t-u)^{\mu-1}g(u)\,du;$$

and let $$H = \overline{\operatorname*{ess.\,bound}_{t\geqslant c}}\, t^{\lambda+1}|h(t)|.$$

Note that $x(t)k(t)\,\varepsilon\,L(1, w)$ for all $w>1$ and so, by Lemma 3, it is sufficient to prove that

$$\int_c^\infty t^{-\lambda-1}\,dt\left|\int_c^t (t-u)^{\lambda-1}g(u)\,\frac{k(u)}{u}\,du\right|$$

* Version (b) of this theorem is slightly more general than Theorem 1 in Sargent (4).

is finite. Further, since $x(t) \varepsilon V_\lambda$, we have, by Lemma 3, that

$$\int_c^\infty t^{-\lambda-2} |g_\lambda(t)|\, dt < \infty.$$

*Case* 1. *Suppose that* $0 < \lambda < 1$, and write, for $t > v > c$,

$$Q(v,\, t) = \frac{1}{\Gamma(\lambda)} \int_c^v (v-u)^{\lambda-1} (t-u)^{\lambda-1} g(u)\, du.$$

It has been shown* that, for almost all $v$ in $(c,\, t)$,

$$|Q(v,\, t)| \leqslant (t-v)^{\lambda-1} |g_\lambda(v)| + (t-v)^{\frac{1}{2}\lambda-1} \int_c^v (t-w)^{\frac{1}{2}\lambda-1} |g_\lambda(w)|\, dw.$$

Hence

$$\int_c^\infty t^{-\lambda-1} dt \left| \int_c^t (t-u)^{\lambda-1} g(u) \frac{k(u)}{u}\, du \right|$$

$$= \frac{1}{\Gamma(\lambda)} \int_c^\infty t^{-\lambda-1} dt \left| \int_c^t (t-u)^{\lambda-1} g(u)\, du \int_u^\infty (v-u)^{\lambda-1} h(v)\, dv \right|$$

$$= \int_c^\infty t^{-\lambda-1} dt \left| \int_c^t h(v)\, Q(v,\, t)\, dv + \int_t^\infty h(v)\, Q(t,\, v)\, dv \right|$$

$$\leqslant H \int_c^\infty v^{-\lambda-1} dv \int_v^\infty t^{-\lambda-1} |Q(v,\, t)|\, dt + H \int_c^\infty t^{-\lambda-1} dt \int_t^\infty v^{-\lambda-1} |Q(t,\, v)|\, dv$$

$$= 2H \int_c^\infty v^{-\lambda-1} dv \int_v^\infty t^{-\lambda-1} |Q(v,\, t)|\, dt$$

$$\leqslant 2H \int_c^\infty v^{-\lambda-1} |g_\lambda(v)|\, dv \int_v^\infty (t-v)^{\lambda-1} t^{-\lambda-1}\, dt$$

$$+ 2H \int_c^\infty v^{-\lambda-1} dv \int_v^\infty (t-v)^{\frac{1}{2}\lambda-1} t^{-\lambda-1} dt \int_c^v (v-w)^{\frac{1}{2}\lambda-1} |g_\lambda(w)|\, dw$$

$$= \frac{2H}{\lambda} \int_c^\infty v^{-\lambda-2} |g_\lambda(v)|\, dv$$

$$+ 2H \int_c^\infty |g_\lambda(w)|\, dw \int_w^\infty (v-w)^{\frac{1}{2}\lambda-1} v^{-\lambda-1}\, dv \int_v^\infty (t-v)^{\frac{1}{2}\lambda-1} t^{-\lambda-1}\, dt$$

$$= 2H\{\lambda^{-1} + B(\tfrac{1}{2}\lambda,\, 1+\tfrac{1}{2}\lambda)\, B(\tfrac{1}{2}\lambda,\, 2+\lambda)\} \int_c^\infty v^{-\lambda-2} |g_\lambda(v)|\, dv < \infty.$$

The result in this case follows.

* Borwein (2), inequality (6.7): note that this differs from the required inequality by a factor $\Gamma(\lambda)$ in the left-hand side and that a suitable value for the constant $M$ is $\max\{\Gamma(\lambda),\, (1-\lambda)^{\frac{1}{2}}\, \Gamma(\lambda+1)\} = \Gamma(\lambda)$ $(0 < \lambda < 1)$.



*Case* 2. *Suppose that* $\lambda = 1$. The required result is now obtained from the following inequality:

$$\int_c^\infty t^{-2} dt \left| \int_c^t g(u) \frac{k(u)}{u}\, du \right| = \int_c^\infty t^{-2} dt \left| \int_c^t g(u)\, du \int_u^\infty h(v)\, dv \right|$$

$$\leqslant H \int_c^\infty t^{-2} dt \int_c^t v^{-2} |g_1(v)|\, dv + H \int_c^\infty t^{-2} |g_1(t)|\, dt \int_t^\infty v^{-2}\, dv$$

$$= 2H \int_c^\infty v^{-3} |g_1(v)|\, dv < \infty.$$

*Case* 3. *Suppose that* $\lambda > 1$, and assume the result with $\lambda$ replaced by $\lambda - 1$. Suppose further, without any loss in generality, that $x(t) = 0$ for $1 \leqslant t \leqslant c$.

Let $p(t) = \dfrac{t}{\Gamma(\lambda-1)} \displaystyle\int_t^\infty (u-t)^{\lambda-2} u h(u)\, du$ when $t > c$, $p(t) = 0$ when $1 \leqslant t \leqslant c$, and note that, for almost all $t \geqslant c$,

$$k(t) = \frac{t}{\Gamma(\lambda-1)} \int_t^\infty (u-t)^{\lambda-2}\, du \int_u^\infty h(v)\, dv.$$

Then it is easily verified that, for $t \geqslant c$,

$$\int_1^t x(u)\, k(u)\, du = \int_1^t g(u)\, u^{-2} k(u)\, du$$

$$= t^{-2} g_1(t)\, k(t) + (2-\lambda) \int_1^t u^{-3} g_1(u)\, k(u)\, du + \int_1^t u^{-3} g_1(u)\, p(u)\, du.$$

Now $u^\lambda \,.\, uh(u) \varepsilon M(c,\, \infty)$, $u^\lambda \displaystyle\int_u^\infty h(v)\, dv \,\varepsilon\, M(c,\, \infty)$ and so both $k(t)$ and $p(t)$ satisfy the hypotheses of $k(t)$ with $\lambda$ replaced by $\lambda - 1$. Further, since $x(u) \varepsilon V_\lambda$, we have, by Lemma 4, that $u^{-3} g_1(u) = u^{-3} \displaystyle\int_1^u v^2 x(v)\, dv \,\varepsilon\, V_{\lambda-1}$. Thus, by the assumption, $u^{-3} g_1(u)\, p(u) \varepsilon V_{\lambda-1}$, $u^{-3} g_1(u)\, k(u) \varepsilon V_{\lambda-1}$ and hence, by Lemma 5, $t^{-2} g_1(t)\, k(t) = o(1) \,|\, C,\, \lambda|$ as $t \to \infty$.

It follows that $x(u)\, k(u) \varepsilon V_\lambda$ and the result in this case is thus established by induction from the two previous cases.

This completes the proof of the theorem.

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