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On Abel-type methods of summability

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1. Introduction

Let

$$\varepsilon_n^\lambda = \binom{n+\lambda}{n},$$

$$s_n = \sum_{r=0}^{n} u_r,$$

$$s_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n,$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda u_n \left(\frac{y}{1+y}\right)^n,$$

$$U_\lambda(y) = \lambda \int_0^y u_\lambda(t)\, dt.$$

The Abel-type summability methods A_λ and A'_λ are defined as follows (see [2] and [3]):

If

$$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$$

is convergent for all x in the open interval (0, 1) and tends to a finite limit l as $x \to 1$ in (0, 1), we say that the sequence $\{s_n\}$ is A_λ-convergent to l and write $s_n \to l(A_\lambda)$. The A_0 method is the ordinary Abel method.

Evidently $s_n \to l(A_\lambda)$ if and only if the series defining $s_\lambda(y)$ is convergent for all $y > 0$ and $s_\lambda(y) \to l$ as $y \to \infty$.

If $u_\lambda(y)$ is defined for all $y > 0$ and $U_\lambda(y)$ tends to a finite limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is A'_λ-convergent to l and write $s_n \to l(A'_\lambda)$.

The methods A_λ and $A'_{\lambda+1}$ are regular for $\lambda > -1$. (See [2], Theorem 1; and [4], § 4. 13.)

We state next the definition of a Hausdorff method H_χ and the product method $A_\lambda H_\chi$:

Given a real function $\chi(t)$ of bounded variation in the interval [0, 1], let

$$h_n = \sum_{r=0}^{n} \binom{n}{r} s_r \int_0^1 t^r (1-t)^{n-r} d\chi(t). \tag{1. 1}$$

If $h_n \to l$ as $n \to \infty$, we write $s_n \to l(H_\chi)$.

If $h_n \to l(A_\lambda)$, we write $s_n \to l(A_\lambda H_\chi)$.

We note that the conditions $\chi(0+) = \chi(0)$ and $\chi(1) - \chi(0) = 1$ are necessary and sufficient for the regularity of the method H_χ.

The following two results are proved in [3]:

Theorem A. *For $\lambda > 0$, $s_n \to l(A_\lambda)$ if and only if $s_n \to l(A'_\lambda)$ and $nu_n \to 0(A_{\lambda-1})$.*

Theorem B. *For $\lambda > 0$, $s_n \to l(A_{\lambda-1})$ if and only if $s_n \to l(A'_\lambda)$.*

It is also known that:

Theorem C. *If $\lambda > -1$ and H_χ is a regular Hausdorff method, then $s_n \to l(A_\lambda H_\chi)$ whenever $s_n \to l(A_\lambda)$.*

The case $\lambda = 0$ of Theorem C was proved by Szasz in [6] and the general case by Amir (Jakimovski) in [1]. See also [2] for a shorter proof of the general case.

In this note we prove, *interalia*, the absolute summability analogues of the above three theorems.

2. Definitions

We now define absolute summability based upon the Abel-type methods A_λ and A'_λ and the product method $A_\lambda H_\chi$.

Absolute Abel-type summability $|A_\lambda|$.

If $s_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is absolutely A_λ-convergent, or $|A_\lambda|$-convergent, to l, and write $s_n \to l \mid A_\lambda \mid$.

Absolute Abel-type summability $|A'_\lambda|$.

If $U_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is absolutely A'_λ-convergent, or $|A'_\lambda|$-convergent, to l, and write $s_n \to l \mid A'_\lambda \mid$.

Absolute summability $|A_\lambda H_\chi|$.

If $h_n \to l \mid A_\lambda \mid$, we say that the sequence $\{s_n\}$ is absolutely $A_\lambda H_\chi$-convergent, or $|A_\lambda H_\chi|$-convergent, to l, and write $s_n \to l \mid A_\lambda H_\chi \mid$.

Remarks.

1. The definition of $|A_\lambda|$-convergence is the case $p = 1$ of the definition of $|A_\lambda|_p$-convergence as given by Mishra in [5].

2. The function $\psi(x) = (1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ is of bounded variation in $[0, 1)$ if and only if $s_\lambda(y)$ is of bounded variation in $[0, \infty)$.

3. Preliminary Results

We require the following lemmas.

Lemma 1. *If $\lambda > \mu > -1$, $y > 0$ and $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

$$(3.1)\qquad s_\mu(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu s_\lambda(t)\, dt.$$

This lemma has been proved in [2] (Lemma 2 (i)).

Lemma 2. *If $\lambda > -1$, $y > 0$ and $\Sigma\, \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

$$(3.2)\qquad u_\lambda(y) = (1+y)^{-1} s_\lambda(y) - \lambda(1+y)^{-\lambda-1} \int_0^y (1+t)^{\lambda-1} s_\lambda(t)\, dt,$$

$$(3.3)\qquad u_\lambda(y) = (1+y)^{-\lambda-1} s_\lambda(0) + (1+y)^{-\lambda-1} \int_0^y (1+t)^\lambda s'_\lambda(t)\, dt,$$

$$(3.4)\qquad U_\lambda(y) = \lambda(1+y)^{-\lambda} \int_0^y (1+t)^{\lambda-1} s_\lambda(t)\, dt,$$

$$(3.5)\qquad s_\lambda(y) = U_\lambda(y) + (1+y) u_\lambda(y),$$

$$(3.6)\qquad s_\lambda(y) = U_{\lambda+1}(y) + u_\lambda(y),$$

$$(3.7)\qquad U_{\lambda+1}(y) = (\lambda+1) y^{-\lambda-1} \int_0^y t^\lambda U_{\lambda+2}(t)\, dt.$$

The proofs are elementary and are omitted. Some of these identities, in essence, are proved in [3], pp. 73—74.

Lemma 3. *If $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ is convergent for $0 \leqq x < 1$ and h_n is defined by (1.1), then, for $y > 0$,*

$$(3.8)\qquad (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda h_n \left(\frac{y}{1+y}\right)^n = \int_0^1 s_\lambda(yt)\, d\chi(t).$$

This lemma is proved in [2] (Lemma 5). See also [1], p. 376.

Lemma 4. *If $\lambda > -1$, a is real, $s_\lambda(y) = O(1)$ for $y > 0$ and $(n+a)v_n = s_n$ for $n = 0, 1, 2, \ldots$, then $v_n \to 0 \mid A_\lambda \mid$.*

Proof. Let

$$\varphi(x) = (1-x)^{\lambda+1} \sum_{n=m}^{\infty} \varepsilon_n^\lambda v_n x^n$$

and

$$\psi(x) = \sum_{n=m}^{\infty} \varepsilon_n^\lambda s_n x^{n+a-1},$$

where $m > |a| + 1$. Because of the convergence of the series defining $\varphi(x)$ in $(0, 1)$ and Lemma 4 in [2], it suffices to show that $x^a \varphi(x)$ is of bounded variation in $[1/2, 1)$. By hypothesis, we have for $0 \leqq x < 1$,

$$x^a \varphi(x) = (1-x)^{\lambda+1} \int_0^x \psi(t)\, dt$$

and

$$|\psi(x)| < K(1-x)^{-\lambda-1}$$

where K is a positive constant.

Hence

$$\int_{\frac{1}{2}}^{1}\left|\frac{d}{dx}(x^a\varphi(x))\right|dx \leqq (\lambda+1)\int_{\frac{1}{2}}^{1}(1-x)^{\lambda}dx\int_{o}^{x}|\psi(t)|\,dt+\int_{\frac{1}{2}}^{1}(1-x)^{\lambda+1}|\psi(x)|\,dx$$

$$\leqq K(\lambda+1)\int_{o}^{1}(1-t)^{-\lambda-1}dt\int_{t}^{1}(1-x)^{\lambda}dx+K=2K.$$

The lemma follows.

An immediate consequence of the above lemma is the following:

Lemma 5. *If $\lambda>-1$, p and q are real and $s_n\to l\,|A_\lambda|$, then*

$$\frac{n+p}{n+q}s_n\to l\,|A_\lambda|.$$

4. Main Results

Theorem 1. *The method $|A_\lambda|$ is translative for $\lambda>-1$.*

By this we mean that $s_n\to l\,|A_\lambda|$ if and only if $s_{n+1}\to l\,|A_\lambda|$.

Proof. Suppose that $s_n\to l\,|A_\lambda|$. We have, for $0\leqq x<1$,

(4. 1) $$(1-x)^{\lambda+1}\sum_{n=1}^{\infty}\varepsilon_n^\lambda s_{n-1}x^n=(1-x)^{\lambda+1}x\sum_{n=o}^{\infty}\varepsilon_n^\lambda s_n x^n+\lambda(1-x)^{\lambda+1}x\sum_{n=o}^{\infty}\varepsilon_n^\lambda\frac{s_n}{n+1}x^n,$$

(4. 2) $$(1-x)^{\lambda+1}x\sum_{n=o}^{\infty}\varepsilon_n^\lambda s_{n+1}x^n=(1-x)^{\lambda+1}\sum_{n=1}^{\infty}\varepsilon_n^\lambda s_n x^n-\lambda(1-x)^{\lambda+1}\sum_{n=1}^{\infty}\varepsilon_n^\lambda\frac{s_n}{n+\lambda}x^n.$$

Applying Lemma 4, we deduce from (4. 1) that $s_{n-1}\to l\,|A_\lambda|$, and from (4. 2) that $s_{n+1}\to l\,|A_\lambda|$. The theorem follows.

Theorem 2. *If $\lambda>\mu>0$ and $s_n\to l\,|A'_\lambda|$, then $s_n\to l\,|A'_\mu|$.*

Proof. We have, by (3. 1) with u in place of s, that

$$\int_{o}^{\infty}|u_\mu(y)|\,dy\leqq\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}\int_{o}^{\infty}y^{-\lambda}dy\int_{o}^{y}(y-t)^{\lambda-\mu-1}t^\mu\,|u_\lambda(t)|\,dt$$

$$=\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}\int_{o}^{\infty}t^\mu\,|u_\lambda(t)|\,dt\int_{t}^{\infty}(y-t)^{\lambda-\mu-1}y^{-\lambda}dy$$

$$=\frac{\lambda}{\mu}\int_{o}^{\infty}|u_\lambda(t)|\,dt<\infty.$$

It follows that $\{s_n\}$ is $|A'_\mu|$-convergent to l' (say). That $l'=l$ is a consequence of a known result ([2], Theorem 2).

We next obtain a sufficient condition for $|A'_\lambda|$-convergence.

Theorem 3. *If $\lambda>0$ and*

$$\int_{o}^{\infty}y^{-1}\,|U_{\lambda+1}(y)-l|\,dy<\infty.$$

then $s_n\to l\,|A'_\lambda|$.

Proof. We have, by (3. 7), that

$$\int_{o}^{\infty}y^{-1}\,|U_\lambda(y)-l|\,dy\leqq\lambda\int_{o}^{\infty}y^{-\lambda-1}dy\int_{o}^{y}t^{\lambda-1}\,|U_{\lambda+1}(t)-l|\,dt$$

$$=\int_{o}^{\infty}t^{\lambda-1}\,|U_{\lambda+1}(t)-l|\,dt\int_{t}^{\infty}\lambda y^{-\lambda-1}dy$$

$$=\int_{o}^{\infty}t^{-1}\,|U_{\lambda+1}(t)-l|\,dt<\infty.$$

Hence, by (3. 5) and (3. 6), we have that

$$\int_{o}^{\infty}|u_\lambda(y)|\,dy=\int_{o}^{\infty}y^{-1}\,|U_{\lambda+1}(y)-U_\lambda(y)|\,dy$$

$$\leqq\int_{o}^{\infty}y^{-1}\,|U_{\lambda+1}(y)-l|\,dy+\int_{o}^{\infty}y^{-1}\,|U_\lambda(y)-l|\,dy$$

$$\leqq 2\int_{o}^{\infty}y^{-1}\,|U_{\lambda+1}(y)-l|\,dy<\infty.$$

It follows that the sequence $\{s_n\}$ is $|A'_\lambda|$-convergent to l' (say). That $l'=l$ follows from the convergence of the integral

$$\int_{o}^{\infty}y^{-1}\,|U_\lambda(y)-l|\,dy.$$

Theorem 4. *For $\lambda>0$, $s_n\to l\,|A_\lambda|$ if and only if $s_n\to l\,|A'_\lambda|$ and $nu_n\to 0\,|A_{\lambda-1}|$.*

Proof. (i) Suppose that $s_n\to l\,|A_\lambda|$, i. e., that $s_\lambda(y)$ is of bounded variation in $[0,\infty)$.

By (3. 3), we have that

$$\int_{o}^{\infty}|u_\lambda(y)|\,dy\leqq|s_\lambda(0)|\int_{o}^{\infty}(1+y)^{-\lambda-1}dy+\int_{o}^{\infty}(1+y)^{-\lambda-1}dy\int_{o}^{y}(1+t)^\lambda\,|s'_\lambda(t)|\,dt$$

$$=\frac{1}{\lambda}|s_\lambda(0)|+\int_{o}^{\infty}(1+t)^\lambda\,|s'_\lambda(t)|\,dt\int_{t}^{\infty}(1+y)^{-\lambda-1}dy$$

$$=\frac{1}{\lambda}|s_\lambda(0)|+\frac{1}{\lambda}\int_{o}^{\infty}|s'_\lambda(t)|\,dt<\infty,$$

i. e., $U_\lambda(y)$ is of bounded variation in $[0,\infty)$. Hence, by Theorem A,

$$s_n\to l\,|A'_\lambda|.$$

Further, by (3. 5), we have that $(1+y)u_\lambda(y)$ is of bounded variation in $[0,\infty)$.

But

$$(1+y)u_\lambda(y)=(1+y)^{-\lambda}\sum_{n=o}^{\infty}\varepsilon_n^\lambda u_n\left(\frac{y}{1+y}\right)^n$$

$$=(1+y)^{-\lambda}\sum_{n=1}^{\infty}\varepsilon_n^{\lambda-1}\frac{\lambda+n}{\lambda n}\cdot nu_n\left(\frac{y}{1+y}\right)^n+u_o(1+y)^{-\lambda},$$

and so, by Theorem A, we have that

$$\frac{\lambda + n}{\lambda n}\, n u_n \to 0 \mid A_{\lambda-1} \mid .$$

Consequently, by Lemma 5,

$$n u_n \to 0 \mid A_{\lambda-1} \mid .$$

(ii) Conversely, suppose that $s_n \to l \mid A'_\lambda \mid$ and $n u_n \to 0 \mid A_{\lambda-1} \mid$. By reversing the argument in the last part of (i), we have that $(1 + y) u_\lambda(y)$ is of bounded variation in $[0, \infty)$.

Hence, by (3. 5), $s_\lambda(y)$ is of bounded variation in $[0, \infty)$. The theorem follows now by Theorem A.

Theorem 5. *For $\lambda > 0$, $s_n \to l \mid A'_\lambda \mid$ if and only if $s_n \to l \mid A_{\lambda-1} \mid$.*

Proof. (i) Suppose that $s_n \to l \mid A'_\lambda \mid$, i. e., $U_\lambda(y)$ is of bounded variation in $[0, \infty)$. In virtue of Theorem B and (3. 6), it suffices to show that $u_{\lambda-1}(y)$ is of bounded variation in $[1, \infty)$.

Now, by (3. 1) with u in place of s, we have that

$$\int_1^\infty \mid u'_{\lambda-1}(y) \mid dy \leqq \lambda^2 \int_1^\infty y^{-\lambda-1} dy \int_0^y t^{\lambda-1} \mid u_\lambda(t) \mid dt + \lambda \int_1^\infty y^{-1} \mid u_\lambda(y) \mid dy$$

$$\leqq \lambda \int_0^1 t^{\lambda-1} \mid u_\lambda(t) \mid dt + 2\lambda \int_0^\infty \mid u_\lambda(y) \mid dy < \infty .$$

Thus $s_n \to l \mid A_{\lambda-1} \mid$.

(ii) Conversely, suppose that $s_n \to l \mid A_{\lambda-1} \mid$. Again by Theorem B and (3. 6) it suffices to show that $u_{\lambda-1}(y)$ is of bounded variation in $[0, \infty)$. Now, by (3. 3), we have that

$$\int_0^\infty \mid u'_{\lambda-1}(y) \mid dy = \int_0^\infty \left| \frac{d}{dy} \left\{ (1 + y)^{-\lambda} s_{\lambda-1}(0) + (1 + y)^{-\lambda} \int_0^y (1 + t)^{\lambda-1} s'_{\lambda-1}(t)\, dt \right\} \right| dy$$

$$\leqq \mid s_{\lambda-1}(0) \mid + \int_0^\infty \lambda (1 + y)^{-\lambda-1} dy \int_0^y (1 + t)^{\lambda-1} \mid s'_{\lambda-1}(t) \mid dt$$

$$+ \int_0^\infty (1 + y)^{-1} \mid s'_{\lambda-1}(y) \mid dy$$

$$\leqq \mid s_{\lambda-1}(0) \mid + 2 \int_0^\infty \mid s'_{\lambda-1}(t) \mid dt < \infty .$$

The theorem follows.

Theorem 6. *If $\lambda > -1$, H_χ is a regular Hausdorff method and $s_n \to l \mid A_\lambda \mid$, then $s_n \to l \mid A_\lambda H_\chi \mid$.*

Proof. Let

$$h_\lambda(y) = (1 + y)^{-\lambda-1} \sum_{n=0}^\infty \varepsilon_n^\lambda h_n \left(\frac{y}{1 + y} \right)^n$$

where h_n is defined by (1. 1) and let V be the total variation of $s_\lambda(y)$ in $[0, \infty)$. We have, for $0 \leqq y_0 < y_1 < \cdots < y_n$, by (3. 8), that

$$\sum_{r=1}^n \mid h_\lambda(y_r) - h_\lambda(y_{r-1}) \mid = \sum_{r=1}^n \left| \int_0^1 \{ s_\lambda(y_r t) - s_\lambda(y_{r-1} t) \}\, d\chi(t) \right|$$

$$\leqq \int_0^1 \sum_{r=1}^n \mid s_\lambda(y_r t) - s_\lambda(y_{r-1} t) \mid \mid d\chi(t) \mid \leqq V \int_0^1 \mid d\chi(t) \mid < \infty .$$

The theorem follows from Theorem C.

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