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ADDENDUM TO "TAUBERIAN THEOREMS FOR BOREL-TYPE METHODS OF SUMMABILITY"

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We will use the notation and definitions given in [2, p. 167 and p. 173]. In addition, we say that $s_n = 0(1)(B', \alpha, \beta)$ if $A_{\alpha,\beta}(x)$ exists for all $x \geq 0$ and $\alpha^{-1} \int_0^x A_{\alpha,\beta}(t)\,dt$ is bounded on $[0, \infty)$. We will also use the notations "$s_n \to s(B', \alpha, \beta)$" and "$\sum_0^\infty a_n = s(B', \alpha, \beta)$" interchangeably.

We need the following lemma, the proof of which is readily obtained by use of [2, Lemma 1(i)].

LEMMA 1. *Let*

$$F_{\alpha,\beta}(x) = \alpha^{-1} \int_0^x A_{\alpha,\beta}(t)\,dt$$

exist for all $x \geq 0$. *Then*

$$F_{\alpha,\beta+\delta}(x) = \int_0^x h(x-t) F_{\alpha,\beta}(t)\,dt$$

where $\delta > 0$ *and* $h(u) = u^{\delta-1} e^{-u}/\Gamma(\delta)$.

The following result is due to Borwein [1, Theorem 2].

THEOREM A. $s_n \to s(B, \alpha, \beta+1)$ *if and only if* $s_n \to s(B', \alpha, \beta)$.

A simplified version of the proof of Theorem A yields the following.

THEOREM B. $s_n = 0(1)(B, \alpha, \beta+1)$ *if and only if* $s_n = 0(1)(B', \alpha, \beta)$.

It is now immediate, in view of Theorems A and B, that the following theorems are equivalent to the corresponding theorems in [2] with $\beta+1$ and $\mu+1$ in place of β and μ.

THEOREM 1. *If* $\sum_0^\infty a_n = s(B', \alpha, \mu)$ *and* $a_n \to 0$ (B', α, β), *then* $\sum_0^\infty a_n = s(B', \alpha, \beta)$.

THEOREM 2. *If* $s_n \to s(B', \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $s_n = 0(1)(B', \alpha, \beta)$, *then* $s_n \to s(B', \alpha, \beta+\delta)$ *for any* $\delta > 0$.

THEOREM 2*. *If* $\sum_0^\infty a_n = s(B', \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $a_n = 0(1)(B', \alpha, \beta)$, *then* $\sum_0^\infty a_n = s(B', \alpha, \beta+\delta)$ *for any* $\delta > 0$.

THEOREM 3. *If* $s_n \to s(B', \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $S_{\alpha,\beta+1}(x)$ *is slowly decreasing, then* $s_n \to s(B', \alpha, \beta)$.

THEOREM 3*. *If* $\sum_0^\infty a_n = s(B', \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $A_{\alpha,\beta+1}(x)$ *is slowly decreasing, then* $\sum_0^\infty a_n = s(B', \alpha, \beta)$.

THEOREM 4. *If* $s_n = 0(1)(B', \alpha, \mu)$ *and* $s_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $s_n = 0(1)(B', \alpha, \beta)$.

THEOREM 5. *If* $s_n \to s(B', \alpha, \mu)$ *and* $s_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $s_n \to s(B', \alpha, \beta)$.

THEOREM 5*. *If* $\sum_0^\infty a_n = s(B', \alpha, \mu)$ *and* $a_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $\sum_0^\infty a_n = s(B', \alpha, \beta)$.

THEOREM 6. *If* $s_n \to s(B', \alpha, \mu)$ *and if there are positive real numbers* A, a, δ *such that* $|S_{\alpha,\mu+1}(z)| \leq A \exp(a|z|)$ *whenever* $\operatorname{Re} z \geq \delta$, *then* $s_n \to s(B', \alpha, \beta)$.

THEOREM 6*. *If* $\sum_0^\infty a_n = s(B', \alpha, \mu)$ *and if there are positive real numbers* A, a, δ *such that* $|A_{\alpha,\mu+1}(z)| \leq A \exp(a|z|)$ *whenever* $\operatorname{Re} z \geq \delta$, *then* $\sum_0^\infty a_n = s(B', \alpha, \beta)$.

THEOREM 7. *If* $\sum_0^\infty a_n = s(B', \alpha, \mu)$ *and* $|a_n| \leq K^n$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $\sum_0^\infty a_n = s(B', \alpha, \beta)$.

In addition, we have the following result which is a more appropriate analogue to [2, Theorem 3] than the above Theorem 3.

THEOREM 8. *If* $s_n \to s(B', \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $\alpha^{-1}\int_0^x A_{\alpha,\beta}(t)\,dt$ *is slowly decreasing, then* $s_n \to s(B', \alpha, \beta)$.

Proof. Let

$$F_{\alpha,\beta}(x) = \alpha^{-1}\int_0^x A_{\alpha,\beta}(t)\,dt, \qquad F_{\alpha,\beta+\varepsilon}(x) = \alpha^{-1}\int_0^x A_{\alpha,\beta+\varepsilon}(t)\,dt.$$

In view of Lemma 1 and [2, lemma 3], we have, by [2, Theorem 9] (with $F(x) = F_{\alpha,\beta+\varepsilon}(x)$, $f(x) = F_{\alpha,\beta}(x)$, $h(u) = u^{\varepsilon-1}e^{-u}/\Gamma(\varepsilon)$), that $F_{\alpha,\beta}(x)$ is bounded on $[0, \infty]$. Hence, by Theorem 2, $F_{\alpha,\beta+1}(x) \to s$ as $x \to \infty$. Thus, in view of Lemma 1, it follows by [2, Theorem 8] (with $F(x) = F_{\alpha,\beta+1}(x)$, $f(x) = F_{\alpha,\beta}(x)$), that $F_{\alpha,\beta}(x) \to s$ as $x \to \infty$.

REFERENCES

1. D. Borwein, *Relations between Borel-type methods of summability*, Journal London Math. Soc., **35** (1960), 65–70.

2. D. Borwein and E. Smet, *Tauberian theorems for Borel-type methods of summability*, Canad. Math. Bull., **17** (1974), 167–173.

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