

ON THE ZEROS OF THE POWER SERIES

$$\sum_{n=0}^{\infty} (-1)^n (1 - c^{-n-1})^\kappa z^n$$

WITH AN APPLICATION TO DISCONTINUOUS RIESZ-SUMMABILITY

BY

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1. On the zeros of $\sum_{n=0}^{\infty} (-1)^n (1 - c^{-n-1})^\kappa z^n$. If not stated otherwise, we assume throughout that $\kappa > 0$, $c > 1$, and that $k < \kappa \leq k + 1$ where $k = 0, 1, 2, \dots$. We reserve the symbol x to denote real numbers, and define $\mathbf{C}^* = \mathbf{C} - \{x : x < -1\}$, \mathbf{C} being the complex plane. Let

$$\phi(z) = \phi(z, c, \kappa) = \sum_{n=0}^{\infty} (-1)^n (1 - c^{-n-1})^\kappa z^n.$$

The series defining $\phi(z)$ is only convergent for $|z| < 1$, but Lemma 1 (1) (below) shows that $\phi(z)$ is a meromorphic function in \mathbf{C} with simple poles at $z = -c^n$, $n = 0, 1, 2, \dots$. The zeros of $\phi(z)$ have been investigated by Peyerimhoff [3], and the following theorem is due to him.

THEOREM P. $\phi(z)$ has exactly k zeros in the region \mathbf{C}^* , and they are all positive and simple. [3, Theorem 5].

REMARK. We denote the zeros of $\phi(z) = \phi(z, c, \kappa)$ by $r_i(c, \kappa)$, $i = 1, \dots, k$ with $0 < r_1(c, \kappa) < \dots < r_k(c, \kappa)$. Since the zeros are simple, we have $\phi'(r_i(c, \kappa)) \neq 0$; and therefore every $r_i(c, \kappa)$ is an analytic function of c and κ for $c > 1$, $\kappa > 0$, by implicit function theory [1, 10.2].

In this part of the paper we prove the following theorem on the monotonicity of the zeros $r_i(c, \kappa)$.

THEOREM 1. Every zero $r_i(c, \kappa)$ is a strictly increasing, unbounded function of c with $(\partial/\partial c)r_i(c, \kappa) > 0$.

Wirsing [4] proved:

THEOREM W. Every zero $r_i(c, \kappa)$ is a strictly decreasing function of κ with $(\partial/\partial \kappa)r_i(c, \kappa) < 0$.

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We shall use the following notation:

$$A_n^{-\kappa-1} = \binom{n-\kappa-1}{n} = (-1)^n \binom{\kappa}{n}$$

for $n = 0, 1, 2, \dots$, where $\binom{\kappa}{n}$ denotes the binomial coefficient;

$$\psi(z) = \psi(z, c, \kappa) = -z\phi(z, c, \kappa);$$

$$\psi'(z) = \psi_1(z, c, \kappa) = \frac{\partial}{\partial z} \psi(z, c, \kappa);$$

$$\psi_2(z, c, \kappa) = \frac{\partial}{\partial c} \psi(z, c, \kappa);$$

$$\theta(z) = \theta(z, c, \kappa) = \psi'(z) \prod_{\nu=0}^{k+1} (c^\nu + z).$$

We need some auxiliary results:

LEMMA 1. For $z \neq -c^n$, $n = 0, 1, 2, \dots$,

$$(1) \quad \phi(z, c, \kappa) = \sum_{n=0}^{\infty} A_n^{-\kappa-1} \frac{1}{c^n + z};$$

$$(2) \quad \psi(z, c, \kappa + 1) = \psi(z, c, \kappa) - \psi\left(\frac{z}{c}, c, \kappa\right);$$

$$(3) \quad \psi_2(z, c, \kappa + 1) = \frac{(\kappa + 1)z}{c^2} \psi_1\left(\frac{z}{c}, c, \kappa\right).$$

Proof. Expanding $(1 - c^{-n-1})^\kappa$ into a binomial series we get (1). We can derive (2) and (3) directly from the power series representation of $\psi(z, c, \kappa)$.

The proof of Theorem 1 is based largely on the following lemma:

LEMMA 2. For all $x > -1$, $(-1)^k \theta^{(k+1)}(x) > 0$.

Proof. Using formula (1) we get

$$\theta(x) = - \sum_{n=0}^{\infty} A_n^{-\kappa-1} c^n \mu_n(x)$$

where

$$\mu_n(x) = \frac{1}{(c^n + x)^2} \prod_{\nu=0}^{k+1} (c^\nu + x) = \frac{1}{w^2} \prod_{\nu=0}^{k+1} (w + c^\nu - c^n)$$

with $w = c^n + x$. We consider two cases.

First, let $n \leq k + 1$. Then

$$\mu_n(x) = (-1)^n \frac{M_n}{w} + P_k(w)$$

where $P_k(w)$ is a polynomial of degree k in w , and

$$M_n = \prod_{\nu=0}^{n-1} (c^n - c^\nu) \prod_{\nu=n+1}^{k+1} (c^\nu - c^n) > 0.$$

Hence

$$(-1)^{k+1} A_n^{-\kappa-1} c^n \left(\frac{d}{dx}\right)^{k+1} \mu_n(x) = (-1)^n A_n^{-\kappa-1} c^n (k+1)! \frac{M_n}{w^{k+1}} > 0$$

for $x > -1$.

Next, let $n > k + 1$. Expanding $\prod_{\nu=0}^{k+1} (w + c^\nu - c^n)$ in powers of w , we get

$$\mu_n(x) = (-1)^{k+2} M_n \left(\frac{1}{w^2} - \frac{1}{w} \sum_{\nu=0}^{k+1} \frac{1}{c^n - c^\nu} \right) + P_k(w)$$

where $P_k(w)$ is a polynomial of degree k in w and

$$M_n = \prod_{\nu=0}^{k+1} (c^n - c^\nu) > 0.$$

Hence

$$\begin{aligned} (-1)^{k+1} A_n^{-\kappa-1} c^n \left(\frac{d}{dx}\right)^{k+1} \mu_n(x) \\ = (-1)^{k+1} A_n^{-\kappa-1} c^n \frac{(k+1)!}{w^{k+3}} M_n \left(\sum_{\nu=0}^{k+1} \frac{c^n + x}{c^n - c^\nu} - k - 2 \right) \geq 0 \end{aligned}$$

for $x > -1$, since $(-1)^{k+1} A_n^{-\kappa-1} \geq 0$ when $n > k + 1$.

It follows that

$$(-1)^{k+1} \theta^{(k+1)}(x) = (-1)^{k+1} \left(- \sum_{n=0}^{\infty} A_n^{-\kappa-1} c^n \left(\frac{d}{dx}\right)^{k+1} \mu_n(x) \right) < 0 \text{ for } x > -1.$$

Proof of Theorem 1. By Lemma 2, $\theta(x)$ has at most $k + 1$ zeros in the range $x > -1$, and consequently the same holds for $\psi'(x)$. By (2), with $z = r_i(c, \kappa + 1) = r_i$, we have $\psi(r_i, c, \kappa) = \psi(r_i/c, c, \kappa)$ and hence $\psi'(x_i) = 0$ for some $x_i \in (r_i/c, r_i)$, $i = 1, \dots, k + 1$.

Peyerimhoff has shown that $0 < r_1/c < r_1 < r_2/c < \dots < r_k < r_{k+1}/c < r_{k+1}$ [3, p. 210]. Thus the $k + 1$ numbers x_1, x_2, \dots, x_{k+1} are distinct and they yield all the zeros of $\psi'(x)$ in the range $x > -1$. Hence

$$(4) \quad \psi_1(r_i(c, \kappa + 1)/c, c, \kappa) \neq 0.$$

Next, by (3) and (4) with $\kappa - 1$ in place of κ , we have

$$c^2 \psi_2(r_i(c, \kappa), c, \kappa) = \kappa r_i(c, \kappa) \psi_1(r_i(c, \kappa)/c, c, \kappa - 1) \neq 0.$$

We also have that $\psi_1(r_i(c, \kappa), c, \kappa) \neq 0$, since the zeros of $\phi(z)$ are simple in \mathbb{C}^*

by Theorem P. Hence

$$\frac{\partial r_i(c, \kappa)}{\partial c} = -\frac{\psi_2(r_i(c, \kappa), c, \kappa)}{\psi_1(r_i(c, \kappa), c, \kappa)} \neq 0 \text{ for } c > 1, i = 1, \dots, k.$$

In order to prove that $(\partial/\partial c)r_i(c, \kappa) > 0$ it suffices to show that

$$\lim_{c \rightarrow \infty} r_i(c, \kappa) = \infty.$$

For fixed $\kappa > 0$, we have

$$\lim_{c \rightarrow \infty} \phi(x, c, \kappa) = \lim_{c \rightarrow \infty} \sum_{n=0}^{\infty} A_n^{-\kappa-1} \frac{1}{c^n + x} = \frac{1}{1+x}$$

uniformly for $x \geq 0$. Therefore, given $r > 0$, there exists s such that

$$\phi(x, c, \kappa) \geq \frac{1}{2(1+r)} > 0$$

whenever $c > s$ and $0 \leq x \leq r$. It follows that $r_i(c, \kappa) > r$ whenever $c > s$, and hence that $\lim_{c \rightarrow \infty} r_i(c, \kappa) = \infty$.

2. On the equivalence of discontinuous Riesz-summability with convergence. Let $\{\lambda_n\}$ be an unbounded increasing sequence of non-negative numbers. Given a series $\sum_1^{\infty} a_n$, and a number $\kappa \geq 0$, let

$$A_{\lambda}^{\kappa}(x) = \sum_{\lambda_n < x} (x - \lambda_n)^{\kappa} a_n.$$

If $x^{-\kappa} A_{\lambda}^{\kappa}(x) \rightarrow s$ as $x \rightarrow \infty$, the series $\sum_1^{\infty} a_n$ is said to be summable (R, λ_n, κ) to s . The series is said to be summable by the discontinuous Riesz method (R^*, λ_n, κ) to s if $\lambda_n^{-\kappa} A_{\lambda}^{\kappa}(\lambda_n) \rightarrow s$ as $n \rightarrow \infty$.

We shall discuss the equivalence of (R^*, λ_n, κ) with convergence in the special case $\lambda_n = c^n$ for some $c > 1$. The following results on the equivalence of (R^*, λ_n, κ) with convergence are known.

THEOREM K 1. *If $\liminf_{n \rightarrow \infty} \lambda_{n+1}/\lambda_n > 1$, then (R^*, λ_n, κ) is equivalent to convergence for $0 \leq \kappa \leq 1$ and for $\kappa = 2$; so that (R^*, c^n, κ) is equivalent to convergence for every $c > 1$ when $0 \leq \kappa \leq 1$ and when $\kappa = 2$. (See Kuttner [2, Theorem 2].)*

In the same paper Kuttner proved the following results:

THEOREM K 2. *If $1 < \kappa < 2$, then (R^*, c^n, κ) is equivalent to convergence for every $c > 1$. [2, Theorem 4].*

THEOREM K 3. *If $\kappa > 2$, then there is a $c_0 = c_0(\kappa)$ such that (R^*, c^n, κ) is not equivalent to convergence whenever $1 < c \leq c_0$. [2, Theorem 3].*

THEOREM K 4. *In order that (R^*, c^n, κ) be equivalent to convergence for $\kappa > 1$, $c > 1$, it is necessary and sufficient that $\phi(z, c, \kappa) \neq 0$ for $|z| \leq 1, z \neq -1$. [2, Lemma 3].*

We shall prove the following theorem.

THEOREM 2. *There exists a function $c(\kappa)$, defined on $[0, \infty)$, such that*

- (a) (R^*, c^n, κ) is equivalent to convergence if and only if $c > c(\kappa)$;
- (b) $c(\kappa)$ is continuous and monotonic non-decreasing on $[0, \infty)$ with $c(\kappa) = 1$ for $0 \leq \kappa \leq 2$, and $c'(\kappa) > 0$ for $\kappa > 2$;
- (c) $c(\kappa)$ is analytic for $\kappa > 2$, and for sufficiently large κ ,

$$c(\kappa) = \sum_{n=-1}^{\infty} c_n \kappa^{-n},$$

where $c_{-1} = 1/\log 2$, $c_0 = -\frac{3}{2}$ and $c_1 = -6 + (\frac{73}{12} - \log 2)\log 2$; so that

$$c(\kappa) = \frac{\kappa}{\log 2} - \frac{3}{2} + O(1) \text{ as } \kappa \rightarrow \infty.$$

We write $f_{\kappa}(c) = \phi(1, c, \kappa)$. Since

$$\phi(z, c, \kappa) = \sum_{n=0}^{\infty} (-1)^n ((1 - c^{-n-1})^{\kappa} - 1) z^n + \frac{1}{1+z} \text{ for } |z| < 1,$$

we have

$$(5) \quad f_{\kappa}(c) = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n ((1 - c^{-n-1})^{\kappa} - 1).$$

For $\kappa \geq 2$, let

$$S_{\kappa} = \{c \geq 1: \phi(r, c, \kappa) = 0 \text{ for some } r \in [0, 1]\};$$

$$c(\kappa) = \sup S_{\kappa};$$

$$\tilde{c}(\kappa) = \sup\{c \geq 1: f_{\kappa}(c) = 0\}.$$

Note that $S_{\kappa} \neq \emptyset$ (since $1 \in S_{\kappa}$ for every $\kappa \geq 2$) and that $\tilde{c}(\kappa) \leq c(\kappa)$.

For the proof of Theorem 2 we need two lemmas.

LEMMA 3.

- (6) $S_{\kappa} = [1, c(\kappa)]$;
- (7) $c(\kappa) > 1$ for all $\kappa > 2$ and $\lim_{\kappa \rightarrow 2^+} c(\kappa) = 1$;
- (8) $r_1(c(\kappa), \kappa) = 1$;
- (9) $c(\kappa) = \tilde{c}(\kappa)$.

Proof. Since

$$(10) \quad \phi(r, c, 2) = \frac{(c-1)^2(c-r)}{(1+r)(c+r)(c^2+r)},$$

it follows that $c(2) = 1$. By Theorem K3, we have $c(\kappa) > 1$ for all $\kappa > 2$. Assume $\kappa > 2$. Since $\phi(r, c, \kappa)$ is continuous in r and c , it follows that $c(\kappa) \in S_\kappa$, i.e., $\phi(r, c(\kappa), \kappa) = 0$ for some $r \in [0, 1]$. Hence $r_1(c(\kappa), \kappa) \leq 1$. Since $r_1(c, \kappa) > 0$ for $c > 1, \kappa > 1$ by Theorem P, and $r_1(c, \kappa)$ is an increasing function of c by Theorem 1, we have

$$0 < r_1(c, \kappa) \leq r_1(c(\kappa), \kappa) \leq 1 \text{ for } 1 < c \leq c(\kappa).$$

Therefore $S_\kappa = [1, c(\kappa)]$.

Let $c' = \limsup_{\kappa \rightarrow 2^+} c(\kappa)$. Then $\phi(r, c', 2) = 0$ for some $r \in [0, 1]$, and hence $c' \leq 1$ by (10). Since $c(\kappa) > 1$ for $\kappa > 2$, it follows that $\lim_{\kappa \rightarrow 2^+} c(\kappa) = 1$.

If $r_1(c(\kappa), \kappa) < 1$, then $r_1(c(\kappa) + \varepsilon, \kappa) < 1$ for some $\varepsilon > 0$, since $r_1(c, \kappa)$ is a continuous function of c . It follows that $c(\kappa) + \varepsilon \in S_\kappa$, which contradicts the definition of $c(\kappa)$. Hence $r_1(c(\kappa), \kappa) = 1$. This implies that $\bar{c}(\kappa) = c(\kappa)$.

LEMMA 4. Let $c^* = (\kappa/\log 2) - \frac{3}{2} + \varepsilon$ for complex ε and κ . Then

$$(11) \quad f_\kappa(c^*) = \frac{\varepsilon \log^2 2}{2\kappa} + O(1/\kappa^2),$$

$$(12) \quad f'_\kappa(c^*) = \frac{\log^2 2}{2\kappa} + O(1/\kappa^2),$$

as $\kappa \rightarrow \infty$ uniformly for $|\varepsilon| \leq 1$; and

$$(13) \quad f'_\kappa(c) > 0 \text{ when } c \text{ and } \kappa \text{ are real, } c \geq \kappa \text{ and } \kappa \text{ is sufficiently large.}$$

Proof. We have

$$\begin{aligned} (1 - c^{*-1})^\kappa &= \exp\left(-\frac{\kappa}{c^*} - \frac{\kappa}{2c^{*2}} + O(1/\kappa^2)\right) \\ &= \frac{1}{2} \left(1 - \frac{\log^2 2}{\kappa} (2 - \varepsilon)\right) + O(1/\kappa^2), \\ (1 - c^{*-2})^\kappa &= 1 - \frac{\log^2 2}{\kappa} + O(1/\kappa^2), \end{aligned}$$

and

$$(1 - c^{*-n-1})^\kappa = 1 + O(1/\kappa^n)$$

uniformly for $|\varepsilon| \leq 1$ and $n = 2, 3, \dots$ as $\kappa \rightarrow \infty$.

From this and (5) we obtain

$$f_\kappa(c^*) = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n ((1 - c^{*-n-1})^\kappa - 1) = \frac{\varepsilon \log^2 2}{2\kappa} + O(1/\kappa^2)$$

and

$$f'_\kappa(c^*) = \frac{\kappa}{c^{*2}} \sum_{n=0}^{\infty} (-1)^n (n+1) c^{*-n} (1 - c^{*-n-1})^{\kappa-1} = \frac{\log^2 2}{2\kappa} + O(1/\kappa^2)$$

as $\kappa \rightarrow \infty$, uniformly for $|\varepsilon| \leq 1$.

For real $c, \kappa, c \geq \kappa, \kappa$ sufficiently large we have

$$f'_\kappa(c) = \frac{\kappa}{c^2} ((1 - c^{-1})^{\kappa-1} + O(1/\kappa)) > 0,$$

since $(1 - c^{-1})^{\kappa-1} \geq (1 - 1/\kappa)^{\kappa-1} \geq 1/e$.

Proof of Theorem 2. Conclusion (a) of the theorem follows from Theorem K4 and (6). Since $r_1(c(\kappa), \kappa) = 1$ for $\kappa > 2$ by (8), we obtain, by implicit function theory [1, 10.2], that $c(\kappa)$ is analytic and $c'(\kappa) = -\{(\partial/\partial\kappa)r_1(c(\kappa), \kappa)\}/\{(\partial/\partial c)r_1(c(\kappa), \kappa)\}$ for $\kappa > 2$. Since $(\partial/\partial c)r_1(c, \kappa) > 0$ by Theorem 1, and $(\partial/\partial\kappa)r_1(c, \kappa) < 0$ by Theorem W, we have $c'(\kappa) > 0$. This, together with (7), Theorems K 2 and K 3 establishes (b).

Let $\varepsilon > 0$ and $\gamma(\kappa) = (\kappa/\log 2) - \frac{3}{2}$. By (11) and (13), we have $f_\kappa(\gamma(\kappa) - \varepsilon) < 0, f_\kappa(\gamma(\kappa) + \varepsilon) > 0$ and $f'_\kappa(c) > 0$ for $c \geq \kappa$ and $\kappa \geq \kappa_0(\varepsilon)$. Hence

$$(14) \quad c(\kappa) = \frac{\kappa}{\log 2} - \frac{3}{2} + O(1) \text{ as } \kappa \rightarrow \infty.$$

Now consider $f_\kappa(c)$ for complex c, κ with $|c| > 1$. For κ sufficiently large, we have, by (11) and (12), that

$$(15) \quad f_\kappa(\gamma(\kappa) + \varepsilon) \neq 0 \text{ whenever } |\varepsilon| = 1$$

and

$$(16) \quad f'_\kappa(\gamma(\kappa) + \varepsilon) \neq 0 \text{ whenever } |\varepsilon| \leq 1.$$

Suppose in what follows that ρ is a sufficiently large positive number. Let $K_\rho = \{\kappa \in \mathbf{C} : |\kappa| = \rho\}$, and let $C_\rho = \{\kappa : \kappa \in K_\rho, f_\kappa(c) = 0 \text{ for some } c \text{ such that } |c - \gamma(\kappa)| \leq 1\}$. Since $|c(\rho) - \gamma(\rho)| < 1$, by (14), we have $\rho \in C_\rho$, and hence $C_\rho \neq \emptyset$. By the continuity of $f_\kappa(c)$ in c and κ, C_ρ is closed. For $\kappa_1 \in C_\rho$, we have $f_{\kappa_1}(c) = 0$ for some c such that $|c - \gamma(\kappa_1)| \leq 1$; and, for the same $c, f'_{\kappa_1}(c) \neq 0$ by (16). By implicit function theory [1, 10.2], we can conclude that there exists a neighbourhood of κ_1 and an analytic function $c(\kappa)$ such that $f_\kappa(c(\kappa)) = 0$ throughout this neighbourhood; moreover, $|c(\kappa) - \gamma(\kappa)| < 1$ by (15). This shows that C_ρ is non-empty, and is open and closed relative to K_ρ . Therefore $C_\rho = K_\rho$.

We show next that, for every $\kappa \in C_\rho$, there exists a unique $c = c(\kappa)$ such that $f_\kappa(c(\kappa)) = 0$ and $|c(\kappa) - \gamma(\kappa)| < 1$. Assume $f_\kappa(c_1) = f_\kappa(c_2) = 0, |c_1 - \gamma(\kappa)| < 1, i = 1, 2$. Then $0 = f_\kappa(c_2) - f_\kappa(c_1) = \int_{c_1}^{c_2} f'_\kappa(c) dc = (c_2 - c_1) \int_0^1 f'_\kappa(u(t)) dt$, where $u(t) = c_1 + t(c_2 - c_1)$. Since $|u(t) - \gamma(\kappa)| < 1$, we have

$$f'_\kappa(u(t)) = ((\log^2 2)/(2\kappa))(1 + O(1/\kappa))$$

as $\kappa \rightarrow \infty$, uniformly for $t \in [0, 1]$, by (12). Therefore $\int_0^1 f'_\kappa(u(t)) dt \neq 0$ for large κ , and this implies that $c_2 = c_1$. We thus have a unique function $c(\kappa)$ for $\kappa \in K_\rho$,

which is analytic on K_p by implicit function theory [1, 10.2]. Therefore

$$c(\kappa) = \sum_{n=-1}^{\infty} c_n \kappa^{-n}$$

for large κ . By (14), $c_{-1} = 1/\log 2$ and $c_0 = -\frac{3}{2}$.

Calculations similar to those used in the proof of Lemma 4 show that $c_1 = -6 + (\frac{73}{12} - \log 2)\log 2 < 0$, and therefore $c(\kappa)$ is convex (i.e., $c''(\kappa) < 0$) for large κ .

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