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On strong summability

By *D. Borwein* and *J. H. Rizvi* in London (Canada)

In this note we consider some strong Abel-type summability methods, establishing the strong summability analogues of the results proved in [4].

§ 1. Introduction

Let

$$\varepsilon_n^\lambda = \binom{n+\lambda}{n} = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+n)}{n!}, \quad n = 1, 2, \dots,$$

$$\varepsilon_0^\lambda = 1,$$

$$s_n = \sum_{r=0}^{n} u_r,$$

$$s_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n,$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda u_n \left(\frac{y}{1+y}\right)^n,$$

$$U_\lambda(y) = \lambda \int_0^y u_\lambda(t)\, dt.$$

The Abel-type methods A_λ and A'_λ, introduced in [2] and [3] are defined as follows: If

$$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$$

is convergent for all x in the open interval $(0, 1)$ and tends to a finite limit l as $x \to 1$ in the open interval, we say that the sequence $\{s_n\}$ is A_λ-convergent to l and write $s_n \to l(A_\lambda)$. It is evident that $s_n \to l(A_\lambda)$ if and only if the series defining $s_\lambda(y)$ is convergent for all $y > 0$ and $s_\lambda(y) \to l$ as $y \to \infty$. The A_0 method is the ordinary Abel method.

If the series defining $u_\lambda(y)$ is convergent for all $y > 0$ and $U_\lambda(y)$ tends to a finite limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is A'_λ-convergent to l and write $s_n \to l(A'_\lambda)$.

It is known that the methods A_λ and $A'_{\lambda+1}$ are regular for $\lambda > -1$ (see [2], Theorem 1 and [5], Theorem 34).

The Hausdorff method H_χ and the product method $A_\lambda H_\chi$ are defined as under: Let $\chi(t)$ be a real function of bounded variation in the interval [0, 1], and

$$h_n = \sum_{r=0}^{n} \binom{n}{r} s_r \int_0^1 t^r (1-t)^{n-r} d\chi(t). \tag{1.1}$$

If $h_n \to l$ as $n \to \infty$, we say that the sequence $\{s_n\}$ is H_χ-convergent to l and write $s_n \to l(H_\chi)$.

If $h_n \to l(A_\lambda)$, we say that the sequence $\{s_n\}$ is $A_\lambda H_\chi$-convergent to l and write $s_n \to l(A_\lambda H_\chi)$.

We recall that the conditions

$$\chi(0+) = \chi(0),$$

and

$$\chi(1) - \chi(0) = 1$$

are necessary and sufficient for the regularity of the method H_χ ([5], Theorem 208).

The absolute Abel-type summability methods, considered in [4], are defined as under (see also [7]):

If $s_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is absolutely A_λ-convergent, or $|A_\lambda|$-convergent to l and write $s_n \to l \mid A_\lambda \mid$.

If $U_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit l as $y \to \infty$, we say that the sequence $\{s_n\}$ is absolutely A'_λ-convergent, or $|A'_\lambda|$-convergent to l and write $s_n \to l \mid A'_\lambda \mid$.

If $h_n \to l \mid A_\lambda \mid$, we say that the sequence $\{s_n\}$ is absolutely $A_\lambda H_\chi$-convergent, or $|A_\lambda H_\chi|$-convergent to l and write $s_n \to l \mid A_\lambda H_\chi \mid$.

The following two theorems are proved in [3]:

Theorem A. *For $\lambda > 0$, $s_n \to l(A_\lambda)$ if and only if $s_n \to l(A'_\lambda)$ and $nu_n \to 0(A_{\lambda-1})$.*

Theorem B. *For $\lambda > 0$, $s_n \to l(A_{\lambda-1})$ if and only if $s_n \to l(A'_\lambda)$.*

It is also known that:

Theorem C. *If $\lambda > -1$, H_χ is a regular Hausdorff method and $s_n \to l(A_\lambda)$, then $s_n \to l(A_\lambda H_\chi)$.*

For complete references to this result, see [4].

In [4], the absolute summability analogues of these results are proved.

§ 2. Definitions

We now define strong summability methods based upon the Abel-type methods A_λ and A'_λ and the product method $A_\lambda H_\chi$ (see also [6]).

Strong Abel-type summability $[A_\lambda]$.

If

$$\int_0^y | s_{\lambda+1}(t) - l \,| \, dt = o(y) \qquad \text{as } y \to \infty,$$

we say that the sequence $\{s_n\}$ is strongly A_λ-convergent or $[A_\lambda]$-convergent to l and write $s_n \to l[A_\lambda]$.

Strong Abel-type summability $[A'_\lambda]$.

If

$$\int_0^y | U_{\lambda+1}(t) - l \,| \, dt = o(y) \qquad \text{as } y \to \infty,$$

we say that the sequence $\{s_n\}$ is strongly A'_λ-convergent, or $[A'_\lambda]$-convergent to l and write $s_n \to l[A'_\lambda]$.

Strong summability $[A_\lambda H_\chi]$.

If $h_n \to l[A_\lambda]$, we say that the sequence $\{s_n\}$ is strongly $A_\lambda H_\chi$-convergent, or $[A_\lambda H_\chi]$-convergent to l and write $s_n \to l[A_\lambda H_\chi]$.

Strong boundedness.

If

$$\int_0^y | s_{\lambda+1}(t) \,| \, dt = O(y) \qquad \text{as } y \to \infty,$$

the sequence $\{s_n\}$ is said to be strongly A_λ-bounded or $[A_\lambda]$-bounded and is written $s_n = O(1)\,[A_\lambda]$.

§ 3. Preliminary results

The following results are required.

Lemma 1. *If $\lambda > \mu > -1$, $y > 0$ and $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

$$s_\mu(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu s_\lambda(t)\, dt. \tag{3.1}$$

For proof see [2], Lemma 2 (i).

Lemma 2. *If $\lambda > -1$, $y > 0$ and $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

$$u_\lambda(y) = (1+y)^{-1} s_\lambda(y) - \lambda(1+y)^{-\lambda-1} \int_0^y (1+t)^{\lambda-1} s_\lambda(t)\, dt, \tag{3.2}$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} s_\lambda(0) + (1+y)^{-\lambda-1} \int_0^y (1+t)^\lambda s'_\lambda(t)\, dt, \tag{3.3}$$

$$U_\lambda(y) = \lambda(1+y)^{-\lambda} \int_0^y (1+t)^{\lambda-1} s_\lambda(t)\, dt, \tag{3.4}$$

$$s_\lambda(y) = U_\lambda(y) + (1+y)\, u_\lambda(y), \tag{3.5}$$

$$s_\lambda(y) = U_{\lambda+1}(y) + u_\lambda(y), \tag{3.6}$$

$$y\, u_\lambda(y) = U_{\lambda+1}(y) - U_\lambda(y), \tag{3.7}$$

$$y \frac{d}{dy}(U_{\lambda+1}(y)) = \frac{1}{\lambda+1}[U_{\lambda+2}(y) - U_{\lambda+1}(y)], \tag{3.8}$$

$$U_{\lambda+1}(y) = (\lambda+1) y^{-\lambda-1} \int_0^y t^\lambda U_{\lambda+2}(t)\, dt. \tag{3.9}$$

For complete proofs, see [8].

Lemma 3. *If $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^{\lambda} s_n x^n$ is convergent for $0 \leqq x < 1$ and h_n is defined by (1. 1), then, for $y > 0$,*

$$h_{\lambda}(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n = \int_0^1 s_{\lambda}(yt)\, d\chi(t). \tag{3. 10}$$

This lemma is proved in [2]. See also [1], p. 376.

Lemma 4. *If $\lambda > -1$, a is real and $s_{\lambda}(y) = O(1)$ for $y > 0$ and $(n+a)v_n = s_n$ for $n = 0, 1, 2, \ldots$, then $v_n \to 0 \mid A_{\lambda} \mid$.*

This lemma is proved in [4] (Lemma 4).

We also require the following result of Mishra ([7], Theorem 6).

Lemma 5. *If $\lambda > -1$ and $s_n \to l \mid A_{\lambda} \mid$, then $s_n \to l[A_{\lambda}]$.*

In view of the above two lemmas we have the following:

Lemma 6. *If $\lambda > -1$, a is real, $s_{\lambda}(y) = O(1)$ for $y > 0$ and $(n+a)v_n = s_n$ for $n = 0, 1, 2, \ldots$, then $v_n \to 0[A_{\lambda}]$.*

An immediate consequence of the above lemma is the following:

Lemma 7. *If $\lambda > -1$, p and q are real and $s_n \to l[A_{\lambda}]$, then*

$$\frac{n+p}{n+q} s_n \to l[A_{\lambda}].$$

§ 4. Main results

Theorem 1. *If $\lambda > 0$ and $s_n \to l[A'_{\lambda}]$, then $s_n \to l(A'_{\lambda})$.*

Proof. We have, by (3. 9), that

$$\begin{aligned} | U_{\lambda}(y) - l | &\leqq \lambda y^{-\lambda} \int_0^y t^{\lambda-1} | U_{\lambda+1}(t) - l |\, dt \\ &= \lambda y^{-\lambda}\left[\{t^{\lambda-1} o(t)\}_0^y - (\lambda-1)\int_0^y t^{\lambda-2} o(t)\, dt\right] \\ &= o(1) \qquad \text{as } y \to \infty. \end{aligned}$$

The theorem follows.

Remark. If we assume only $[A'_{\lambda}]$-boundedness in the hypothesis of the theorem, we obtain A'_{λ}-boundedness as the conclusion.

This remark also applies to other theorems.

Theorem 2. *If $\lambda > 0$ and $s_n \to l(A'_{\lambda})$, then $s_n \to l[A'_{\lambda-1}]$.*

Proof. We have, by hypothesis, that

$$| U_{\lambda}(t) - l | = o(1) \qquad \text{as } t \to \infty.$$

It follows, by the regularity of the (C. 1) method, that

$$\int_0^y |U_{\lambda}(t) - l|\, dt = o(y) \qquad \text{as } y \to \infty;$$

i. e., $s_n \to l[A'_{\lambda-1}]$.

The next theorem gives necessary and sufficient conditions for $[A'_{\lambda}]$-convergence.

Theorem 3. *For $\lambda > 0$, the necessary and sufficient conditions for the $[A'_{\lambda}]$-convergence of the sequence $\{s_n\}$ to l are:*

$$s_n \to l\,(A'_{\lambda}), \tag{4. 1}$$

and

$$\int_0^y \left| t \frac{d}{dt} U_{\lambda}(t) \right| dt = o(y) \qquad \text{as } y \to \infty. \tag{4. 2}$$

Proof. (i) *Necessity.*

(4. 1) follows by Theorem 1.

Now, by (3. 8), we have that

$$\begin{aligned} \int_0^y \left| t \frac{d}{dt} U_{\lambda}(t) \right| dt &= \frac{1}{\lambda} \int_0^y | U_{\lambda+1}(t) - U_{\lambda}(t) |\, dt \\ &\leqq \frac{1}{\lambda} \int_0^y | U_{\lambda+1}(t) - l |\, dt + \frac{1}{\lambda} \int_0^y | U_{\lambda}(t) - l |\, dt \\ &= o(y) \qquad \text{as } y \to \infty, \end{aligned}$$

by Theorem 2.

(ii) *Sufficiency.*

By (4. 1) and Theorem 2, we have that

$$\int_0^y | U_{\lambda}(t) - l |\, dt = o(y) \qquad \text{as } y \to \infty.$$

Hence, it follows by (4. 2) and (3. 8) that

$$\begin{aligned} \int_0^y | U_{\lambda+1}(t) - l |\, dt &\leqq \lambda \int_0^y \left| t \frac{d}{dt} U_{\lambda}(t) \right| dt + \int_0^y | U_{\lambda}(t) - l |\, dt \\ &= o(y) \qquad \text{as } y \to \infty. \end{aligned}$$

This completes the proof of the theorem.

Theorem 4. *For $\lambda > 0$, $s_n \to l[A_{\lambda}]$ if and only if $s_n \to l[A'_{\lambda}]$ and $nu_n \to 0[A_{\lambda-1}]$.*

Proof. (i) Suppose that $s_n \to l[A_{\lambda}]$, i. e.

$$\int_0^y | s_{\lambda+1}(t) - l |\, dt = o(y) \qquad \text{as } y \to \infty.$$

By (3. 4), we have that

$$U_{\lambda+1}(t) - l = (\lambda+1)(1+t)^{-\lambda-1} \int_0^t (1+z)^{\lambda}[s_{\lambda+1}(z) - l]\, dz - l(1+t)^{-\lambda-1}.$$

Hence

$$\begin{aligned} &\int_0^y | U_{\lambda+1}(t) - l |\, dt \\ &\leqq (\lambda+1) \int_0^y (1+t)^{-\lambda-1} dt \int_0^t (1+z)^{\lambda} | s_{\lambda+1}(z) - l |\, dz + |l| \int_0^y (1+t)^{-\lambda-1} dt \\ &= I(y) + o(y) \qquad \text{as } y \to \infty, \end{aligned}$$

where

$$I(y) = (\lambda + 1) \int_0^y (1 + t)^{-\lambda-1} dt \int_0^t (1 + z)^{\lambda} \mid s_{\lambda+1}(z) - l \mid dz$$

$$= (\lambda + 1) \int_0^y (1 + z)^{\lambda} \mid s_{\lambda+1}(z) - l \mid dz \int_z^y (1 + t)^{-\lambda-1} dt$$

$$= o(y) - \frac{\lambda + 1}{\lambda}(1 + y)^{-\lambda} \int_0^y (1 + z)^{\lambda} \mid s_{\lambda+1}(z) - l \mid dz$$

$$= o(y) \qquad \text{as } y \to \infty,$$

i. e., $s_n \to l[A'_\lambda]$.

Further, by (3. 5), we have that

$$(1 + t) u_{\lambda+1}(t) = s_{\lambda+1}(t) - U_{\lambda+1}(t).$$

But

$$(1 + t) u_{\lambda+1}(t) = (1 + t)^{-\lambda-1} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda} \frac{\lambda + 1 + n}{(\lambda + 1) n} n u_n \left(\frac{t}{1 + t}\right)^n + u_0 (1 + t)^{-\lambda-1}.$$

Thus we have

$$\int_0^y \left| (1 + t)^{-\lambda-1} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda} \frac{\lambda + 1 + n}{(\lambda + 1) n} n u_n \left(\frac{t}{1 + t}\right)^n \right| dt$$

$$\leqq \int_0^y \mid s_{\lambda+1}(t) - l \mid dt + \int_0^y \mid U_{\lambda+1}(t) - l \mid dt + \mid u_0 \mid \int_0^y (1 + t)^{-\lambda-1} dt$$

$$= o(y) \qquad \text{as } y \to \infty,$$

i. e.,

$$\frac{\lambda + 1 + n}{(\lambda + 1) n} n u_n \to 0[A_{\lambda-1}].$$

Consequently, by Lemma 7

$$n u_n \to 0[A_{\lambda-1}].$$

(ii) Suppose that $s_n \to l[A'_\lambda]$ and $n u_n \to 0[A_{\lambda-1}]$. It follows from the last part of (i) that

$$\int_0^y \mid s_{\lambda+1}(t) - l \mid dt = o(y) \qquad \text{as } y \to \infty,$$

i. e., $s_n \to l[A_\lambda]$.

This completes the proof of the theorem.

Theorem 5. *For $\lambda > 0$, $s_n \to l[A'_\lambda]$ if and only if $s_n \to l[A_{\lambda-1}]$.*

Proof. (i) Suppose that $s_n \to l[A'_\lambda]$, i. e.,

$$\int_0^y \mid U_{\lambda+1}(t) - l \mid dt = o(y) \qquad \text{as } y \to \infty.$$

By (3. 6), we have that

$$\int_0^y \mid s_\lambda(t) - l \mid dt \leqq \int_0^y \mid U_{\lambda+1}(t) - l \mid dt + \int_0^y \mid u_\lambda(t) \mid dt = o(y) + I(y)$$

as $y \to \infty$. Now, by (3. 7), it follows that

$$I(y) = \int_0^y \mid u_\lambda(t) \mid dt = \int_0^1 \mid u_\lambda(t) \mid dt + \int_1^y \frac{1}{t} \mid U_{\lambda+1}(t) - U_\lambda(t) \mid dt,$$

$$\leqq o(y) + \int_1^y \mid U_{\lambda+1}(t) - l \mid dt + \int_1^y \mid U_\lambda(t) - l \mid dt = o(y) \qquad \text{as } y \to \infty,$$

in view of Theorems 3 and 2.

Hence $s_n \to l[A_{\lambda-1}]$.

(ii) Suppose that $s_n \to l[A_{\lambda-1}]$, i. e.,

$$\int_0^y \mid s_\lambda(t) - l \mid dt = o(y) \qquad \text{as } y \to \infty.$$

Again, by (3. 6), it suffices to show that

$$\int_0^y \mid u_\lambda(t) \mid dt = o(y) \qquad \text{as } y \to \infty.$$

But, by (3. 2), we have that

$$\int_0^y \mid u_\lambda(t) \mid dt \leqq \int_0^y (1 + t)^{-1} \mid s_\lambda(t) - l \mid dt + \lambda \int_0^y (1 + t)^{-\lambda-1} dt \int_0^t (1 + z)^{\lambda-1} \mid s_\lambda(z) - l \mid dz$$

$$+ \mid l \mid \int_0^y (1 + t)^{-\lambda-1} dt$$

$$= o(y) + I(y) \qquad \text{as } y \to \infty$$

where

$$I(y) = \lambda \int_0^y (1 + t)^{-\lambda-1} dt \int_0^t (1 + z)^{\lambda-1} \mid s_\lambda(z) - l \mid dz$$

$$= \int_0^y (1 + z)^{\lambda-1} \mid s_\lambda(z) - l \mid dz \int_z^y \lambda (1 + t)^{-\lambda-1} dt$$

$$= \int_0^y (1 + z)^{-1} \mid s_\lambda(z) - l \mid dz - (1 + y)^{-\lambda} \int_0^y (1 + z)^{\lambda-1} \mid s_\lambda(z) - l \mid dz$$

$$= o(y) \qquad \text{as } y \to \infty.$$

The theorem follows.

Theorem 6. *If $\lambda > -1$, H_χ is a regular Hausdorff method and $s_n \to l[A_\lambda]$, then $s_n \to l[A_\lambda H_\chi]$.*

Proof. We have, by (3. 10), that

$$h_{\lambda+1}(y) = (1 + y)^{-\lambda-2} \sum_{n=0}^{\infty} \varepsilon_n^{\lambda+1} h_n \left(\frac{y}{1 + y}\right)^n = \int_0^1 s_{\lambda+1}(yt) d\chi(t).$$

Since H_χ is regular, it follows that

$$\int_0^y |h_{\lambda+1}(z)-l|\,dz = \int_0^y dz \left| \int_0^1 \{s_{\lambda+1}(zt)-l\} d\chi(t) \right| \leqq \int_0^y dz \int_0^1 |s_{\lambda+1}(zt)-l|\,|d\chi(t)|$$

$$= y \int_0^1 |d\chi(t)| \frac{1}{yt} \int_0^{yt} |s_{\lambda+1}(x)-l|\,dx.$$

Hence

$$\frac{1}{y}\int_0^y |h_{\lambda+1}(z)-l|\,dz \leqq \int_0^1 f(yt)\,|d\chi(t)|$$

where

$$f(t) = \frac{1}{t}\int_0^t |s_{\lambda+1}(x)-l|\,dx = o(1) \qquad \text{as } t \to \infty.$$

It follows by the regularity of the continuous Hausdorff transformation ([5], Theorem 217) that

$$\int_0^y |h_{\lambda+1}(z)-l|\,dz = o(y) \qquad \text{as } y \to \infty.$$

The theorem follows.

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Department of Mathematics, University of Western Ontario, London, Canada

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