

## ON FUNCTIONAL CESÀRO AND HÖLDER METHODS OF SUMMABILITY

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**1. Cesàro and Hölder-type methods of summability.** Suppose that  $f(x)$  is integrable  $L$  in every finite interval  $[0, X]$ , and that  $\delta > 0$ . Define

$$f_\delta(x) = \{\Gamma(\delta)\}^{-1} \int_0^x (x-t)^{\delta-1} f(t) dt, \quad \text{and} \quad g(x) = e^x f(x).$$

*Definition.* If  $\Gamma(\delta + 1)x^{-\delta} f_\delta(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ , then we say that the  $(C, \delta)$  limit of  $f(x)$  is  $\sigma$ , and write  $f(x) \rightarrow \sigma(C, \delta)$ .

*Definition.* If  $e^{-x} g_\delta(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ , then we say that the  $(\hat{C}, \delta)$  limit of  $f(x)$  is  $\sigma$ , and write  $f(x) \rightarrow \sigma(\hat{C}, \delta)$ .

Note that  $(C, \delta)$  is the standard Cesàro method of summability, that

$$\begin{aligned} e^{-x} g_\delta(x) &= \{\Gamma(\delta)\}^{-1} e^{-x} \int_0^x (x-t)^{\delta-1} e^t f(t) dt \\ &= \{\Gamma(\delta)\}^{-1} X^{-1} \int_1^X (\log X/T)^{\delta-1} f(\log T) dT, \end{aligned}$$

and that this final integral is the functional Hölder transform of  $f(\log T)$ .

It is well known [4] that  $f(x) \rightarrow \sigma(C, \delta)$  if and only if  $f(e^x) \rightarrow \sigma(\hat{C}, \delta)$ . Our primary objective is to prove that if  $f(x) \rightarrow \sigma(\hat{C}, \delta)$  then  $f(x) \rightarrow \sigma(C, \delta)$ , and that there is a function whose  $(C, \delta)$  limit exists but whose  $(\hat{C}, \delta)$  limit does not exist.

We need two lemmas. The first is due to M. Riesz [3].

LEMMA 1. For  $x > t > 0$  and  $0 < \delta < 1$ ,

$$\Gamma(1-\delta) \int_0^t (x-v)^{\delta-1} f(v) dv = \delta \int_0^t f_\delta(v) dv \int_t^x (x-w)^{\delta-1} (w-v)^{-\delta-1} dw.$$

LEMMA 2. If  $0 < \delta \leq 1$  and  $e^{-x} g_\delta(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ , then

$$\delta x^{-\delta} \int_0^x (x-t)^{\delta-1} e^{-t} g(t) dt \rightarrow \sigma \quad \text{as } x \rightarrow \infty.$$

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*Proof.* Suppose first that  $0 < \delta < 1$ . Using the result of Lemma 1, we have

$$\begin{aligned} &\int_0^x (x-t)^{\delta-1} e^{-t} g(t) dt - \Gamma(\delta) e^{-x} g_\delta(x) \\ &= \int_0^x e^{-t} dt \int_0^t (x-u)^{\delta-1} g(u) du \\ &= \frac{\delta}{\Gamma(1-\delta)} \int_0^x e^{-t} dt \int_0^t g_\delta(v) dv \int_t^x (x-w)^{\delta-1} (w-v)^{-\delta-1} dw \\ &= \frac{\delta}{\Gamma(1-\delta)} \int_0^x e^{-v} g_\delta(v) dv \int_v^x (x-w)^{\delta-1} (w-v)^{-\delta-1} dw \int_v^x e^{-(t-v)} dt \\ &= \int_0^x J(x-v) e^{-v} g_\delta(v) dv, \quad \text{where} \end{aligned}$$

$$J(y) = \frac{\delta}{\Gamma(1-\delta)} \int_0^y (y-u)^{\delta-1} u^{-\delta-1} (1-e^{-u}) du.$$

It now suffices to show that

$$\delta x^{-\delta} \int_0^x J(x-v) \phi(v) dv \rightarrow \sigma$$

whenever  $\phi(x)$  is integrable  $L$  in every finite interval  $[0, X]$  and tends to  $\sigma$  as  $x \rightarrow \infty$ . This is true since (see, for example [2, Theorem 6])

$$\begin{aligned} \delta x^{-\delta} \int_0^x J(x-v) dv &= \frac{\delta}{\Gamma(1-\delta)} x^{-\delta} \int_0^x (x-u)^{\delta} u^{-\delta-1} (1-e^{-u}) du \\ &\rightarrow \frac{\delta}{\Gamma(1-\delta)} \int_0^\infty u^{-\delta-1} (1-e^{-u}) du = 1 \quad \text{as } x \rightarrow \infty, \end{aligned}$$

and since, for each fixed  $y > 0$ ,

$$\delta x^{-\delta} \int_0^y J(x-v) dv \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

When  $\delta = 1$ , we have

$$x^{-1} \int_0^x e^{-t} g(t) dt - x^{-1} e^{-x} g_1(x) = x^{-1} \int_0^x e^{-t} g_1(t) dt,$$

and the desired result now follows from the regularity of the  $(C, 1)$  method.

We now prove two theorems which show the relation between the methods  $(\hat{C}, \alpha)$  and  $(C, \alpha)$ .

THEOREM 1. For  $\alpha > 0$ , if  $f(x) \rightarrow \sigma(\hat{C}, \alpha)$  then  $f(x) \rightarrow \sigma(C, \alpha)$ .

*Proof.* First suppose that  $0 < \alpha \leq 1$ . Since

$$\Gamma(\alpha) f_\alpha(x) = \int_0^x (x-t)^{\alpha-1} e^{-t} g(t) dt,$$

the result follows from Lemma 2.

Now suppose that  $\alpha > 1$ . Set  $\alpha = k + \delta$  where  $0 < \delta \leq 1$  and  $k = 1, 2, \dots$ . Integration by parts  $k$  times yields

$$\begin{aligned} \Gamma(\alpha)f_\alpha(x) &= (-1)^k \int_0^x g_k(t) \left(\frac{d}{dt}\right)^k [e^{-t}(x-t)^{\alpha-1}] dt \\ &= a_0 \int_0^x g_k(t) e^{-t} (x-t)^{\delta-1} dt + \sum_{r=1}^k a_r \int_0^x g_k(t) e^{-t} (x-t)^{\delta-1+r} dt \\ &= a_0 \int_0^x g_k(t) e^{-t} (x-t)^{\delta-1} dt + \sum_{r=0}^{k-1} b_r \int_0^x g_{k+1}(t) e^{-t} (x-t)^{\delta-1+r} dt \\ &\quad + \int_0^x g_{k+1}(t) e^{-t} (x-t)^{\alpha-1} dt, \end{aligned}$$

where the  $a_r$  and  $b_r$  are constants.

By assumption,  $e^{-x}g_\alpha(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ , and so since  $k+1 \geq \alpha$ , it is easy to show that  $e^{-x}g_{k+1}(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ . Thus using Lemma 1 for the term involving  $a_0$  and the regularity of the Cesàro methods for the other terms, it follows that  $\Gamma(\alpha+1)x^{-\alpha}f_\alpha(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ . This completes the proof of Theorem 1.

**THEOREM 2.** For  $\alpha > 0$ ,  $e^{ix} \rightarrow 0(C, \alpha)$ , but the  $(\hat{C}, \alpha)$  limit of  $e^{ix}$  does not exist.

*Proof.* By the Riemann-Lebesgue theorem,

$$x^{-\alpha} \int_0^x (x-t)^{\alpha-1} e^{it} dt = \int_0^1 (1-u)^{\alpha-1} e^{iux} du \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

so that  $e^{ix} \rightarrow 0(C, \alpha)$ . On the other hand,

$$\begin{aligned} e^{-x} \int_0^x (x-t)^{\alpha-1} e^{it} dt &= e^{ix} \int_0^x t^{\alpha-1} e^{-t(1+i)} dt \\ &= e^{ix} \int_0^\infty t^{\alpha-1} e^{-t(1+i)} dt + o(1) = \frac{e^{ix} \Gamma(\alpha)}{(1+i)^\alpha} + o(1), \end{aligned}$$

which does not tend to a limit as  $x \rightarrow \infty$ ; that is, the  $(\hat{C}, \alpha)$  limit of  $e^{ix}$  does not exist.

**2. Application to the Borel-type methods of summability.** Suppose that  $\lambda > 0$ , that  $\mu$  is real and that  $N$  is a non-negative integer greater than  $-\mu/\lambda$ . Let  $\rho, s_n$  ( $n = 0, 1, \dots$ ) be complex numbers. Define

$$S_{\lambda, \mu}(x) = \lambda e^{-x} \sum_{n=N}^{\infty} \frac{s_n x^{\lambda n + \mu - 1}}{\Gamma(\lambda n + \mu)}.$$

*Definition* [1]. If  $S_{\lambda, \mu}(x) \rightarrow \rho$  as  $x \rightarrow \infty$ , then we say that the  $(B, \lambda, \mu)$  limit of the sequence  $\{s_n\}$  is  $\rho$ , and writes  $s_n \rightarrow \rho(B, \lambda, \mu)$ .

The following two theorems are known.

**THEOREM 3** [5]. The  $(\hat{C}, \alpha)$   $(B, \lambda, \mu)$  transform of the sequence  $\{s_n\}$  is equal to the  $(B, \lambda, \mu + \delta)$  transform of the sequence  $\{s_n\}$ ; that is

$$e^{-x} \int_0^x (x-t)^{\delta-1} e^{t} S_{\lambda, \mu}(t) dt = S_{\lambda, \mu+\delta}(x).$$

From this it follows that  $s_n \rightarrow \rho(\hat{C}, \delta)$   $(B, \lambda, \mu)$  if and only if  $s_n \rightarrow \rho(B, \lambda, \mu + \delta)$ .

**THEOREM 4.** If  $s_n \rightarrow \rho(B, \lambda, \mu)$  then  $s_n \rightarrow \rho(C, \delta)$   $(B, \lambda, \mu)$ .

This is trivial since  $(C, \delta)$  is a regular method. See also [6].

The following theorem, which follows immediately from Theorem 3 and the results of §1, extends Theorem 4.

**THEOREM 5.** (i) If  $s_n \rightarrow \rho(B, \lambda, \mu + \delta)$  then  $s_n \rightarrow \rho(C, \delta)$   $(B, \lambda, \mu)$ ;

(ii) There is a sequence whose  $(C, \delta)$   $(B, \lambda, \mu)$  limit exists but whose  $(B, \lambda, \mu + \delta)$  limit does not exist.

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