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TAUBERIAN CONDITIONS FOR THE EQUIVALENCE OF WEIGHTED MEAN AND POWER SERIES METHODS OF SUMMABILITY

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1. Introduction. Suppose throughout that $\{p_n\}$ is a sequence of non-negative numbers with $p_0>0$, that

$$P_n=\sum_{k=0}^{n} p_k \to \infty,$$

and that $\{s_n\}$ is a sequence of real numbers. Let

$$p(x)=\sum_{k=0}^{\infty} p_k x^k, \qquad P(x)=\sum_{k=0}^{\infty} P_k x^k,$$

$$t_n=\frac{1}{P_n}\sum_{k=0}^{n} p_k s_k, \qquad \sigma(x)=\frac{1}{p(x)}\sum_{k=0}^{\infty} p_k s_k x^k,$$

and suppose that

(1) $$p(x)<\infty \quad \text{for} \quad 0<x<1.$$

Then

(2) $$(1-x)P(x)=p(x) \quad \text{for} \quad 0<x<1.$$

The weighted mean summability method M_p and the power series method J_p are defined as follows:

$s_n \to s(M_p)$ if $t_n \to s$,

$s_n \to s(J_p)$ if $\sigma(x)$ is convergent for $0<x<1$ and $\sigma(x)\to s$ as $x\to 1-$.

Both methods are known to be regular (see [3, pp. 57, 81]). It is also known (see [4]) that $s_n \to s(M_p)$ implies $s_n \to s(J_p)$.

The purpose of this paper is to establish results concerning Tauberian conditions sufficient for $s_n \to s(J_p)$ to imply $s_n \to s(M_p)$. In §2 we prove the following two theorems:

THEOREM 1. *Let* $s_n \to s(J_p)$, *let* $s_n > -H$ *for* $n=0,1,\ldots,$ *where* H *is a*

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constant; and let $p(x)$ *satisfy either*

(3) $$\lim_{x\to 1-}\frac{p(x^2)}{p(x)}=1$$

or

(4) $\lim_{x\to 1-}\frac{p(x^{m+1})}{p(x)}=\mu_m>0$ *for* $m=0,1,\ldots,$ *where* $\{\mu_m\}$ *is totally monotone.*

Then $s_n\to s(M_p)$.

THEOREM 2. *Let*

(5) $$np_n=o(P_n),$$

let $s_n\to s(J_p)$ *and let* $s_n>-\gamma_n$, *where* $\gamma_n\geq 0$ *for* $n=0,1,\ldots,$ *and*

(6) $$np_n\gamma_n=O(P_n).$$

Then $s_n\to s(M_p)$.

Note that (1) is a consequence of (5) and (2), since (5) implies that $P_{n-1}/P_n\to 1$.

The Abel-type method A_α $(\alpha>-1)$ is the J_p method given by $p(x)=(1-x)^{-\alpha-1}$ (see [1] and [2]) which satisfies (4) with $\mu_m=(m+1)^{-\alpha-1}$. Theorem 1 thus yields a Tauberian result for A_α. The case $\alpha=0$ of this result, which is well-known (see [3, Theorem 13] and [6, Theorem $2(A_\alpha)$]), states that

if $s_n\to s(A)$ and $s_n>-H$ for $n=0,1,\ldots,$ then $s_n\to s(C,1)$,

A being the standard Abel method and $(C,1)$ the Cesàro method of order 1.

The logarithmic methods L and l are respectively the methods J_p and M_p given by $p_n=(n+1)^{-1}$. Since $p_n=(n+1)^{-1}$, $\gamma_n=-\mu\log(n+1)$ satisfy the conditions of Theorem 2, we get as a corollary of that theorem a result proved by Kochanovski [5], namely

if $s_n\to s(L)$ and $s_n>-\mu\log(n+1)$ for $n=0,1,\ldots,$ then $s_n\to s(l)$.

In §3 we prove two theorems which set out simple conditions sufficient for (3) or (4) to hold.

2. Proofs of Theorems 1 and 2. We introduce some additional notation. Let

$$\phi(x)=\begin{cases}\dfrac{1}{x} & \text{for } c\leq x\leq 1,\\ 0 & \text{otherwise,}\end{cases}$$

where $0<c<1$, and let

$$\psi(x)=\frac{1}{p(x)}\sum_{k=0}^{\infty}p_ks_kx^k\phi(x^k).$$

Observe that

(7) $$-\frac{c}{1-c}+\frac{x}{1-c}\leq\phi(x)\leq 1+\frac{1}{c}-\frac{x}{c}\quad\text{for}\quad 0\leq x\leq 1.$$

Proof of Theorem 1. Suppose without loss in generality that $H=0$, i.e., that $s_n\geq 0$ for $n=0,1,\ldots.$

CASE 1. Suppose (3) holds. Then, by (7),

$$\limsup_{x\to 1-}\psi(x)\leq\left(1+\frac{1}{c}\right)\lim_{x\to 1-}\sigma(x)-\frac{1}{c}\lim_{x\to 1-}\frac{p(x^2)}{p(x)}\sigma(x^2)=\left(1+\frac{1}{c}\right)s-\frac{s}{c}=s;$$

and similarly $\liminf_{x\to 1-}\psi(x)\geq s$. It follows that $\lim_{x\to 1-}\psi(x)=s$, and therefore that

$$\psi(c^{1/n})=\frac{1}{p(c^{1/n})}\sum_{k=0}^{n}p_ks_k\to s.$$

Taking $s_k=1$ for $k=0,1,\ldots,$ we obtain

(8) $$\frac{P_n}{p(c^{1/n})}\to 1.$$

Consequently $t_n\to s$.

CASE 2. Suppose (4) holds. Then (see [3, Theorem 207]),

$$\mu_m=\int_0^1 t^m\,d\chi(t)\quad\text{for}\quad m=0,1,\ldots,$$

where the function χ is non-decreasing and bounded on $[0,1]$. Further, since $\mu_1>0$, we can choose $c\in(0,1)$ to be such that χ is continuous at c and

$$\alpha=\int_c^1\frac{d\chi(t)}{t}>0.$$

Then, for $m=0,1,\ldots,$

$$\frac{1}{p(x)}\sum_{k=0}^{\infty}p_ks_kx^kx^{mk}=\frac{p(x^{m+1})}{p(x)}\sigma(x^{m+1})\to\mu_ms\quad\text{as}\quad x\to 1-;$$

and so, for any polynomial $a(x)=a_0+a_1x+\cdots+a_mx^m$,

$$\frac{1}{p(x)}\sum_{k=0}^{\infty}p_ks_kx^ka(x^k)\to(a_0\mu_0+a_1\mu_1+\cdots+a_m\mu_m)s$$

$$=s\int_0^1 a(t)\,d\chi(t)\quad\text{as}\quad x\to 1-.$$

Since χ is continuous at c it is readily demonstrated that given $\varepsilon>0$, there are polynomials $a(x)$, $b(x)$ such that

$$a(x)\le\phi(x)\le b(x) \quad \text{for} \quad 0\le x\le 1 \quad \text{and} \quad \int_0^1 (b(t)-a(t))\,d\chi(t)<\varepsilon.$$

It follows that

$$\lim_{x\to 1-} \psi(x)=s\int_0^1 \phi(t)\,d\chi(t)=s\int_c^1 \frac{d\chi(t)}{t}=s\alpha.$$

Hence

$$\psi(c^{1/n})=\frac{1}{p(c^{1/n})}\sum_{k=0}^{n} p_k s_k \to s\alpha$$

and, taking $s_k=1$ for $k=0,1,\ldots,$

$$\frac{P_n}{p(c^{1/n})}\to\alpha.$$

Thus $t_n\to s$.

This completes the proof of Theorem 1.

Proof of Theorem 2. First we note that, for $0<x<1$, $m\ge 1$ we have, by (5) and (2),

$$\begin{aligned}0<p(x)-p(x^{m+1})&=\sum_{k=0}^{\infty} p_k x^k(1-x^{km})\le m(1-x)\sum_{k=0}^{\infty} kp_kx^k\\&=o((1-x)P(x))\\&=o(p(x)) \quad \text{as} \quad x\to 1-.\end{aligned}$$

Since $p(x)\to\infty$ as $x\to 1-$, it follows that

(9) $$\lim_{x\to 1-}\frac{p(x^{m+1})}{p(x)}=1 \quad \text{for} \quad m\ge 1.$$

Further, by (7), we have that, for $0<x<1$,

$$\begin{aligned}\psi(x)&=\frac{1}{p(x)}\sum_{k=0}^{\infty} p_k(s_k+\gamma_k)x^k\phi(x^k)-\frac{1}{p(x)}\sum_{k=0}^{\infty} p_k\gamma_kx^k\phi(x^k)\\&\le\Big(1+\frac{1}{c}\Big)\sigma(x)-\frac{1}{c}\,\sigma(x^2)\,\frac{p(x^2)}{p(x)}+\frac{1}{c(1-c)p(x)}\sum_{k=0}^{\infty} p_k\gamma_kx^k(1-x^k)\\&\le\Big(1+\frac{1}{c}\Big)\sigma(x)-\frac{1}{c}\,\sigma(x^2)\,\frac{p(x^2)}{p(x)}+\frac{1-x}{c(1-c)p(x)}\sum_{k=0}^{\infty} kp_k\gamma_kx^k.\end{aligned}$$

Therefore, by (2), (6), and (9), there is a constant M such that

$$\limsup_{x\to 1-}\psi(x)\le\Big(1+\frac{1}{c}\Big)s-\frac{s}{c}+M=s+M<\infty.$$

Similarly

$$\liminf_{x\to 1-}\psi(x)>-\infty;$$

and hence $\psi(x)=O(1)$ for $0<x<1$. It follows that

$$\psi(c^{1/n})=\frac{1}{p(c^{1/n})}\sum_{k=0}^{n} p_ks_k=O(1),$$

and hence, since (8) is a consequence of (9), that

(10) $$t_n=O(1);$$

Since

$$(1-x)\sum_{k=0}^{\infty} P_kt_kx^k=\sum_{k=0}^{\infty} p_ks_kx^k \quad \text{for} \quad 0<x<1,$$

we have, by (2), that

(11) $$\sigma(s)=\frac{1}{P(x)}\sum_{k=0}^{\infty} P_kt_kx^k\to s \quad \text{as} \quad x\to 1-.$$

Next, by (2) and (9),

(12) $$\lim_{x\to 1-}\frac{P(x^{m+1})}{P(x)}=\lim_{x\to 1-}\frac{p(x^{m+1})}{p(x)}\,\frac{1-x}{1-x^{m+1}}=\frac{1}{m+1} \quad \text{for } m=0,1,\ldots.$$

It follows from (10), (11) and (12), by Case 2 of Theorem 1, that

(13) $$u_n=\frac{1}{Q_n}\sum_{k=0}^{n} P_kt_k\to s, \quad \text{where} \quad Q_n=\sum_{k=0}^{n} P_k.$$

Further, by (5), (6) and (10), we have that, for $n\ge 1$,

$$t_n-t_{n-1}=s_n\frac{p_n}{P_n}-t_{n-1}\frac{p_n}{P_n}>-\frac{\gamma_np_n}{P_n}-t_{n-1}\frac{p_n}{P_n}>-\frac{\gamma}{n}$$

for some positive constant γ. Thus, for $m>n>1$,

$$t_m-t_n\ge-\gamma\sum_{k=n+1}^{m}\frac{1}{k}\ge-\gamma\log\frac{m}{n},$$

and so

(14) $$\liminf(t_m-t_n)\ge 0 \quad \text{when} \quad m>n\to\infty \quad \text{and} \quad \frac{m}{n}\to 1.$$

Now, by (5),

$$nP_n-(n-1)P_{n-1}=P_n+(n-1)p_n\sim P_n$$

and therefore $nP_n\sim Q_n$.

It follows that, for $m>n(1+\delta)$, $\delta>0$,

$$(15)\quad \frac{Q_m}{Q_n}=\frac{1}{Q_n}\sum_{k=n+1}^{m}P_k+1\geq\frac{P_n}{Q_n}(m-n)+1\geq\frac{\delta nP_n}{Q_n}+1\to 1+\delta \quad \text{as} \quad n\to\infty.$$

Suppose without loss in generality that $s=0$, i.e., $u_n\to 0$. It follows from (14) that, given $\varepsilon>0$, there are positive numbers n_0, δ such that $t_m-t_n>-\varepsilon$ when $m>n>n_0$ and $(m/n)<1+2\delta$. Consequently if m, n satisfy these conditions we have, by (13), that

$$(t_n-\varepsilon)\sum_{k=n+1}^{m}P_k\leq\sum_{k=n+1}^{m}P_kt_k=u_mQ_m-u_nQ_n\leq(t_m+\varepsilon)\sum_{k=n+1}^{m}P_k$$

and hence that

$$(16)\quad t_n-\varepsilon\leq\frac{u_mQ_m-u_nQ_n}{Q_m-Q_n}=u_m+\frac{u_m-u_n}{(Q_m/Q_n)-1}\leq t_m+\varepsilon.$$

Letting $m,n\to\infty$ subject to $1+\delta<(m/n)<1+2\delta$, it follows from (15) that

$$\frac{1}{(Q_m/Q_n)-1}=O(1),$$

and hence from (16) that

$$\limsup t_n\leq\varepsilon \quad \text{and} \quad \liminf t_m\geq-\varepsilon.$$

Therefore $t_n\to 0$.

This completes the proof of Theorem 2.

REMARK. A trivial modification of the proof of Theorem 1, and of the part of the proof of Theorem 2 up to and including (10), shows that the theorems remain valid if in each the hypothesis "$s_n\to s(J_p)$" is replaced by "$\sigma(x)=O(1)$ for $0<x<1$" and the conclusion "$s_n\to s(M_p)$" by "$t_n=O(1)$".

3. Other theorems. The following two theorems give simple conditions sufficient for (3) or (4) to hold.

THEOREM 3. *Let*

$$(17)\quad P_n\sim P_{n+1}.$$

(i) *If*

$$(18)\quad P_n\sim P_{2n},$$

then (3) *holds.*

(ii) *If*

$$(19)\quad \lim_{n\to\infty}\frac{P_n}{P_{nm}}=\mu_{m-1}>0 \quad \textit{for } m=1,2,\ldots, \textit{ where } \{\mu_m\} \textit{ is totally montone,}$$

then (4) *holds.*

THEOREM 4. *Let* $p_n>0$ *for* $n=0,1,\ldots,$ *and let*

$$(20)\quad p_n\sim p_{n+1}.$$

(i) *If*

$$(21)\quad p_n\sim 2p_{2n},$$

then (3) *holds.*

(ii) *If*

$$(22)\quad \lim_{n\to\infty}\frac{p_n}{p_{nm}}=m\mu_{m-1}>0 \quad \textit{for } m=1,2,\ldots, \textit{ where } \{\mu_m\} \textit{ is totally monotone,}$$

then (4) *holds.*

Proof of Theorem 3. We shall prove Part (ii). By (17) and (19), we have that, when $x\to 1-$,

$$P(x^m)=\sum_{n=0}^{\infty}\frac{P_n}{P_{nm}}P_{nm}x^{nm}\sim\mu_{m-1}\sum_{n=0}^{\infty}P_{nm}x^{nm}$$

$$\sim\frac{\mu_{m-1}}{m}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}P_{nm+k}x^{nm+k}=\frac{\mu_{m-1}}{m}P(x).$$

Hence, by (2),

$$\lim_{x\to 1-}\frac{p(x^m)}{p(x)}=\lim_{x\to 1-}\frac{P(x^m)}{P(x)}\cdot\frac{1-x^m}{1-x}=\mu_{m-1}.$$

This establishes Part (ii). The proof of Part (i) is similar but simpler.

Theorem 4 can be proved in the same way, or by first establishing the following simple implications: (20) $\Rightarrow$ (17); (20) and (21) $\Rightarrow$ (18); (20) and (22) $\Rightarrow$ (19).

Added December 15, 1980. It has been brought to the author's attention that Case 1 of Theorem 1 appears as Theorem 6 in a paper by B. Kwee, "On generalized logarithmic methods of summation", J. Math. Anal. Appl. 35 (1971), 83–89. His proof is somewhat more complicated than the one herein and should be corrected by the replacement of certain instances of "lim" by "lim sup" or "lim inf".

REFERENCES

1. A. Amir (Jakimovski), Some relations between the methods of summability of Abel, Borel, Cesàro, Hölder and Hausdorff, *J. d'Analyse Math.*, **3** (1954), 346–381.

2. D. Borwein, On a scale of Abel-type summability methods, *Proc. Cambridge Phil. Soc.*, **53** (1957), 318–322.

3. G. H. Hardy, *Divergent Series*, Oxford (1949).

4. K. Ishiguro, A Tauberian theorem for (J,p_n) summability, *Proc. Japan Acad.*, **40** (1964), 807–812.

5. A. P. Kochanovski, A condition for the equivalence of logarithmic methods of summability, *Ukrainian Math. J.*, **27** (1975), 229–234.

6. C. T. Rajagopal, On Tauberian theorems for some standard methods of summability, *J. Indian Math. Soc.* **39** (1975), 69–82.

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