SCALES OF LOGARITHMIC METHODS OF SUMMABILITY

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1. Introduction. We suppose throughout that p is a non-negative integer, and use the following notations:

$$\pi_p(x) = \begin{cases} \dfrac{1}{\log_0 x \cdot \log_1 x \cdot \cdots \cdot \log_p x} & \text{for } x \geq e_p, \\ 0 & \text{otherwise}, \end{cases}$$

where $\log_0 x = x$ for $x \geq e_0 = 1$, and $\log_{n+1} x = \log(\log_n x)$ for $x \geq e_{n+1} = e^{e_n}$ $(n = 0, 1, 2, \cdots)$;

$$\sigma_p(x) = \sum_{n=0}^{\infty} \pi_p(n)\, x^n \qquad (-1 < x < 1);$$

$$s_n = \sum_{k=0}^{n} a_k \qquad (n = 0, 1, 2, \cdots);$$

$$t_p(n) = \frac{1}{\log_{p+1} n} \sum_{k=0}^{n} \pi_p(k) s_k \quad (n \geq e_{p+1}).$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable L_p to s, and we write $\sum_{n=0}^{\infty} a_n = s$ (L_p) or $s_n \to s$ (L_p), if

$$\lim_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n)\, s_n x^n = s.$$

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If $t_p(n) \to s$ as $n \to \infty$ the series $\sum_{n=0}^{\infty} a_n$ is said to be summable ℓ_p to s, and we write $\sum_{n=0}^{\infty} a_n = s\ (\ell_p)$ or $s_n \to s(\ell_p)$ (see [5]).

Given two summability methods A, B we write $A \supseteq B$ if any series summable B is summable A to the same sum; if in addition there is a series summable A but not summable B we write $A \supset B$. If $A \supseteq B$ and $B \supseteq A$ the two methods are said to be equivalent and we write $A \simeq B$. It is known [5] that the L_p and ℓ_p methods are regular and that $L_0 \simeq L$, $\ell_0 \simeq \ell$ where L and ℓ are standard logarithmic methods (for definitions see [3]). The aim of this paper is to establish various inclusion theorems for the two scales of methods.

2. Lemmas. We require four lemmas.

LEMMA 1. *If* $s_n \to s(\ell_p)$, *then* $s_n = o\left(\frac{1}{\pi_{p+1}(n)}\right)$ *and*

$$a_n = o\left(\frac{1}{\pi_{p+1}(n)}\right).$$

Proof. The case $p = 0$ of this lemma is due to Ishiguro [3, Theorem 4]. For $n - 1 \geq e_{p+1}$ we have that

$$s_n = \frac{1}{\pi_p(n)}[t_p(n)\log_{p+1} n - t_p(n-1)\log_{p+1}(n-1)];$$

hence

$$\pi_{p+1}(n)s_n = t_p(n) - t_p(n-1)\frac{\log_{p+1}(n-1)}{\log_{p+1} n} \to 0,$$

and so,

$$\pi_{p+1}(n)a_n = \pi_{p+1}(n)s_n - \frac{\pi_{p+1}(n)}{\pi_{p+1}(n-1)}\pi_{p+1}(n-1)s_{n-1} \to 0.$$

LEMMA 2. $L_p \supseteq \ell_p$.

Proof. Since $\ell_p \simeq (\bar{N}, q_n)$ with $q_n = \pi_p(n)$, the lemma follows from a known result [4, Theorem 1].

LEMMA 3. *If* $x \geq e_p$, $y > 0$, *then*

$$(\log_p x)^{-y} = \int_0^{\infty} e^{-xt}\lambda_{p,y}(t)dt,$$

where $\lambda_{p,y}(t)$ *is defined by the recursive formulae:*

$$\lambda_{0,y}(t) = \frac{t^{y-1}}{\Gamma(y)},$$

$$\lambda_{r+1,y}(t) = \frac{1}{\Gamma(y)}\int_0^{\infty} u^{y-1}\lambda_{r,u}(t)du \qquad (r = 0, 1, 2, \cdots).$$

Proof. The lemma is true for $p = 0$, since, when $x \geq e_0 = 1$,

$$(\log_0 x)^{-y} = x^{-y} = \frac{1}{\Gamma(y)}\int_0^{\infty} e^{-xt}t^{y-1}dt = \int_0^{\infty} e^{-xt}\lambda_{0,y}(t)dt.$$

Assume the lemma is true for $p = r$. Then, for $x \geq e_{r+1}$ we have

$$(\log_{r+1} x)^{-y} = \frac{1}{\Gamma(y)}\int_0^{\infty} e^{-u\log_{r+1} x}u^{y-1}du$$

$$= \frac{1}{\Gamma(y)}\int_0^{\infty} (\log_r x)^{-u}u^{y-1}du$$

$$= \frac{1}{\Gamma(y)} \int_0^\infty u^{y-1} du \int_0^\infty e^{-xt} \lambda_{r,u}(t) dt$$

$$= \int_0^\infty e^{-xt} dt \frac{1}{\Gamma(y)} \int_0^\infty u^{y-1} \lambda_{r,u}(t) du$$

$$= \int_0^\infty e^{-xt} \lambda_{r+1,y}(t) dt,$$

the inversion in the order of integration being justified by Fubini's theorem since all the functions concerned are non-negative and Lebesgue measurable. The lemma is thus established by induction.

The case $p = 1$ of the next lemma is due to Hardy [2, page 268].

LEMMA 4. *If* $n \geq e_p$, $y > 0$, *then*

$$(\log_p n)^{-y} = \int_0^1 t^n \phi(t) dt,$$

where the function ϕ *is non-negative and independent of* n.

Proof. By Lemma 3,

$$(\log_p n)^{-y} = \int_0^\infty e^{-nx} \lambda_{p,y}(x) dx = \int_0^1 t^n \phi(t) dt,$$

where $\phi(t) = \frac{1}{t} \lambda_{p,y}(\log \frac{1}{t})$.

3. *Inclusion Theorems.*

THEOREM 1. *There is a series summable* ℓ_{p+1} *but not summable* L_p *i.e.* $L_p \not\supseteq \ell_{p+1}$.

Proof. Let N be the integer such that $N - 1 < e_{p+1} \leq N$, and, with $i = \sqrt{-1}$, let

$$a_n = \begin{cases} \pi_p(n)(\log_{p+1} n)^{-1-i} & \text{for } n \geq e_{p+1}, \\ 0 & \text{for } n < e_{p+1}. \end{cases}$$

Then

$$s_{n-1} - \left((\log_{p+1} n)^{-i} - (\log_{p+1} N)^{-1}\right)$$

$$= \sum_{k=N}^{n-1} \pi_p(k)(\log_{p+1} k)^{-1-i} - \int_N^n (\log_{p+1} t)^{-1-i} \pi_p(t) dt$$

$$= \sum_{k=N}^{n-1} \Phi_k,$$

where

$$\Phi_k = \int_k^{k+1} \left(\int_k^t \left(- \frac{d}{dx} \pi_p(x)(\log_{p+1} x)^{-1-i}\right) dx \right) dt$$

$$= \int_k^{k+1} \left(\int_k^t (\pi_p(x))^2 (\log_{p+1} x)^{-1-i} \left(\sum_{r=0}^{p} \frac{\pi_r(x)}{\pi_p(x)} + (1+i)(\log_{p+1} x)^{-1} \right) dx \right) dt$$

$$= \int_k^{k+1} \left(\int_k^t O\left(\frac{1}{x^2}\right) dx \right) dt$$

$$= O\left(\frac{1}{k^2}\right).$$

Hence $\sum_{k=N}^{\infty} \Phi_k$ converges, and so $s_{n-1} - i(\log_{p+1} n)^{-1}$ tends to a finite limit as $n \to \infty$. Since $s_n = s_{n-1} + \pi_p(n)(\log_{p+1} n)^{-1-i}$, we have that $s_n = i(\log_{p+1} n)^{-1} + k_n$ where k_n tends to a finite limit as $n \to \infty$.

Consequently $\{s_n\}$ is bounded but does not converge, and as $a_n = O(\pi_{p+1}(n))$, it follows from a known tauberian theorem [5, Corollary] that $\sum_{n=0}^{\infty} a_n$ is not L_p summable.

We now show that $\sum_{n=0}^{\infty} a_n$ is ℓ_{p+1} summable. For $m \geq N$, we have that

$$t_{p+1}(m) = \frac{1}{\log_{p+2} m} \sum_{n=N}^{m} \pi_{p+1}(n) \left(\frac{(\log_{p+1} n)^{-i}}{i} + k_n \right)$$

$$= \frac{1}{i \log_{p+2} m} \sum_{n=N}^{m} \pi_p(n)(\log_{p+1} n)^{-1-i}$$

$$+ \frac{1}{\log_{p+2} m} \sum_{n=0}^{m} \pi_{p+1}(n) k_n$$

$$= \frac{1}{i \log_{p+2} m} s_m + \frac{1}{\log_{p+2} m} \sum_{n=0}^{m} \pi_{p+1}(n) k_n ,$$

and hence $t_{p+1}(m)$ tends to a finite limit as $m \to \infty$.

THEOREM 2. $L_{p+1} \supset L_p$.

Proof. By Lemma 4, for $n \geq e_{p+1}$,

$$\frac{\pi_{p+1}(n)}{\pi_p(n)} = (\log_{p+1} n)^{-1} = \int_0^1 t^n \phi(t)dt ,$$

where $\phi(t)$ is non-negative and independent of n, and hence, by a result due to Borwein [1, Theorem A], $L_{p+1} \supseteq L_p$. The stronger inclusion follows immediately from Theorem 1 and Lemma 2.

THEOREM 3. $L_p \supset \ell_p$.

Proof. We consider a series used to show the existence of a series summable by the Abel method A, but not summable by any Cesàro method [2, Theorem 56].

Let

$$e^{1/(1+x)} = \sum_{n=0}^{\infty} a_n x^n .$$

It is known that a_n is not $O(n^r)$ for any r, and hence, by Lemma 1, $\sum_{n=0}^{\infty} a_n$ is not summable ℓ_p. Since the series is summable A, and [2, page 81] $A \subset L \simeq L_0 \subseteq L_p$, the theorem can now be deduced from Lemma 2.

THEOREM 4. $\ell_{p+1} \supset \ell_p$.

Proof. The inclusion $\ell_{p+1} \supseteq \ell_p$ follows immediately from a a known theorem for $\overline{N}$ methods [2, Theorem 14]. The stronger inclusion may be deduced from Theorem 1. However a direct proof is easy.

Consider

$$s_n = (-1)^n \frac{1}{\pi_{p+1}(n)} \quad (n \geq e_{p+1}).$$

Then $s_n \to 0\ (\ell_{p+1})$, i.e. $\sum_{n=0}^{\infty} a_n$ is summable ℓ_{p+1}, but $s_n \neq o(\frac{1}{\pi_{p+1}(n)})$; hence, by Lemma 1, $\sum_{n=0}^{\infty} a_n$ is not ℓ_p summable.

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