

In general, however, we are interested in short papers dealing with material familiar to a reasonable number of MONTHLY readers. We are able to afford more space than is common in research journals, and authors are urged to add an occasional paragraph if that will increase readability. But most important, there must be a sound mathematical reason for an article. One theorem with real content and an elegant proof is worth twenty-five new definitions or minor generalizations. Results without significant examples are always judged inferior to those with meaningful applications.

**II. Classroom Notes.** Contributions to this section are few in number. This is not surprising since many undergraduate fields have been so worked over that new insights useful in the classroom are not easy to produce. Good Classroom Notes are greatly valued and will be published promptly.

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#### A PROPERTY OF GRADIENTS

D. BORWEIN, University of Western Ontario, Canada, and A. MEIR, University of Alberta, Canada

The object of this note is to prove the following:

**THEOREM.** *Let  $f$  be a continuous real-valued function on the unit Euclidean  $n$ -ball  $B = \{x: \|x\| \leq 1\}$ , let the first order partial derivatives of  $f$  exist at every point in the interior of  $B$ , and let  $|f(x)| \leq 1$ , whenever  $\|x\| = 1$ . Then there is a point  $v$  in the interior of  $B$  for which  $\|\text{grad } f(v)\| \leq 1$ .*

The function  $f$ , defined by  $f(x) = a \cdot x$  where  $\|a\| = 1$ , shows that the final inequality in the Theorem cannot be sharpened. The Theorem thus provides the answer to H. S. Shapiro's question concerning the best value of the constant in Problem E 1986 [this MONTHLY, 75 (1968) p. 787]. In addition, the Theorem shows that the hypotheses of the problem, that  $f$  be differentiable on an open set containing  $B$ , and that  $|f(x)| \leq 1$  for every  $x$  in  $B$ , can be relaxed.

*Proof of the Theorem.* Let

$$g(x) = \|x\|^2 - f(x)^2, \quad \lambda = \min_{\|x\| \leq 1} g(x).$$

Then  $g(x) \geq 0$  whenever  $\|x\| = 1$ , and  $\lambda \leq g(0) \leq 0$ . There is thus a point  $v$  such that  $\|v\| < 1$  and  $g(v) = \lambda$ . Consequently

$$\text{grad } g(v) = 2v - 2f(v) \text{ grad } f(v) = 0,$$

and so

$$f(v)^2 \|\text{grad } f(v)\|^2 = \|v\|^2 = f(v)^2 + \lambda \leq f(v)^2.$$

The required conclusion follows either if  $\lambda < 0$ , or if  $\lambda = 0$  and  $g(v) = 0$  for some point  $v$  such that  $0 < \|v\| < 1$ .

In the one remaining case we have  $g(x) > \lambda = 0$  whenever  $0 < \|x\| < 1$ ; so that  $|f(x)| < \frac{1}{2}$  whenever  $\|x\| = \frac{1}{2}$  and, consequently, there is a positive  $\epsilon$  for which

$$\max_{\|x\|=1/2} |f(x) + \epsilon| < \frac{1}{2}.$$

Setting

$$h(x) = \|x\|^2 - \{f(x) + \epsilon\}^2, \quad \mu = \min_{\|x\| \leq 1/2} h(x),$$

we observe that  $h(x) > 0$  whenever  $\|x\| = \frac{1}{2}$ , and  $\mu \leq h(0) = -\epsilon^2 < 0$ . Hence there is a point  $w$  such that  $\|w\| < \frac{1}{2}$  and  $h(w) = \mu$ . As above we deduce that

$$\{f(w) + \epsilon\}^2 \|\text{grad } f(w)\|^2 = \|w\|^2 = \{f(w) + \epsilon\}^2 + \mu < \{f(w) + \epsilon\}^2,$$

from which it follows that  $\|\text{grad } f(w)\| < 1$ .

### PIVOTAL ROLE OF THE TRIPLE CROSS PRODUCT

FRANCIS P. CALLAHAN, Pennsylvania State University

This paper gives a proof of the following:

**THEOREM.** *Let  $(R, \cdot)$  be a real inner product space and let  $(x \circ y)$  be a vector product defined in  $R$ . If  $R$  is at least two dimensional and if the vector product satisfies the identity*

$$(1) \quad (x \circ y) \circ z = (x \cdot z)y - (y \cdot z)x$$

for all  $x, y, z$  in  $R$ , then

- (A)  $(x \circ y)$  is linear in both  $x$  and  $y$ ,
- (B)  $(x \circ y) + (y \circ x) = 0$  for all  $x, y$  in  $R$ ,
- (C)  $R$  is exactly three dimensional, and
- (D) if  $(i, j, k)$  is an orthonormal basis for  $R$ , then  $(i \circ j) = ck$ ,  $(j \circ k) = ci$ , and  $(k \circ i) = cj$ , where  $c$  is either  $+1$  or  $-1$ .

By a vector product,  $(x \circ y)$ , is meant a not necessarily linear function defined for all ordered pairs,  $(x, y)$ , of elements of  $R$  and having values again in  $R$ . The identity (1) is the "triple cross product identity" and the conclusion of the Theorem is that  $(x \circ y)$  can only be the usual three-space cross product (either right handed or left handed). That is, the triple cross product identity is sufficient to single out the usual cross product(s) from among all possible vector products, and it even forces the dimension of the space on which it operates to be three.

The proof of the Theorem makes use of some lemmas.

**LEMMA 1.**  $x \circ y = y \circ (-x)$  for all  $x, y$  in  $R$ .

*Proof.* Let  $x \circ y = u$  and  $y \circ (-x) = u'$  and apply (1) to show that  $u \circ v = u' \circ v$  for all  $v$  in  $R$ , so that