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A Tauberian theorem for Borel-type methods of summability

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1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a series of complex numbers. Let A_n denote the partial sum

$$a_0 + \cdots + a_n$$

of the series if $n \geqq 0$ and let $A_n = 0$ if $n < 0$. Suppose throughout that $\alpha > 0$, β is real, and N is a non-negative integer such that $\alpha N + \beta > 0$. The series is said to be summable (B, α, β) to A $\left(\text{we write } \sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)\right)$ if, as $x \to \infty$, $\alpha e^{-x} \sum_{n=0}^{\infty} \frac{A_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to A$. The actual choice of N is clearly immaterial. The Borel-type summability method (B, α, β) is regular, and $(B, 1, 1)$ is the standard Borel exponential method B.

Our aim in this paper is to prove the following Tauberian theorem.

Theorem 1. *If* $\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)$ *and* $a_n = O\left(n^{-\frac{1}{2}}\right)$ *then* $\sum_{n=0}^{\infty} a_n = A$.

The case $\alpha = \beta = 1$ of the theorem is known ([4], Theorem 156) as is the corresponding result with "O" replaced by "o".

Borwein [2] has proved:

If $J(z) = \sum_{n=N}^{\infty} \frac{z^n}{h(n)}$ *where* $h(z)$ *is an analytic function of* $z = x + iy$ *in the region* $x > x_0$, *such that*

(i) *when* $x > x_0$ *and* $|z|$ *is large* $h(z) = z^{\alpha z+\beta} e^{\gamma z}\left\{C + O\left(\frac{1}{|z|}\right)\right\}$ *where* $C > 0$, $\alpha > 0$, β *and* γ *are real, and*

(ii) $h(x)$ *is real for* $x > x_0$,

then

$$\sum_{n=0}^{\infty} a_n = A(J) \qquad \text{i. e.,} \quad \left(\frac{1}{J(x)} \sum_{n=0}^{\infty} \left(\frac{1}{h(n)}\right) A_n x^n \to A \text{ as } x \to \infty\right)$$

if and only if $\sum_{n=0}^{\infty} a_n = A\left(B, \alpha, \beta + \frac{1}{2}\right)$.

In particular, taking $h(z) = \{\Gamma(az+b)\}^c (z+p)^{qz+r}$ *where* b, c, p, q, *and* r *are real,* $a > 0$ *and* $ac + q > 0$, *so that*

$$J(z) = \sum_{n=N}^{\infty} \frac{z^n}{\{\Gamma(an+b)\}^c (n+p)^{qn+r}},$$

then $\sum_{n=0}^{\infty} a_n = A(J)$ *if and only if* $\sum_{n=0}^{\infty} a_n = A\left(B, ac+q, bc+r+\frac{1-c}{2}\right)$.

It follows that Theorem 1 is in fact a Tauberian theorem for quite a wide class of summability methods.

Theorem 1 remains true if $\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)$ is replaced by $\sum_{n=0}^{\infty} a_n = A(B', \alpha, \beta)$, by which it is meant that, as $y \to \infty$,

$$\int_0^y e^{-x} dx \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to A - A_{N-1}.$$

This is a consequence of the following known result ([1], Theorem 2):

$$\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta+1) \quad \text{if and only if} \quad \sum_{n=0}^{\infty} a_n = A(B', \alpha, \beta).$$

2. Preliminary results

Lemma 1. *Let* $x > 0$, $h = n - \frac{x}{\alpha}$, $\frac{1}{2} < \zeta < \frac{2}{3}$, *and* $0 < \eta < 2\zeta - 1$.

Then for $n = N, N+1, \ldots$

$$(2.1) \qquad e^{-x} \sum_{|h| > x^{\zeta}} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = O(e^{-x^{\eta}})$$

and

$$(2.2) \qquad e^{-x} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = \frac{1}{\sqrt{2\pi x}} e^{-\frac{\alpha^2 h^2}{2x}} \left\{1 + O\left(\frac{|h|+1}{x}\right) + O\left(\frac{|h|^3}{x^2}\right)\right\}$$

if $|h| \leqq x^{\zeta}$.

Formulae (2.1) *and* (2.2) *also hold with* $h = n - \left[\frac{x}{\alpha}\right]$.

Proof. Formula (2.1) is Lemma 2, part (d) of Borwein [3], while Borwein would have obtained (2.2) instead of part (e) of the same lemma if he had not used

$$\frac{|h|+1}{x} = O(x^{3\zeta-2}) \quad \text{and} \quad \frac{|h|^3}{x^2} = O(x^{3\zeta-2})$$

in simplifying near the end of his proof.

(We write $O\left(\frac{|h|+1}{x}\right)$ instead of $O\left(\frac{|h|}{x}\right)$ in order to include the case $h = 0$.)

Taking $h = n - \left[\frac{x}{\alpha}\right]$ would have necessitated only minor changes in his proof.

Lemma 2. $\frac{n^{\frac{1}{2}}}{\Gamma(\alpha n+\beta)} = O\left(\frac{1}{\Gamma(\alpha n+\beta-\frac{1}{2})}\right)$, *for* $n = N, N+1, \ldots.$

Proof. It follows from Stirling's theorem that

$$\Gamma(\alpha n+\beta) = (2\pi)^{\frac{1}{2}} e^{-\alpha n} (\alpha n)^{\alpha n+\beta-\frac{1}{2}} \left\{1 + O\left(\frac{1}{n}\right)\right\}.$$

Thus $\frac{(\alpha n)^{\frac{1}{2}} \Gamma\left(\alpha n+\beta-\frac{1}{2}\right)}{\Gamma(\alpha n+\beta)} \to 1$ and the lemma follows.

Lemma 3. *Let* $\frac{1}{2} < \zeta < \frac{2}{3}$. *If* $a_n = o(1)$ *and* $|h| \leqq n^{\zeta}$, *then*

$$A_{n+h} - A_n = o(|h|) \qquad \text{as } n \to \infty,$$

uniformly for $|h| \leqq n^{\zeta}$.

Proof. The result is a special case of a known result ([4], Theorem 144); it may easily be verified directly.

Lemma 4. *Let* $\frac{1}{2} < \zeta < \frac{2}{3}$. *If* $A_n = o(n^{\frac{1}{2}})$ *and* $|h| \leqq (\alpha n)^{\zeta}$, *then*

$$A_{n+h} = o(n^{\frac{1}{2}}) \qquad \text{as } n \to \infty,$$

uniformly for $|h| \leqq (\alpha n)^{\zeta}$.

Proof. $\frac{|A_{n+h}|}{n^{\frac{1}{2}}} = \left(1 + \frac{h}{n}\right)^{\frac{1}{2}} \frac{|A_{n+h}|}{(n+h)^{\frac{1}{2}}} \leqq \left(1 + \frac{(\alpha n)^{\zeta}}{n}\right)^{\frac{1}{2}} \frac{|A_{n+h}|}{(n+h)^{\frac{1}{2}}}$. Since

$$n + h \geqq n - (\alpha n)^{\zeta} \quad \text{and} \quad n - (\alpha n)^{\zeta} \to \infty \qquad \text{as } n \to \infty,$$

we have

$$\frac{A_{n+h}}{(n+h)^{\frac{1}{2}}} = o(1) \qquad \text{as } n \to \infty,$$

uniformly for $|h| \leqq (\alpha n)^{\zeta}$. Since

$$\left(1 + \frac{(\alpha n)^{\zeta}}{n}\right)^{\frac{1}{2}} \to 1 \qquad \text{as } n \to \infty,$$

the result follows.

Lemma 5. *Let* $\frac{1}{2} < \zeta < \frac{2}{3}$. *If* $A_n = o(n^{\frac{1}{2}})$ *and* $|h| > (\alpha n)^{\zeta} > 0$, *then*

$$A_{n+h} = O(|h|).$$

Proof. It follows from $A_n = o(n^{\frac{1}{2}})$ that $A_{n+h} = O((n+|h|)^{\frac{1}{2}})$. Note that

$$\frac{3}{2} < \frac{1}{\zeta} < 2 \quad \text{and} \quad \frac{3}{4} < \frac{1}{2\zeta} < 1.$$

Since $|h| > (\alpha n)^{\zeta}$,

$$n + |h| = O(|h|^{\frac{1}{\zeta}}) + |h| = O(|h|^{\frac{1}{\zeta}}),$$

so that $(n+|h|)^{\frac{1}{2}} = O(|h|^{\frac{1}{2\zeta}}) = O(|h|)$, and the result follows.

Lemma 6. *If n is a positive integer and $a_n = O(n^{-\frac{1}{2}})$, then, for $|h| \leqq n^{\zeta}$,*

$$A_{n+h} - A_n = O\left(\frac{|h|}{\sqrt{n}}\right).$$

Proof. This may be readily verified.

The remaining results of this section give estimates of certain sums in terms of integrals. The proofs are elementary. Part of Lemma 8 is proved to give an idea of what is involved. Otherwise the proofs are omitted. Throughout the rest of this section n is a positive integer and $c > 0$.

Lemma 7.

$$\left| \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} - 2\int_0^{\infty} e^{-\frac{ct^2}{n}}\,dt \right| < 1.$$

Whence

$$\sqrt{\frac{c}{\pi n}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} = 1 + O(n^{-\frac{1}{2}}), \tag{2. 3}$$

uniformly in any finite interval $0 \leqq c \leqq k$ (cf. [4], Theorem 140).

Lemma 8.

$$\sum_{h=-\infty}^{\infty} |h|\, e^{-\frac{ch^2}{n}} \geqq 2\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t\,dt - \left(\frac{2n}{ec}\right)^{\frac{1}{2}} = \left(\frac{n}{c}\right) - \left(\frac{2n}{ec}\right)^{\frac{1}{2}}.$$

For n sufficiently large,

$$\sum_{h=-\infty}^{\infty} |h|\, e^{-\frac{ch^2}{n}} \leqq 2\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t\,dt + \left(\frac{2n}{ec}\right)^{\frac{1}{2}} = \left(\frac{n}{c}\right) + \left(\frac{2n}{ec}\right)^{\frac{1}{2}}.$$

Whence

$$\sum_{h=-\infty}^{\infty} |h|\, e^{-\frac{ch^2}{n}} = O\left(\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t\,dt\right). \tag{2. 4}$$

Proof. Let $S = \sum_{h=1}^{\infty} h e^{-\frac{ch^2}{n}}$ so that $\sum_{h=-\infty}^{\infty} |h|\, e^{-\frac{ch^2}{n}} = 2S$. Let $f(t) = t e^{-\frac{ct^2}{n}}$. Then

$$f'(t) = e^{-\frac{ct^2}{n}}\left(1 - \frac{2ct^2}{n}\right) \quad \text{and} \quad f''(t) = \left(\frac{2ct}{n}\right) e^{-\frac{ct^2}{n}} \left(\frac{2ct^2}{n} - 3\right).$$

It is easily verified that f is monotone increasing for $0 \leqq t \leqq \left(\frac{n}{2c}\right)^{\frac{1}{2}}$, monotone decreasing for $t \geqq \left(\frac{n}{2c}\right)^{\frac{1}{2}}$, and takes a maximum value of $\left(\frac{n}{2ec}\right)^{\frac{1}{2}}$ when $t = \left(\frac{n}{2c}\right)^{\frac{1}{2}}$. Also f is concave downward for $0 \leqq t \leqq \left(\frac{3n}{2c}\right)^{\frac{1}{2}}$. Choose an integer h_0 such that

$$h_0 - 1 < \left(\frac{n}{2c}\right)^{\frac{1}{2}} \leqq h_0.$$

Since $\left(\frac{3n}{2c}\right)^{\frac{1}{2}} \geqq \left(\frac{n}{2c}\right)^{\frac{1}{2}} + 1 > h_0$ if $n > (2 + \sqrt{3})c$, it follows that f is concave downward for $h_0 - 1 \leqq t \leqq h_0$ when $n > (2 + \sqrt{3})c$. Set

$$a(t) = h e^{-\frac{ch^2}{n}}, \qquad h \leqq t < h + 1$$

$$b(t) = (h+1) e^{-\frac{c(h+1)^2}{n}}, \qquad h \leqq t \leqq h + 1.$$

Then

$$b(t) < f(t) \leqq a(t) \qquad \text{for } t \geqq h_0$$

while

$$a(t) \leqq f(t) < b(t) \qquad \text{for } 0 \leqq t \leqq h_0 - 1.$$

Then

$$\begin{aligned}
\int_0^{\infty} f(t)\,dt &\geqq \int_0^{h_0-1} a(t)\,dt + \int_{h_0-1}^{h_0} f(t)\,dt + \int_{h_0}^{\infty} b(t)\,dt \\
&= \int_0^{\infty} b(t)\,dt - \int_0^{h_0} \{b(t) - a(t)\}\,dt - \int_{h_0-1}^{h_0} a(t)\,dt + \int_{h_0-1}^{h_0} f(t)\,dt \\
&= S - f(h_0) - f(h_0 - 1) + \int_{h_0-1}^{h_0} f(t)\,dt \\
&= S - \int_{h_0-1}^{h_0} f(t)\,dt + 2\left(\int_{h_0-1}^{h_0} f(t)\,dt - \frac{f(h_0 - 1) + f(h_0)}{2}\right) \\
&\geqq S - \int_{h_0-1}^{h_0} f(t)\,dt \qquad \text{if } n > (2 + \sqrt{3})c \\
&\geqq S - \left(\frac{n}{2ec}\right)^{\frac{1}{2}}.
\end{aligned}$$

The second inequality of the lemma follows. The first inequality can be proved in a similar way.

Lemma 9. *For* $\frac{1}{2} < \zeta < \frac{2}{3}$, $\sum_{|h| > (\alpha n)^{\zeta}} |h|\, e^{-\frac{ch^2}{n}} = O\left(\int_{(\alpha n)^{\zeta}}^{\infty} e^{-\frac{ct^2}{n}}\, t\,dt\right)$.

Lemma 10.

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-\frac{ch^2}{n}} \geqq 2\left(\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t^3\,dt - \left(\frac{3n}{2ec}\right)^{\frac{3}{2}}\right) = \frac{n^2}{c^2} - \frac{1}{\sqrt{2}}\left(\frac{3n}{ec}\right)^{\frac{3}{2}}.$$

For n sufficiently large,

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-\frac{ch^2}{n}} \leqq 2\left(\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t^3\,dt + \left(\frac{3n}{2ec}\right)^{\frac{3}{2}}\right) = \frac{n^2}{c^2} + \frac{1}{\sqrt{2}}\left(\frac{3n}{ec}\right)^{\frac{3}{2}}.$$

Whence

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-\frac{ch^2}{n}} = O\left(\int_0^{\infty} e^{-\frac{ct^2}{n}}\, t^3\,dt\right). \tag{2. 5}$$

3. Summability (e, c)

Let $c > 0$. Then

$$\sum_{n=0}^{\infty} a_n = A(e, c) \quad \text{if} \quad \sqrt{\frac{c}{\pi n}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} A_{n+h} \to A \quad \text{when } n \to \infty$$

(cf. [4], § 9. 10). Note that (2. 3) implies that summability (e, c) is a regular method.

Lemma 11. *If $a_n = o(1)$, and either*

$$\text{(i)} \quad \sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta) \quad \textit{or} \quad \text{(ii)} \quad \sum_{n=0}^{\infty} a_n = A(e, c),$$

then $A_n = o(n^{\frac{1}{2}})$.

Proof. The result for case (i) is just Lemma 5 of Borwein [3] with $k = 0$, $\mu = 1$, and $\lambda = 0$, while case (ii) is Theorem 150 of Hardy [4].

In each of the next four lemmas, we let $h = m - n = m - \frac{x}{\alpha}$, choose $\frac{1}{2} < \zeta < \frac{2}{3}$, and assume the condition

$$A_n = o(n^{\frac{1}{2}}). \tag{3. 1}$$

Lemma 12. $$e^{-\alpha n} \sum_{|h| > (\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m + \beta - 1}}{\Gamma(\alpha m + \beta)} = o(1) \quad \textit{as } n \to \infty.$$

Proof. We have

$$e^{-\alpha n} \sum_{|h| > (\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$$

$$= O\left(e^{-\alpha n} \sum_{|h| > (\alpha n)^{\zeta}} m^{\frac{1}{2}} \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}\right) \qquad \text{(by (3. 1))}$$

$$= O\left((\alpha n)^{\frac{1}{2}} e^{-\alpha n} \sum_{|h| > (\alpha n)^{\zeta}} \frac{(\alpha n)^{\alpha m+\beta-\frac{3}{2}}}{\Gamma\left(\alpha m - \beta - \frac{1}{2}\right)}\right) \qquad \text{(by Lemma 2)}$$

$$= O((\alpha n)^{\frac{1}{2}} e^{-(\alpha n)^{\eta}}) \text{ (where } 0 < \eta < 2\zeta - 1, \text{ by (2. 1) with } x = \alpha n)$$

$$= o(1) \qquad \text{as } n \to \infty.$$

Lemma 13. $$\sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h|+1}{n}\right) = o(n^{\frac{1}{2}}) \quad \textit{as } n \to \infty.$$

Proof. $$\sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h|+1}{n}\right)$$

$$= \sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} o(n^{\frac{1}{2}}) O\left(\frac{|h|+1}{n}\right) \qquad \text{(by Lemma 4)}$$

$$= o(n^{-\frac{1}{2}}) \sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} O(|h|+1) \qquad \text{(by Lemma 4).}$$

$$= o\left(n^{-\frac{1}{2}} \sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (|h|+1)\right)$$

$$= o\left(n^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{\alpha t^2}{2n}} (t+1)\, dt\right) \qquad \text{(by Lemma 7 and (2. 4))}$$

$$= o\left(n^{-\frac{1}{2}}\left\{\frac{n}{\alpha} + \frac{1}{2}\sqrt{\frac{2\pi n}{\alpha}}\right\}\right) = o(n^{\frac{1}{2}}).$$

Lemma 14. $$\sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h|^3}{n^2}\right) = o(n^{\frac{1}{2}}) \quad \textit{as } n \to \infty.$$

Proof. $$\sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h|^3}{n^2}\right) = o\left(n^{-\frac{3}{2}} \sum_{|h| \le (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} |h|^3\right) \qquad \text{(by Lemma 4)}$$

$$= o\left(n^{-\frac{3}{2}} \int_0^{\infty} e^{-\frac{\alpha t^2}{2n}} t^3\, dt\right) \qquad \text{(by (2. 5))}$$

$$= o\left\{n^{-\frac{3}{2}} \frac{2n^2}{\alpha^2}\right\} = o(n^{\frac{1}{2}}).$$

Lemma 15. $$\sum_{|h| > (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = o(1) \quad \textit{as } n \to \infty.$$

Proof. $$\sum_{|h| > (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = \sum_{|h| > (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} O(|h|) \qquad \text{(by Lemma 5)}$$

$$= O\left(\sum_{|h| > (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} |h|\right)$$

$$= O\left(\int_{(\alpha n)^{\zeta}}^{\infty} e^{-\frac{\alpha t^2}{2n}} t\, dt\right) \qquad \text{(by Lemma 9)}$$

$$= O\left(\frac{n}{\alpha} e^{-\frac{\alpha^{2\zeta+1} n^{2\zeta-1}}{2}}\right)$$

$$= O(n e^{-\varrho n^a}) \qquad \text{where } \varrho = \frac{\alpha^{2\zeta+1}}{2} \text{ and } a = 2\zeta - 1 > 0$$

$$= o(1) \qquad \text{as } n \to \infty.$$

Lemma 16. *If* $e^{-x} \sum_{m=N}^{\infty} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \to 0$ *as $x \to \infty$ through integer multiples of α, then it approaches zero as $x \to \infty$ (without restriction).*

Proof. Letting $h = m - n = m - \left[\frac{x}{\alpha}\right]$, it follows just as in Lemmas 12, 13, and 14 respectively, that

$$e^{-x} \sum_{|h| > x^{\zeta}} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = o(1) \quad \text{as } x \to \infty, \tag{3. 2}$$

$$\sum_{|h| \le x^{\zeta}} e^{-\frac{\alpha^2 h^2}{2x}} A_{n+h} O\left(\frac{|h|+1}{x}\right) = o(x^{\frac{1}{2}}) \quad \text{as } x \to \infty, \tag{3. 3}$$

and

$$\sum_{|h| \le x^{\zeta}} e^{-\frac{\alpha^2 h^2}{2x}} A_{n+h} O\left(\frac{|h|^3}{x^2}\right) = o(x^{\frac{1}{2}}) \quad \text{as } x \to \infty. \tag{3. 4}$$

Thus, using (3. 2), (2. 2), (3. 3), and (3. 4), we obtain:

$$e^{-x} \sum_{m=N}^{\infty} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = \frac{1}{\sqrt{2\pi x}} \sum_{|h| \le x^{\zeta}} A_{n+h} e^{-\frac{\alpha^2 h^2}{2x}} + o(1) \quad \text{as } x \to \infty. \tag{3. 5}$$

Letting $x = \alpha n + k$ where $0 \leq k < \alpha$, it follows from (3. 5) using Lemmas 12, 13, 14, and 15, that it is sufficient to prove that:

$$(\alpha n)^{-\frac{1}{2}} \sum_{|h| \leq x^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = o(1) \quad \text{as } n \to \infty$$

implies

$$x^{-\frac{1}{2}} \sum_{|h| \leq x^{\zeta}} e^{-\frac{\alpha^2 h^2}{2x}} A_{n+h} = o(1) \quad \text{as } x \to \infty.$$

Since

$$x^{-\frac{1}{2}} = (\alpha n)^{-\frac{1}{2}} + O(n^{-\frac{3}{2}})$$

and

$$e^{-\frac{\alpha^2 h^2}{2(\alpha n + k)}} = e^{-\frac{\alpha^2 h^2}{2n}} \left\{ 1 + O\left(\frac{h^2}{n^2}\right) \right\} = e^{-\frac{\alpha^2 h^2}{2n}} \left\{ 1 + O\left(\frac{|h| + 1}{n}\right) \right\},$$

the result follows.

Theorem 2. *If $A_n = o(n^{\frac{1}{2}})$, then*

$$\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta) \quad \textit{if and only if} \quad \sum_{n=0}^{\infty} a_n = A\left(e, \frac{\alpha}{2}\right) \qquad \text{(cf. [4], Theorem 151).}$$

Proof. We assume as we may without loss of generality (because both methods are regular) that $A = 0$. We further assume, without loss in generality, that

$$a_0 = \cdots = a_{N-1} = 0,$$

so that $A_m = 0$ if $m < N$.

Let $x = \alpha n$, $\frac{1}{2} < \zeta < \frac{2}{3}$, and $h = m - n = m - \frac{x}{\alpha}$. Then

$$
\begin{aligned}
&e^{-\alpha n} \sum_{m=N}^{\infty} A_m \frac{(\alpha n)^{\alpha m + \beta - 1}}{\Gamma(\alpha m + \beta)} \\
&= e^{-\alpha n} \sum_{|h| > (\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m + \beta - 1}}{\Gamma(\alpha m + \beta)} + e^{-\alpha n} \sum_{|h| \leq (\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m + \beta - 1}}{\Gamma(\alpha m + \beta)} \\
&= o(1) + n^{-\frac{1}{2}} \left\{ \sum_{|h| \leq (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} + \sum_{|h| \leq (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h| + 1}{n}\right) \right. \\
&\qquad \left. + \sum_{|h| \leq (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} O\left(\frac{|h|^3}{n^2}\right) \right\} \\
&\qquad \text{(by Lemma 12 and (2. 2) with } x = \alpha n) \\
&= n^{-\frac{1}{2}} \sum_{|h| \leq (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} + o(1) \qquad \text{(by Lemmas 13 and 14).}
\end{aligned}
$$

Since

$$\sum_{|h| > (\alpha n)^{\zeta}} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = o(1) \qquad \text{(by Lemma 15),}$$

it follows that

$$e^{-\alpha n} \sum_{m=N}^{\infty} A_m \frac{(\alpha n)^{\alpha m + \beta - 1}}{\Gamma(\alpha m + \beta)} = o(1) \quad \text{as } n \to \infty$$

if and only if

$$n^{-\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = o(1) \quad \text{as } n \to \infty;$$

i. e., if and only if $\sum_{n=0}^{\infty} a_n = 0\left(e, \frac{\alpha}{2}\right)$.

The result follows in view of Lemma 16.

Corollary. *If $a_n = o(1)$, then $\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)$ if and only if $\sum_{n=0}^{\infty} a_n = A\left(e, \frac{\alpha}{2}\right)$.*

Proof. Use Lemma 11 and Theorem 2.

Theorem 3. *If $A_n = o(n^{\frac{1}{2}})$ and $\sum_{n=0}^{\infty} a_n = A(e, c)$ then for $0 < d < c$, $\sum_{n=0}^{\infty} a_n = A(e, d)$.*

The result remains true if $A_n = o(n^{\frac{1}{2}})$ is replaced by $a_n = o(1)$.

Proof. With hypothesis $a_n = O(1)$ the result is Theorem 155 of Hardy [4]. Minor modifications of his proof yield the result with the hypothesis $A_n = o(n^{\frac{1}{2}})$.

4. Proof of Theorem 1

It is convenient to establish two preliminary results.

Theorem 4. *If $a_n = O(n^{-\frac{1}{2}})$ and $\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)$ then*

$$A_n = O(1) \qquad \text{(cf. [4], Theorem 156).}$$

Proof. Since $a_n = O(n^{-\frac{1}{2}})$, we have $a_n = o(1)$, and thus $A_n = o(n^{\frac{1}{2}})$ by Lemma 11. Therefore $\sum_{n=0}^{\infty} a_n = A\left(e, \frac{\alpha}{2}\right)$ by Theorem 2. Thus

$$\left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{\alpha h^2}{2n}} A_{n+h} = A + o(1) \quad \text{as } n \to \infty.$$

Since

$$\left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{\alpha h^2}{2n}} = 1 + o(1)$$

by (2. 3), it follows that

$$A_n\{1 + o(1)\} = \left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{\alpha h^2}{2n}} \{A_{n+h} + (A_n - A_{n+h})\}.$$

Thus

$$
\begin{aligned}
\text{(4. 1)} \qquad A_n\{1 + o(1)\} &= A + o(1) + \left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h| > n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}) \\
&\quad + \left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h| \leq n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}).
\end{aligned}
$$

Since $a_n = o(1)$, it follows that $A_n - A_{n+h} = O(|h|)$. Thus

$$
\begin{aligned}
\left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h| > n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}) &= O\left\{ n^{-\frac{1}{2}} \sum_{|h| > n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} |h| \right\} \\
&= O\left\{ n^{-\frac{1}{2}} \int_{n^{\zeta}}^{\infty} e^{-\frac{\alpha t^2}{2n}} t\, dt \right\} \qquad \text{(by Lemma 9)} \\
&= O\left\{ n^{\frac{1}{2}} e^{-\frac{\alpha n^{2\zeta - 1}}{2}} \right\} = o(1)
\end{aligned}
$$

as $n \to \infty$, since $\alpha > 0$ and $2\zeta - 1 > 0$.

That is

$$(4.2)\qquad \left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h|>n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}) = o(1) \quad \text{as } n\to\infty.$$

Now

$$\left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h|\le n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}) = O\left\{\frac{1}{n}\sum_{|h|\le n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} |h|\right\} \quad \text{(by Lemma 6)}$$

$$= O\left\{\frac{1}{n}\int_0^{\infty} e^{-\frac{\alpha t^2}{2n}} t\,dt\right\} \quad \text{(by (2.4))}$$

$$= O(1).$$

That is

$$(4.3)\qquad \left(\frac{\alpha}{2\pi n}\right)^{\frac{1}{2}} \sum_{|h|\le n^{\zeta}} e^{-\frac{\alpha h^2}{2n}} (A_n - A_{n+h}) = O(1).$$

It follows from (4.1), (4.2), and (4.3) that

$$A_n\{1 + o(1)\} = A + o(1) + o(1) + O(1) = O(1).$$

Therefore $A_n = O(1)$.

Theorem 5. *If* $\sum_{n=0}^{\infty} a_n = A(B, \alpha, \beta)$ *and* $A_n = O(1)$ *then* $\sum_{n=0}^{\infty} a_n = A(e, c)$ *for all positive* c.

Proof. Since $A_n = O(1)$, $A_n = o(n^{\frac{1}{2}})$ and thus $\sum_{n=0}^{\infty} A_n = A\left(e, \frac{\alpha}{2}\right)$ by Theorem 2. Therefore

$$(4.4)\qquad \sum_{n=0}^{\infty} a_n = A(e, c) \quad \text{for } 0 < c \le \frac{\alpha}{2}$$

by Theorem 3.

Now let $z = x + iy \in K$ (the complex plane). Choose $y_0 > 0$ and $0 < x_0 < \frac{\alpha}{2}$. Let $D = \{z \in K : x > x_0 \text{ and } |y| < y_0\}$ and

$$\phi_n(z) = \left(\frac{z}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{zh^2}{n}} A_{n+h} = \left(\frac{z}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-n}^{\infty} e^{-\frac{zh^2}{n}} A_{n+h}.$$

Since $\left|e^{-\frac{zh^2}{n}} A_{n+h}\right| \le H\left(e^{-\frac{x_0}{n}}\right)^{h^2}$ for some constant H and all $z \in D$, it follows that

$$\sum_{h=-n}^{\infty} e^{-\frac{zh^2}{n}} A_{n+h}$$

converges uniformly in D and thus that $\phi_n(z)$ is analytic in D $(n = 1, 2, 3, \ldots)$.

In D, $|z| \le x\left\{1 + \left(\frac{y_0}{x_0}\right)^2\right\}^{\frac{1}{2}}$ and therefore $\left(\frac{|z|}{\pi n}\right)^{\frac{1}{2}} = O\left(\sqrt{\frac{x}{\pi n}}\right)$ uniformly for all $z \in D$. Thus

$$|\phi_n(z)| = O\left(\sqrt{\frac{x}{\pi n}} \sum_{h=-\infty}^{\infty} e^{-\frac{xh^2}{n}}\right) = O\left\{1 + O\left(\sqrt{\frac{x}{n}}\right)\right\} \quad \text{(by Lemma 7)}$$

$$= O(1)$$

if x is bounded above (say if $x_0 \le x \le x_1$).

Therefore $\{\phi_n(z)\}$ is almost uniformly bounded in D (i. e., it is uniformly bounded on compact subsets of D).

Since $\phi_n(c) \to A$ as $n \to \infty$ for $x_0 \le c \le \frac{\alpha}{2}$ by (4.4), the sequence $\{\phi_n\}$ satisfies the hypotheses of the following theorem of Vitali (cf. [5], p. 117; [6], p. 168; or [7], chapter 4, § 5).

Let D be a region of the complex plane and suppose:

(i) *$\{\phi_n\}$ is a sequence of functions analytic and almost uniformly bounded in D,*

and

(ii) *there exists a sequence $\{Z_m\}$ (of distinct Z's) in D with at least one limit point $Z_0 \in D$ such that $\lim_{n\to\infty} \phi_n(Z_m)$ exists (necessarily finite because of (i)) for $m = 1, 2, 3, \ldots$. Then there exists a function $\phi(z)$, analytic in D, such that $\{\phi_n(z)\}$ converges almost uniformly to $\phi(z)$ in D.*

It follows that $\phi_n(z) \to A$ as $n \to \infty$ for all $z \in D$. In particular $\sum_{n=0}^{\infty} a_n = A(e, c)$ for $x_0 \le c < +\infty$. Combining this with (4.4) gives the theorem.

We now prove Theorem 1.

Proof. It follows from Theorems 4 and 5 that

$$(4.5)\qquad \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} A_{n+h} = A + o(1) \quad \text{as } n\to\infty \text{ for all } c > 0.$$

Since $a_n = o(1)$, $A_n - A_{n+h} = O(|h|)$ and

$$(4.6)\qquad \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{|h|>n^{\zeta}} e^{-\frac{ch^2}{n}} (A_n - A_{n+h}) = o(1) \quad \text{as } n\to\infty.$$

$\left(\text{Just replace } \frac{\alpha}{2} \text{ by } c \text{ in (4.2).}\right)$

Since for any fixed positive c

$$\left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} = 1 + o(1) \quad \text{as } n\to\infty \text{ by (2.3)},$$

it follows that

$$A_n\{1 + o(1)\} = \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} (A_n - A_{n+h}) + \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{h=-\infty}^{\infty} e^{-\frac{ch^2}{n}} A_{n+h}$$

$$= \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{|h|\le n^{\zeta}} e^{-\frac{ch^2}{n}} (A_n - A_{n+h}) + o(1) + A + o(1)$$

by (4.5) and (4.6). Thus

$$(4.7)\qquad A_n - A = \left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{|h|\le n^{\zeta}} e^{-\frac{ch^2}{n}} (A_n - A_{n+h}) + o(1)$$

since

$$A_n \cdot o(1) = O(1) \cdot o(1) = o(1).$$

Now $|A_n - A_{n+h}| \le \frac{H\,|h|}{\sqrt{n}}$ for some constant H and for all h with

$$|h| \le n^{\zeta} \quad \text{and} \quad n = 1, 2, 3, \ldots,$$

by Lemma 6.

Therefore

$$\left|\left(\frac{c}{\pi n}\right)^{\frac{1}{2}} \sum_{|h| \leq n^{\zeta}} e^{-\frac{ch^2}{n}} (A_n - A_{n+h})\right| \leqq \left(\frac{H}{n}\right)\left(\frac{c}{\pi}\right)^{\frac{1}{2}} \sum_{|h| \leq n^{\zeta}} e^{-\frac{ch^2}{n}} |h|$$

$$\leqq \left(\frac{H}{n}\right)\left(\frac{c}{\pi}\right)^{\frac{1}{2}} \left\{\frac{n}{c} + \left(\frac{2n}{ec}\right)^{\frac{1}{2}}\right\} \qquad \text{(by Lemma 8)}$$

$$= \frac{H}{\sqrt{\pi c}} + o(1) \quad \text{as } n \to \infty.$$

It follows from (4. 7) that

$$|A_n - A| \leqq \frac{H}{\sqrt{\pi c}} + o(1).$$

Thus

$$\limsup_{n \to \infty} |A_n - A| \leqq \limsup_{n \to \infty} \left(\frac{H}{\sqrt{\pi c}} + o(1)\right) = \frac{H}{\sqrt{\pi c}}.$$

Since this holds for all positive c, $\lim_{n \to \infty} A_n = A$ and Theorem 1 is proved.

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