TAUBERIAN THEOREMS FOR STRONG AND ABSOLUTE BOREL-TYPE METHODS OF SUMMABILITY[1]

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1. **Introduction.** Suppose throughout that s, a_n $(n = 0, 1, 2, \ldots)$ are arbitrary complex numbers, that $\alpha > 0$ and β is real and that N is a non-negative integer such that $\alpha N + \beta \geq 1$. Let

$$s_n = \sum_{\nu=0}^{n} a_\nu \ (n \geq 0), \qquad s_{-1} = 0,$$

$$s_{\alpha,\beta}(z) = \sum_{n=N}^{\infty} s_n \frac{z^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}, \qquad a_{\alpha,\beta}(z) = \sum_{n=N}^{\infty} a_n \frac{z^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},$$

$$S_{\alpha,\beta}(z) = \alpha e^{-z} s_{\alpha,\beta}(z), \qquad A_{\alpha,\beta}(z) = \alpha e^{-z} a_{\alpha,\beta}(z)$$

where $z = x + iy$ is a complex variable and the power z^γ is assumed to have its principal value.

Borel-type methods are defined as follows:

(a) Summability: If $S_{\alpha,\beta}(x)$ exists for all $x \geq 0$ and tends to s as $x \to \infty$, we say that $s_n \to s(B, \alpha, \beta)$ or $\sum_0^\infty a_n = s(B, \alpha, \beta)$;

(b) Strong summability with index $p > 0$: If $S_{\alpha,\beta-1}(x)$ exists for all $x \geq 0$ and

$$\int_0^x e^t \left| S_{\alpha,\beta-1}(t) - s \right|^p dt = o(e^x),$$

we say that $s_n \to s[B, \alpha, \beta]_p$;

(c) Absolute summability: If $s_n \to s(B, \alpha, \beta)$ and $S_{\alpha,\beta}(x) \in BV_x[0, \infty)$,[2] we say that $s_n \to s \,|B, \alpha, \beta|$;

(d) Boundedness: If $S_{\alpha,\beta}(x)$ exists and is bounded on $[0, \infty)$, we say that $s_n = 0(1)(B, \alpha, \beta)$;

(e) Strong boundedness with index $p > 0$: If $S_{\alpha,\beta-1}(x)$ exists for all $x \geq 0$ and

$$\int_0^x e^t \left| S_{\alpha,\beta-1}(t) \right|^p dt = 0(e^x),$$

we say that $s_n = 0(1)[B, \alpha, \beta]_p$.

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[2] $f(x) \in BV_x[0, \infty)$ means that $f(x)$ is of bounded variation with respect to x on $[0, \infty)$.

The summability method $(B, 1, 1)$ is the Borel exponential method B (see [7]). The (B, α, β) method is due to Borwein (see [2]) and the $[B, \alpha, \beta]_p$ and $|B, \alpha, \beta|$ methods are due to Borwein and Shawyer (see [4], [3] respectively). Strong Borel-type summability $[B, \alpha, \beta]$ (see [3]) is the $[B, \alpha, \beta]_1$ method.

The actual choice of the integer N in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods, with α fixed and $\beta = \beta_1, \beta_2, \ldots, \beta_k$, are under consideration that N is such that $\alpha N + \beta_r \geq 1$ $(r = 1, 2, \ldots, k)$.

The following known result establishes a natural scale for these summability methods. (Theorem A(i) is [1, (II)]. Theorem A(ii) is [3, Theorem 9] when $p = 1$ and part of [4, Theorem 9*(ii)] when $p \geq 1$. Theorem A(iii) is [8, Lemma].)

THEOREM A. *Let* $\beta > \mu$.

(i) *If* $s_n \rightarrow s(B, \alpha, \mu)$, *then* $s_n \rightarrow s(B, \alpha, \beta)$.
(ii) *If* $p \geq 1$ *and* $s_n \rightarrow s[B, \alpha, \mu]_p$, *then* $s_n \rightarrow s[B, \alpha, \beta]_p$.
(iii) *If* $s_n \rightarrow s|B, \alpha, \mu|$, *then* $s_n \rightarrow s|B, \alpha, \beta|$.

In [5] we established a number of tauberian theorems for the (B, α, β) method. In this paper we investigate all the corresponding results for the $[B, \alpha, \beta]_p$ method with $p \geq 1$ and either prove them or show, by means of counterexamples, that they are false. We also examine some of the corresponding results for the $|B, \alpha, \beta|$ method.

2. **Preliminary results.** We first state some known results.

LEMMA 1.

(i) *If* $p \geq 1$ *and* $s_n \rightarrow s[B, \alpha, \beta]_p$, *then* $a_n \rightarrow 0[B, \alpha, \beta]_p$.
(ii) *If* $s_n \rightarrow s|B, \alpha, \beta|$, *then* $a_n \rightarrow 0|B, \alpha, \beta|$.

LEMMA 2.

(i) *If* $p \geq 1$ *and* $s_n \rightarrow s[B, \alpha, \beta]_p$, *then* $s_n \rightarrow s(B, \alpha, \beta)$.
(ii) *If* $p > 0$ *and* $s_n \rightarrow s(B, \alpha, \beta)$, *then* $s_n \rightarrow s[B, \alpha, \beta+1]_p$.

Lemma 1(i) is included in [3, Theorem 15] when $p = 1$ and in [4, Theorem 15*] when $p > 1$. Lemma 1(ii) is included in [3, Theorem 14]. Lemma 2(i) is [3, Theorem 3] when $p = 1$ while Lemma 2(i) follows from [4, Theorem 3*] and Theorem A(i) when $p > 1$. Lemma 2(ii) is [4, Theorem 5*].

Wherever it occurs in the following lemmas, we suppose that $f(x)$ is bounded and Lebesgue measurable on every finite interval $[0, X]$ and we let $f_\delta(x)$ be defined by

$$f_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t) \, dt$$

where $\delta > 0$.

LEMMA 3. *If* $\delta > 0$ *and* $\gamma > 0$, *then*

$$f_{\delta+\gamma}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f_\delta(t) \, dt.$$

LEMMA 4.

(i) *Let* $f(x) = s_{\alpha,\beta}(x)$ *and let* $\delta > 0$. *Then* $s_{\alpha,\beta+\delta}(x) = f_\delta(x)$.
(ii) $A_{\alpha,\beta}(x) = S_{\alpha,\beta}(x) - S_{\alpha,\beta+\alpha}(x) - \alpha e^{-x} s_{N-1}(x^{\alpha N+\beta-1}/\Gamma(\alpha N + \beta))$.

Lemma 3 is a well-known result the proof of which is straightforward. Lemma 4(i) is [2, Lemma 2]. The proof of Lemma 4(ii) is also straightforward.

LEMMA 5. *If* $s_n = 0(1)(B, \alpha, \beta)$, *then* $s_n = 0(1)(B, \alpha, \beta+\delta)$ *for every* $\delta > 0$.

Lemma 5 is [3, Theorem 8].

LEMMA 6. *Let* $p \geq 1$. *If* $s_n = 0(1)[B, \alpha, \beta]_p$, *then*

(i) $s_n = 0(1)(B, \alpha, \beta)$,
(ii) $s_n = 0(1)[B, \alpha, \beta+\delta]_p$ *where* $0 < \delta < 1$, *and*
(iii) $s_n = 0(1)[B, \alpha, \beta+\delta]_r$ *where* $r > 0$ *and* $\delta \geq 1$.

Proof. (i) When $p = 1$ the result is [3, Theorem 4]. Thus we suppose that $p > 1$ and we let $1/p + 1/q = 1$. Using Hölder's inequality and Lemma 4(i), we have that

$$|S_{\alpha,\beta}(x)| \leq e^{-x} \int_0^x e^t |S_{\alpha,\beta-1}(t)| \, dt$$

$$\leq e^{-x} \left\{ \int_0^x e^t |S_{\alpha,\beta-1}(t)|^p \, dt \right\}^{1/p} \left\{ \int_0^x e^t \, dt \right\}^{1/q}$$

$$\leq e^{-x} \{Ke^x\}^{1/p} \{e^x\}^{1/q} = K^{1/p}$$

for some positive constant K since $s_n = 0(1)[B, \alpha, \beta]_p$.

(ii) When $p = 1$ the result is included in [3, Theorem 10]. Thus we again suppose that $p > 1$ and we let $1/p + 1/q = 1$. Furthermore, we let $f(x) = \alpha s_{\alpha,\beta-1}(x)$, $L = 2^p/\{\Gamma(\delta)\}^p$, and $M = [1/\{\Gamma(\delta)\}^p] \int_0^1 e^{(1-p)t} |f_\delta(t)|^p \, dt$. Then, using Lemma 4(i), Hölder's inequality, and part of the proof of (i), we have for $x \geq 1$ that

$$\int_0^x e^t |S_{\alpha,\beta+\delta-1}(t)|^p \, dt = \frac{1}{\{\Gamma(\delta)\}^p} \int_0^x e^{(1-p)t} \left| \int_0^t (t-u)^{\delta-1} f(u) \, du \right|^p dt$$

$$\leq L \int_1^x e^{(1-p)t} \left\{ \int_0^{t-1} |f(u)| \, du \right\}^p dt$$

$$+ L \int_1^x e^{(1-p)t} \left\{ \int_{t-1}^t (t-u)^{\delta-1} |f(u)|^p \, du \right\}$$

$$\times \left\{ \int_{t-1}^t (t-u)^{\delta-1} \, du \right\}^{p/q} dt + M$$

$$\leq L \int_0^x e^{(1-p)t} \{K^{1/p} e^t\}^p \, dt$$

$$+ \frac{L}{\delta^{p/q}} \int_1^x e^{(1-p)t} \, dt \int_{t-1}^t (t-u)^{\delta-1} |f(u)|^p \, du + M$$

$$\leq LKe^x + \frac{L}{\delta^{p-1}} \int_0^x |f(u)|^p \, du \int_u^{u+1} e^{(1-p)t}(t-u)^{\delta-1} \, dt + M$$

$$\leq LKe^x + \frac{L}{\delta^p} \int_0^x e^{(1-p)u} |f(u)|^p \, du + M$$

$$= LKe^x + \frac{L}{\delta^p} \int_0^x e^u |S_{\alpha,\beta-1}(u)|^p \, du + M$$

$$= 0(e^x) \quad \text{since} \quad s_n = 0(1)[B, \alpha, \beta]_p.$$

This establishes the desired result.

(iii) If $\delta \geq 1$, then

$$\int_0^x e^t |S_{\alpha,\beta+\delta-1}(t)|^r \, dt \leq \int_0^x K^r e^t \, dt \leq K^r e^x$$

for some positive constant K by Lemma 6(i) and Lemma 5.

LEMMA 7. *If*

$$e^{-x} \int_0^x f(t) \, dt = o(1),$$

then

$$e^{-x} \int_0^x f_\delta(t) \, dt = o(1)$$

for every $\delta > 0$.

The proof of Lemma 7 is essentially the same as the proof of [3, Lemma 5].

LEMMA 8. *Let $p \geq 1$. If*

$$e^{-x} \int_0^x e^{(1-p)t} |f(t)|^p \, dt = 0(1) \quad and \quad e^{-x} \int_0^x f(t) \, dt = o(1),$$

then

(i) $e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p \, dt = o(1)$ *where $0 < \delta < 1$ and*

(ii) $e^{-x} \int_0^x e^{(1-r)t} |f_\delta(t)|^r \, dt = o(1)$ *where $r > 0$ and $\delta \geq 1$.*

Proof. (i) Let $\varepsilon > 0$. By hypothesis, there exists a number $Y \geq 0$ such that

$$\left| \int_0^x f(t) \, dt \right| \leq \varepsilon e^x$$

for all $x \geq Y$. Let

$$N(\varepsilon) = \sup_{0 \leq x \leq Y} \left| \int_0^x f(t) \, dt \right| < \infty.$$

Now

$$\limsup_{x \to \infty} e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p \, dt$$

$$= \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \frac{1}{\Gamma(\delta)} \int_0^t (t-u)^{\delta-1} f(u) \, du \right|^p \, dt$$

$$\leq \frac{2^p}{\{\Gamma(\delta)\}^p} \left\{ \limsup_{x \to \infty} I_1 + \limsup_{x \to \infty} I_2 \right\}.$$

where

$$I_1 = e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \int_0^{t-\varepsilon} (t-u)^{\delta-1} f(u) \, du \right|^p \, dt$$

and

$$I_2 = e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \int_{t-\varepsilon}^t (t-u)^{\delta-1} f(u) \, du \right|^p \, dt$$

But, using the Second Mean Value Theorem,

$$\limsup_{x \to \infty} I_1 = \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \varepsilon^{\delta-1} \int_{\mu(t)}^{t-\varepsilon} f(u) \, du \right|^p \, dt$$

$$\leq 2^p \varepsilon^{(\delta-1)p} \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{N(\varepsilon) + \varepsilon e^t\}^p \, dt$$

$$\leq 2^{2p} \varepsilon^{(\delta-1)p} \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{(N(\varepsilon))^p + \varepsilon^p e^{pt}\} \, dt$$

$$= 2^{2p} \varepsilon^{\delta p}$$

since

$$\left| \int_{\mu(t)}^{t-\varepsilon} f(u) \, du \right| \leq 2 \sup_{0 \leq y \leq t-\varepsilon} \left| \int_0^y f(u) \, du \right| \leq 2\{N(\varepsilon) + \varepsilon e^t\}$$

and

$$\lim_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{N(\varepsilon)\}^p \, dt = 0.$$

Also, by hypothesis there is a number $K \geq 0$ such that

$$e^{-x} \int_0^x e^{(1-p)t} |f(t)|^p \, dt \leq K$$

for all $x \geq 0$, and therefore, when $p = 1$,

$$\limsup_{x \to \infty} I_2 \leq \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x dt \int_{t-\varepsilon}^t (t-u)^{\delta-1} |f(u)| \, du$$

$$\leq \limsup_{x \to \infty} e^{-x} \int_0^x |f(u)| \, du \int_u^{u+\varepsilon} (t-u)^{\delta-1} \, dt$$

$$\leq K \frac{\varepsilon^\delta}{\delta},$$

while, when $p > 1$,

$$\limsup_{x \to \infty} I_2 \leq \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \left\{ \int_{t-\varepsilon}^t (t-u)^{\delta-1} |f(u)|^p \, du \right\}$$

$$\times \left\{ \int_{t-\varepsilon}^t (t-u)^{\delta-1} \, du \right\}^{p-1} dt$$

$$= \left\{ \frac{\varepsilon^\delta}{\delta} \right\}^{p-1} \limsup_{x \to \infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \, dt \int_{t-\varepsilon}^t (t-u)^{\delta-1} |f(u)|^p \, du$$

$$\leq \left\{ \frac{\varepsilon^\delta}{\delta} \right\}^{p-1} \limsup_{x \to \infty} e^{-x} \int_0^x |f(u)|^p \, du \int_u^{u+\varepsilon} (t-u)^{\delta-1} e^{(1-p)t} \, dt$$

$$\leq \left\{ \frac{\varepsilon^\delta}{\delta} \right\}^p \limsup_{x \to \infty} e^{-x} \int_0^x e^{(1-p)u} |f(u)|^p \, du \leq K \left\{ \frac{\varepsilon^\delta}{\delta} \right\}^p.$$

Thus for $p \geq 1$ we have that

$$\limsup_{x \to \infty} e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p \, dt \leq \frac{2^p (2^{2p} + K \delta^{-p})}{\{\Gamma(\delta)\}^p} \varepsilon^{\delta p}$$

from which it follows that

$$\limsup_{x \to \infty} e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p \, dt = 0$$

since ε is arbitrary. This establishes the desired result.

(ii) Since $e^{-x} f_1(x) = o(1)$ by hypothesis, we have, when $\delta = 1 + \mu$ where $\mu > 0$, that

$$e^{-x} f_{1+\mu}(x) = e^{-x} \int_0^x f_\mu(t) \, dt = o(1),$$

using Lemma 3 and Lemma 7. Hence, for $\delta \geq 1$,

$$e^{-x} \int_0^x e^{(1-r)t} |f_\delta(t)|^r \, dt = e^{-x} \int_0^x e^t |e^{-t} f_\delta(t)|^r \, dt$$

$$= e^{-x} \int_0^x e^t o(1) \, dt$$

$$= o(1).$$

If b is a real number, we let

$$H_b = \{z \mid \operatorname{Re} z \geq b\}.$$

A function $g(z)$ is said to be of exponential type in H_b if $g(z)$ is analytic in H_b and if there are positive numbers A, a such that $|g(z)| \leq A e^{a|z|}$ for all z in H_b.

LEMMA 9. *If $g(z)$ is of exponential type in H_0 and if*

$$\int_0^\infty |g(x)|^p \, dx < \infty \qquad (p > 0),$$

then

$$\int_0^\infty |g'(x)|^p \, dx < \infty.$$

Lemma 9 is due to Gaier [6, Theorem 2].

LEMMA 10. *If $g(z)$ is of exponential type in H_b and $g(x) \in BV_x[b, \infty)$, then*

$$g^{(k)}(x) \in BV_x[b, \infty)$$

for every non-negative integer k.

Proof. Suppose that $g^{(k)}(x) \in BV_x[b, \infty)$ where k is a non-negative integer. Then

$$\int_0^\infty |g^{(k+1)}(x+b+1)| \, dx < \infty$$

and

$$|g^{(k+1)}(z+b+1)| \leq \frac{(k+1)!}{2\pi} \int_0^{2\pi} |g(z+b+e^{i\theta})| \, d\theta$$

$$\leq (k+1)! \, A e^{a(|z|+|b|+1)}$$

for all z in H_0 where A, a are positive constants. Hence, by Lemma 9,

$$\int_0^\infty |g^{(k+2)}(x+b+1)| \, dx = \int_{b+1}^\infty |g^{(k+2)}(x)| \, dx < \infty$$

i.e. $g^{(k+1)}(x) \in BV_x[b+1, \infty)$. Since $g^{(k+1)}(x) \in BV_x[b, b+1]$, therefore $g^{(k+1)}(x) \in BV_x[b, \infty)$. The desired result now follows by induction.

3. **Tauberian theorems for strong Borel-type summability with index $p \geq 1$.** We first show that the scale in Theorem A(ii) us proper. In [5] we showed that there is a sequence $\{s_n\}$ which tends to a limit (B, α, β) but does not tend to a limit $(B, \alpha, \beta - 1)$. Hence, in view of Lemma 2, there is a sequence $\{s_n\}$ which tends to a limit $[B, \alpha, \beta + 1]_p$ for every $p > 0$ but does not tend to a limit $[B, \alpha, \beta - 1]_p$ for any $p \geq 1$.

THEOREM 1. *Let* $p, r \geq 1$. *If* $s_n \to s[B, \alpha, \mu]_p$ *and* $a_n \to 0[B, \alpha, \beta]_r$, *then* $s_n \to s[B, \alpha, \beta]_r$.

Proof. By Lemma 2(i), $s_n \to s(B, \alpha, \mu)$. The result now follows by [9, Theorem 3] and the note following [9, Theorem 3].

THEOREM 2. *Let* $p \geq 1$. *If* $s_n \to s[B, \alpha, \beta + \varepsilon]_p$ *for some* $\varepsilon > 0$ *and* $s_n = 0(1)[B, \alpha, \beta]_p$, *then* $s_n \to s[B, \alpha, \beta + \delta]_p$ *for every* $\delta > 0$.

Proof. We can suppose without loss of generality that $s = 0$. Then $s_n \to 0(B, \alpha, \beta + \varepsilon)$ and $s_n = 0(1)(B, \alpha, \beta)$ by Lemma 2(i) and Lemma 6(i). Hence $s_n \to 0(B, \alpha, \beta + \delta)$ by [5, Theorem 2] for $\delta > 0$. Also $s_n = 0(1)[B, \alpha, \beta + \delta]_p$ by Lemma 6(ii) or (iii). Therefore, letting $f(x) = \alpha s_{\alpha, \beta + \delta - 1}(x)$, we have that

$$e^{-x} \int_0^x f(t)\, dt = S_{\alpha, \beta + \delta}(x) = o(1)$$

and

$$e^{-x} \int_0^x e^{(1-p)t} |f(t)|^p\, dt = e^{-x} \int_0^x e^t |S_{\alpha, \beta + \delta - 1}(t)|^p\, dt = 0(1)$$

using Lemma 4(i), and consequently,

$$e^{-x} \int_0^x e^t |S_{\alpha, \beta + 2\delta - 1}(t)|^p\, dt = e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p\, dt = o(1)$$

using Lemma 4(i) and Lemma 8, i.e. $s_n \to 0[B, \alpha, \beta + 2\delta]_p$. This establishes the desired result.

THEOREM 2*. *Let* $p \geq 1$. *If* $\sum_0^\infty a_n = s[B, \alpha, \beta + \varepsilon]_p$ *for some* $\varepsilon > 0$ *and* $a_n = 0(1)[B, \alpha, \beta]_p$, *then* $\sum_0^\infty a_n = s[B, \alpha, \beta + \delta]_p$ *for every* $\delta > 0$.

Proof. By Lemma 1(i), $a_n \to 0[B, \alpha, \beta + \varepsilon]_p$ and thus, by Theorem 2, $a_n \to 0[B, \alpha, \beta + \delta]_p$ for every $\delta > 0$. The result now follows by Theorem 1.

A real-valued function $g(x)$, with domain $[0, \infty)$, is slowly decreasing if for every $\varepsilon > 0$ there exist positive numbers X, δ such that $g(x) - g(y) > -\varepsilon$ whenever $x \geq y \geq X$ and $x - y \leq \delta$. The following result is [5, Theorem 3]: If $s_n \to s(B, \alpha, \beta + \varepsilon)$ for some $\varepsilon > 0$ and $S_{\alpha, \beta}(x)$ is slowly decreasing, then $s_n \to s(B, \alpha, \beta)$. We now show that there is no analogue to this result for the $[B, \alpha, \beta]_p$ method.

Let $\{s_n\}$ be the sequence defined by $\sum_{n=0}^\infty s_n(x^n/n!) = e^x \sin e^x$ (cf. [7, p. 183]). Then $S_{1,1}(x) = \sin e^x$ where we choose $N = 0$. Thus, using Lemma 4(i),

$$S_{1,2}(x) = e^{-x} \int_0^x e^t \sin e^t\, dt = e^{-x}(\cos 1 - \cos e^x) = o(1)$$

and therefore $s_n \to 0(B, 1, 2)$. (In fact, by [5, Theorem 2], $s_n \to 0(B, 1, 1 + \delta)$ for every $\delta > 0$.) Hence, by Lemma 2(ii), $s_n \to 0[B, 1, 3]_r$, for every $r > 0$. Furthermore,

$$e^{-x} \int_0^x e^t |S_{1,1}(t) - 0|^r\, dt = e^{-x} \int_0^x e^t |\sin e^t|^r\, dt$$

$$= e^{-x} \int_1^{e^x} |\sin u|^r\, du \to \frac{L(r)}{\pi}$$

as $x \to \infty$ where $L(r) = \int_0^\pi |\sin u|^r\, du$. Therefore $s_n \nrightarrow 0[B, 1, 2]_r$, $s_n \to 0[B, 1, 3]_r$ and both $e^{-x} \int_0^x e^t S_{1,1}(t)\, dt$ and $e^{-x} \int_0^x e^t |S_{1,1}(t)|^r\, dt$ are slowly decreasing (since they both tend to a limit as $x \to \infty$).

THEOREM 3. *Let* $p \geq 1$. *If* $s_n \to s[B, \alpha, \mu]_p$ *and*

(i) $s_n \geq -K$ *for all* $n \geq 0$, *or*
(ii) $a_n \geq -K$ *for all* $n \geq 0$, *or*
(iii) $S_{\alpha, \mu}(z)$ *is of exponential type in* H_δ, *or*
(iv) $A_{\alpha, \mu}(z)$ *is of exponential type in* H_δ, *or*
(v) $|a_n| \leq K^n$ *for all* $n \geq 0$,

where K, δ *are positive constants, then*

$$s_n \to s[B, \alpha, \beta]_r$$

for every $r > 0$.

Proof. By Lemma 2(i), $s_n \to s(B, \alpha, \mu)$. Hence, by [5, Theorem 5, 5*, 6, 6*, or 7], $s_n \to s(B, \alpha, \beta - 1)$. The result now follows by Lemma 2(ii).

4. Tauberian theorems for absolute Borel-type summability. We first show that the scale in Theorem A(iii) is proper in the sense that for each β there is a sequence $\{s_n\}$ which is summable $|B, \alpha, \beta|$ but is not summable $|B, \alpha, \beta - 1|$.

Choose an integer m such that $\alpha m > 1$ and let P be the smallest integer such that $mP \geq N$. Let

$$x^P e^{-x} \sin e^x = \sum_{n=P}^\infty b_n x^n$$

and let

$$s_n = \begin{cases} \Gamma(\alpha n + \beta) b_k & \text{if} \quad n = mk, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$S_{\alpha, \beta}(x) = \alpha x^{\alpha mP + \beta - 1} e^{-x} e^{-x^{\alpha m}} \sin e^{x^{\alpha m}} = o(1)$$

and

$$S'_{\alpha,\beta}(x) = \alpha(\alpha mP + \beta - 1)x^{\alpha mP + \beta - 2}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}$$
$$- \alpha x^{\alpha mP + \beta - 1}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}$$
$$- \alpha(\alpha m)x^{\alpha mP + \alpha m + \beta - 2}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}$$
$$+ \alpha(\alpha m)x^{\alpha mP + \alpha m + \beta - 2}e^{-x}\cos e^{x^{\alpha m}}$$

so that $S'_{\alpha,\beta}(x) = o(1)$ and $S'_{\alpha,\beta}(x) \in L_1[0,\infty)$ since $\alpha mP + \beta - 2 \geq \alpha N + \beta - 2 \geq 0$ by our choice of N. Hence $s_n \to 0 |B, \alpha, \beta|$. However

$$S''_{\alpha,\beta}(x) = f(x) - \alpha(\alpha m)^2 x^{\alpha mP + 2\alpha m + \beta - 3}e^{-x}e^{x^{\alpha m}}\sin e^{x^{\alpha m}}$$

where $f(x) \in L_1[0,\infty)$ and therefore $S''_{\alpha,\beta}(x) \notin L_1[0,\infty)$ since $\alpha m > 1$. Thus, since

$$S_{\alpha,\beta-1}(x) = S_{\alpha,\beta}(x) + S'_{\alpha,\beta}(x)$$

and

$$S'_{\alpha,\beta-1}(x) = S'_{\alpha,\beta}(x) + S''_{\alpha,\beta}(x),$$

we have that

$$s_n \to 0(B, \alpha, \beta - 1) \quad \text{but} \quad s_n \nrightarrow 0 |B, \alpha, \beta - 1|.$$

THEOREM 4. *If $s_n \to s |B, \alpha, \mu|$ and $a_n \to 0 |B, \alpha, \beta|$, then $s_n \to s |B, \alpha, \beta|$.*

Proof. By [5, Theorem 1], $s_n \to s(B, \alpha, \beta)$. Thus it remains only to show that $S_{\alpha,\beta}(x) \in BV_x[0,\infty)$. Let k be a positive integer. Then, in view of Theorem A(iii), $A_{\alpha,\beta+(k-1)\alpha}(x) \in BV_x[0,\infty)$. Moreover, by Lemma 4(ii),

$$S_{\alpha,\beta+(k-1)\alpha}(x) = A_{\alpha,\beta+(k-1)\alpha}(x) + S_{\alpha,\beta+k\alpha}(x) + \alpha e^{-x}s_{N-1}\frac{x^{\alpha N+\beta-1}}{\Gamma(\alpha N+\beta)}.$$

Therefore $S_{\alpha,\beta+(k-1)\alpha}(x) \in BV_x[0,\infty)$ if $S_{\alpha,\beta+k\alpha}(x) \in BV_x[0,\infty)$. Since, in view of Theorem A(iii), $S_{\alpha,\beta+k\alpha}(x) \in BV_x[0,\infty)$ when $\beta + k\alpha \geq \mu$, it readily follows that $S_{\alpha,\beta}(x) \in BV_x[0,\infty)$.

If $\{s_n\}$ is the sequence described in the paragraph preceding Theorem 3, then, using Lemma 4(i),

$$S_{1,3}(x) = e^{-x}\int_0^x (\cos 1 - \cos e^t)\, dt$$

and thus it is readily seen that $s_n \to 0 |B, 1, 3|$ and $s_n \nrightarrow 0 |B, 1, 2|$. Hence there is also no immediate absolute summability analogue to [5, Theorem 3].

Our final results are extensions of a result due to Gaier (see [6]).

THEOREM 5. *If $s_n \to s |B, \alpha, \mu|$ and $S_{\alpha,\mu}(z)$ is of exponential type in H_δ for some $\delta > 0$, then $s_n \to s |B, \alpha, \beta|$.*

Proof. Let k be a positive integer such that $\mu - k \leq \beta$. By [5, Theorem 6] we have that $s_n \to s(B, \alpha, \mu - k)$. Furthermore, since

$$S_{\alpha,\mu-1}(z) = S_{\alpha,\mu}(z) + S^{(1)}_{\alpha,\mu}(z),$$

it is readily seen that

$$S_{\alpha,\mu-k}(z) = S_{\alpha,\mu}(z) + \sum_{j=1}^{k}\binom{k}{j}S^{(j)}_{\alpha,\mu}(z).$$

Since $S_{\alpha,\mu}(z)$ is of exponential type in H_δ and since $S_{\alpha,\mu}(x) \in BV_x[0,\infty)$ by hypothesis, we have, by Lemma 10, that $S^{(j)}_{\alpha,\mu}(x) \in BV_x[\delta,\infty)$ for $j = 1, \ldots, k$; also, since we choose N so that $\alpha N + \mu - k \geq 1$, we have that $S^{(j)}_{\alpha,\mu}(x) \in BV_x[0,\delta]$ for $j = 1, \ldots, k$. Therefore, $S^{(j)}_{\alpha,\mu}(x) \in BV_x[0,\infty)$ for $j = 1, \ldots, k$ and, consequently, $S_{\alpha,\mu-k}(x) \in BV_x[0,\infty)$. Hence $s_n \to s |B, \alpha, \mu - k|$ and, by Theorem A(iii), $s_n \to s |B, \alpha, \beta|$.

THEOREM 5*. *If $s_n \to s |B, \alpha, \mu|$ and $A_{\alpha,\mu}(z)$ is of exponential type in H_δ for some $\delta > 0$, then $s_n \to s |B, \alpha, \beta|$.*

Proof. By Lemma 1(ii), $a_n \to 0 |B, \alpha, \mu|$ and thus, by Theorem 5, $a_n \to 0 |B, \alpha, \beta\|$. The result now follows by Theorem 4.

THEOREM 6. *If $s_n \to s |B, \alpha, \mu|$ and $|a_n| \leq K^n$ for all $n \geq 0$ where K is a positive constant, then $s_n \to s |B, \alpha, \beta|$.*

Proof. Since $|a_n| \leq K^n$ for all $n \geq 0$, we have that

$$|A_{\alpha,\mu}(z)| \leq Ae^{K^{1/\alpha}|z|}$$

for some positive constant A. The desired result now follows by Theorem 5*.

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