MATRIX TRANSFORMATIONS OF WEAKLY MULTIPLICATIVE SEQUENCES OF RANDOM VARIABLES

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1. *Introduction*

Suppose throughout that $\{X_n\}$ $(n = 0, 1, \ldots)$ is a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, and that $\{a_{nk}\}$ $(n, k = 0, 1, \ldots)$ is a (summability) matrix satisfying

$$\sum_{k=0}^{\infty} |a_{nk}| < \infty \quad \text{for} \quad n = 0, 1, 2, \ldots . \tag{1}$$

Let

$$b_{i_1 i_2 \ldots i_n} = E(X_{i_1} X_{i_2} \ldots X_{i_n}),$$

$$B_n(q) = \sum_{0 \leqslant i_1 < i_2 < \ldots < i_n} |b_{i_1 i_2 \ldots i_n}|^q ,$$

where the summation is extended to all integers $i_1, i_2, \ldots, i_n$ satisfying $0 \leqslant i_1 < i_2 < \ldots < i_n$. Let

$$\sigma_n(p) = \left(\sum_{k=0}^{\infty} |a_{nk}|^p \right)^{1/(p-1)},$$

and let

$$T_n = \sum_{k=0}^{\infty} a_{nk} X_k .$$

The primary object of this paper is to establish the following two theorems concerning the almost sure convergence to zero of the sequence $\{T_n\}$.

THEOREM 1. *Let* $1 < p \leqslant 2, \dfrac{1}{p} + \dfrac{1}{q} = 1, 0 < M < \infty$, *let r be an even positive integer, and let*

$$EX_n^r \leqslant M \quad \textit{for} \quad n = 0, 1, \ldots , \tag{2}$$

$$B_r(q) < \infty , \tag{3}$$

$$\sum_{n=0}^{\infty} \sigma_n(p)^{r/q} < \infty . \tag{4}$$

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Then

$$E \sum_{n=0}^{\infty} T_n^r < \infty$$

and, in particular, $T_n \to 0$ a.s.

THEOREM 2. *Let* $1 < p \leqslant 2, \frac{1}{p} + \frac{1}{q} = 1, 0 < M < \infty$, *and let*

$$|X_n| \leqslant M \text{ a.s.} \quad for \quad n = 0, 1, \ldots, \tag{5}$$

$$B_n(q)^{1/n} \leqslant M \quad for \quad n = 1, 2, \ldots, \tag{6}$$

$$\sum_{n=0}^{\infty} e^{-\varepsilon/\sigma_n(p)} < \infty \quad for\ every \quad \varepsilon > 0. \tag{7}$$

Then

$$\sum_{n=0}^{\infty} P[|T_n| > \varepsilon] < \infty \quad for\ every \quad \varepsilon > 0$$

and, in particular, $T_n \to 0$ a.s.

In Theorem 2 the conditions on the sequence $\{X_n\}$ are more restrictive whereas the conditions on the matrix $\{a_{nk}\}$ are less restrictive than in Theorem 1. It is easily demonstrated (see Hill [4]) that condition (7) is implied by either condition (4) or by

$$\sigma_n(p) < \infty \quad \text{for} \quad n = 0, 1, \ldots \quad \text{and} \quad \sigma_n(p) \log n \to 0 \quad \text{as} \quad n \to \infty. \tag{8}$$

Evidently condition (4) becomes less restrictive as p decreases. In §5 it is shown that condition (7) does likewise provided that

$$\sup_{n \geqslant 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$

The sequence $\{X_n\}$ is said to be *multiplicative* if $b_{i_1 i_2 \ldots i_n} = 0$ whenever $0 \leqslant i_1 < i_2 \ldots < i_n$; in particular, it is multiplicative if it is independent with expectation $EX_n = 0$ for $n = 0, 1, \ldots$. The sequence is said to be *weakly multiplicative* if it satisfies condition (3) for some pair of positive numbers r, q.

Hill [3, 4] proved Theorems 1 and 2 for the special case in which $p = q = 2$, $\{X_n\}$ is the sequence of Rademacher functions on $\Omega = [0, 1]$ and P is Lebesgue measure. Azuma [1] proved Theorem 2 for the special case in which $p = q = 2$ and $\{X_n\}$ is multiplicative. Other results of a similar nature appear in Chapter 4 of Stout's book [8].

In §6 it is shown that when P is Lebesgue measure on $\Omega = [0, 1]$ then the sequence $\{\cos(k_n x + \alpha)\}$ on Ω satisfies condition (6) for every $q \geqslant 1$ provided $c > 2$ and $k_{n+1} \geqslant ck_n > 0$ for $n = 0, 1, \ldots$. The sequence is known to be multiplicative if $k_n/(2\pi)$ is an integer and $k_{n+1} \geqslant 2k_n > 0$ for $n = 0, 1, \ldots$.

In §7 it is shown that the standard Cesàro and Euler summability matrices satisfy condition (4) for certain pairs of positive numbers r, p.

2. *Preliminary results*

Two lemmas are required.

LEMMA 1. *Let* $1 < p \leqslant 2$, *let* r *be an even positive integer, let* $\{a_k\}$ *be a sequence of real numbers and let* $\{X_k\}$ *satisfy conditions* (2) *and* (3). *Then*

$$E\left(\sum_{k=0}^{m} a_k X_k\right)^r \leqslant K\left(\sum_{k=0}^{m} |a_k|^p\right)^{r/p} \quad for \quad m = 0, 1, \ldots,$$

where K *is a positive number independent of* $\{a_k\}, \{X_k\}$ *and* m.

This result is due to Móricz [6]; his proof is based on an inequality established by Gapoškin [2]. The case in which $p = 2$ of the following lemma is also due to Móricz [7]; our proof is modelled on his.

LEMMA 2. *Let* $1 < p \leqslant 2, u > 0$, *let* $\{a_k\}$ *be a sequence of real numbers, let* $\{X_k\}$ *satisfy conditions* (5) *and* (6), *and let*

$$t_m = \sum_{k=0}^{m} |a_k|^p, \quad S_m = \sum_{k=0}^{m} a_k X_k.$$

Then

$$Ee^{uS_m} \leqslant Ce^{cu^p t_m} \quad for \quad m = 0, 1, \ldots$$

where C, c *are positive numbers independent of* $\{a_k\}, \{X_k\}, u$ *and* m.

Proof. Let $B_n = B_n(q)$ and let

$$\beta > B = \limsup_{n \to \infty} B_n^{1/n},$$

the finiteness of B being ensured by condition (6).

Because of the convexity of e^{vx} we have, for every real v and $-1 \leqslant x \leqslant 1$, that $e^{vx} \leqslant \cosh v(1 + x \tanh v)$. Thus

$$Ee^{uS_m} = E \prod_{k=0}^{m} \exp(uMa_k X_k/M)$$

$$\leqslant \prod_{k=0}^{m} \cosh uMa_k \cdot E \prod_{k=0}^{m} (1 + \delta_k X_k)$$

where $\delta_k = \frac{1}{M} \tanh uMa_k$.

Next, since $\cosh t \leqslant e^{t^2/2} \leqslant e^{t^2}, \cosh t \leqslant e^{|t|}$, and $1 < p \leqslant 2$, we have that $\cosh t \leqslant e^{|t|^p}$ and so $\prod_{k=0}^{m} \cosh uMa_k \leqslant \prod_{k=0}^{m} \exp(u^p M^p |a_k|^p) = \exp(u^p M^p t_m)$. Further, by Hölder's inequality,

$$E\prod_{k=0}^{m}(1+\delta_k X_k) = 1+\sum_{j=1}^{m}\sum_{0\leqslant i_1<i_2<\ldots<i_j\leqslant n}\delta_{i_1}\delta_{i_2}\ldots\delta_{i_j}b_{i_1 i_2\ldots i_j}$$

$$\leqslant\left(1+\sum_{j=1}^{m}\beta^{jp/q}\sum_{0\leqslant i_1<i_2<\ldots<i_j\leqslant n}|\delta_{i_1}\delta_{i_2}\ldots\delta_{i_j}|^p\right)^{1/p}$$

$$\times\left(1+\sum_{j=1}^{m}\frac{1}{\beta^j}\sum_{0\leqslant i_1<i_2<\ldots<i_j\leqslant n}|b_{i_1 i_2\ldots i_j}|^q\right)^{1/q}.$$

Since $1+t \leqslant e^t$ and $\tanh t \leqslant t$ when $t \geqslant 0$, it follows that

$$E\prod_{k=0}^{m}(1+\delta_k X_k) \leqslant \prod_{k=0}^{m}\left(1+\beta^{p/q}|\delta_k|^p\right)^{1/p}\cdot\left(1+\sum_{j=1}^{m}B_j/\beta^j\right)^{1/q}$$

$$\leqslant C\exp\left(\frac{1}{p}\beta^{p/q}\sum_{k=0}^{m}|\delta_k|^p\right) \leqslant C\exp\left(\frac{1}{p}\beta^{p/q}u^p t_m\right)$$

where $C = \left(1+\sum_{j=1}^{\infty}B_j/\beta^j\right)^{1/q} < \infty$.

Collecting inequalities we arrive at the desired result, namely

$$Ee^{uS_m} \leqslant C\exp\left(\frac{1}{p}\beta^{p/q}u^p t_m\right)\exp(u^p M^p t_m) = Ce^{cu^p t_m}$$

where $c = M^p + \dfrac{\beta^{p/q}}{p}$.

3. *Proof of Theorem 1*

Let

$$T_{nm} = \sum_{k=0}^{m} a_{nk} X_k. \tag{9}$$

By Hölder's inequality and conditions (1) and (2), we have that

$$E\left(\sum_{k=0}^{\infty}|a_{nk}X_k|\right)^r \leqslant E\sum_{k=0}^{\infty}|a_{nk}|\,X_k^r\cdot\left(\sum_{k=0}^{\infty}|a_{nk}|\right)^{r-1}$$

$$\leqslant M\left(\sum_{k=0}^{\infty}|a_{nk}|\right)^r < \infty.$$

It follows that, for $n = 0, 1, \ldots, \sum_{k=0}^{\infty} a_{nk}X_k$ converges a.s. and so

$$\lim_{m\to\infty} T_{nm} = T_n \text{ a.s.}$$

Hence, by Fatou's Lemma and Lemma 1,

$$ET_n^r = E\liminf_{m\to\infty} T_{nm}^r \leqslant \liminf_{m\to\infty} ET_{nm}^r$$

$$\leqslant \liminf_{m\to\infty} K\left(\sum_{k=0}^{m}|a_{nk}|^p\right)^{r/p} \leqslant K\sigma_n^{r/q}.$$

Consequently, by condition (4),

$$E\sum_{n=0}^{\infty}T_n^r \leqslant K\sum_{n=0}^{\infty}\sigma_n^{r/q} < \infty,$$

as desired.

4. *Proof of Theorem 2*

Let u, ε be positive numbers. Then, for T_{nm} given by (9), we have, by Lemma 2, that

$$Ee^{uT_{nm}} \leqslant C\exp(cu^p\sigma_n^{p-1}),$$

where C, c are positive constants and $\sigma_n = \sigma_n(p)$. In view of conditions (1) and (5),

$$\lim_{m\to\infty} T_{nm} = T_n \text{ a.s.},$$

and hence, by Fatou's Lemma,

$$Ee^{uT_n} = E\liminf_{m\to\infty} e^{uT_{nm}} \leqslant \liminf_{m\to\infty} Ee^{uT_{nm}} \leqslant C\exp(cu^p\sigma_n^{p-1}).$$

Consequently, by Chebyshev's inequality,

$$P[|T_n| > \varepsilon] = P[T_n > \varepsilon] + P[T_n < -\varepsilon] \leqslant e^{-u\varepsilon}(Ee^{uT_n} + Ee^{-uT_n})$$

$$\leqslant 2C\exp(cu^p\sigma_n^{p-1} - u\varepsilon).$$

It follows, on taking $u = (\varepsilon/pc\sigma_n^{p-1})^{q/p}$, that

$$P[|T_n| > \varepsilon] \leqslant 2C\exp\left(\frac{-\varepsilon^q}{q(pc)^{q/p}\sigma_n}\right),$$

and hence, by condition (7), that

$$\sum_{n=0}^{\infty} P[|T_n| > \varepsilon] < \infty ,$$

as desired. Since ε is an arbitrary positive number this implies that $T_n \to 0$ a.s. by a corollary of the Borel–Cantelli Lemma (see [**8**; Theorem 2.1.1.]).

5. *Variation in strength of condition* (~~4~~ 7)

We shall prove the following.

PROPOSITION 1. *Let $v > w > 1$, and let*

$$\sup_{n \geqslant 0} \sum_{k=0}^{\infty} |a_{nk}| = M < \infty .$$

Then $\sigma_n(w) \leqslant \sigma_n(v) M^{\delta}$ for $n = 0, 1, \ldots,$ and $\delta = \dfrac{1}{w-1} - \dfrac{1}{v-1}$, hence condition (~~4~~ 7) *holds with $p = w$ whenever it holds with $p = v$.*

Proof. Let $\mu = \dfrac{v-1}{w-1}$, $\dfrac{1}{\lambda} + \dfrac{1}{\mu} = 1$. Then, by Hölder's inequality,

$$\sigma_n(w)^{w-1} = \sum_{k=0}^{\infty} |a_{nk}|^{w-1} |a_{nk}| \leqslant \left(\sum_{k=0}^{\infty} |a_{nk}|^{v-1} |a_{nk}| \right)^{1/\mu} \left(\sum_{k=0}^{\infty} |a_{nk}| \right)^{1/\lambda}$$

$$\leqslant \sigma_n(v)^{w-1} M^{1/\lambda},$$

and the desired inequality follows.

6. *A weakly multiplicative sequence*

Let P be Lebesgue measure in $\Omega = [0, 1]$, let α be any real number and let

$$X_n = \cos(k_n x + \alpha) \quad \text{for} \quad x \in \Omega, \quad n = 0, 1, \ldots .$$

We shall prove the following.

PROPOSITION 2. *Let $c > 2$ and let $k_{n+1} \geqslant c k_n > 0$ for $n = 0, 1, \ldots .$ Then the sequence $\{X_n\}$ satisfies condition* (6) *for every $q \geqslant 1$.*

Proof. By induction we have that

$$k_{n+1} \geqslant k_n + k_{n-1} + \ldots + k_0 + k_0(c-1)^n \quad \text{for} \quad n = 0, 1, \ldots .$$

Let $0 \leqslant i_1 < i_2 \ldots < i_n = m$. Then

$$|b_{i_1 i_2 \ldots i_n}| = \left| E \prod_{r=1}^{n} X_{i_r} \right| = \frac{1}{2^n} \left| E \prod_{r=1}^{n} \left(\exp(ik_{i_r} x + i\alpha) + \exp(-ik_{i_r} x - i\alpha) \right) \right|$$

$$= \frac{1}{2^n} \left| E \sum_{\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \ldots, \varepsilon_n = \pm 1} \exp\left(ix(\varepsilon_1 k_{i_1} + \varepsilon_2 k_{i_2} + \ldots + \varepsilon_n k_{i_n})\right) \right|$$

$$\leqslant \frac{2^n}{2^n} \frac{2}{k_m - k_{m-1} - k_{m-2} - \ldots - k_0} \leqslant \frac{2}{k_0 (c-1)^m} .$$

Hence

$$B_{nm} = \sum_{0 \leqslant i_1 < i_2 < \ldots < i_n = m} |b_{i_1 i_2 \ldots i_n}| \leqslant \binom{m}{n-1} \frac{2}{k_0 (c-1)^m} ,$$

and so, for $0 < t < c-2$,

$$\sum_{n=1}^{\infty} B_n(1) t^n = \sum_{n=1}^{\infty} t^n \sum_{m=n-1}^{\infty} B_{nm} \leqslant \frac{2}{k_0} \sum_{m=0}^{\infty} \frac{1}{(c-1)^m} \sum_{n=1}^{m+1} \binom{m}{n-1} t^n$$

$$= \frac{2}{k_0} \sum_{m=0}^{\infty} \left(\frac{1+t}{c-1} \right)^m < \infty .$$

Therefore $\sup_{n \geqslant 1} B_n(1)^{1/n} < \infty$, and since $B_n(q)^{1/q} \leqslant B_n(1)$ for $q \geqslant 1$, it follows that $\sup_{n \geqslant 1} B_n(q)^{1/n} < \infty$, as desired.

Incidentally, it is evident from the above argument with $c = 2$ that if $k_n/(2\pi)$ is an integer and $k_{n+1} \geqslant k_n > 0$ for $n = 0, 1, \ldots,$ then $b_{i_1 i_2 \ldots i_n} = 0$ whenever $0 \leqslant i_1 < i_2 < \ldots < i_n$, that is $\{X_n\}$ is multiplicative.

7. *Applications of Theorem* 1

(a) *The Cesàro matrix C_α* ($\alpha > 0$). This is the triangular matrix $\{a_{nk}\}$ given by

$$a_{nk} = \binom{n-k+\alpha-1}{\alpha-1} \Big/ \binom{n+\alpha}{\alpha} \quad \text{for} \quad 0 \leqslant k \leqslant n; \quad a_{nk} = 0 \quad \text{for} \quad k > n .$$

We shall prove the following result.

THEOREM 4. *Let $r > q > 1/\alpha$ where r is an even integer and $q \geqslant 2$, and let $\{X_n\}$ satisfy conditions* (2) *and* (3). *Then $X_n \to 0(C_\alpha)$ a.s.*

Proof. It is familiar that, for $\mu > -1$,

$$\binom{n+\mu}{\mu} \sim \frac{n^\mu}{\Gamma(\mu+1)} \quad \text{as} \quad n \to \infty .$$

Hence, for $\frac{1}{p}+\frac{1}{q}=1$,

$$\sigma_n(p)^{p-1} = \frac{1}{\binom{n+\alpha}{\alpha}^p}\sum_{k=0}^{n}\binom{k+\alpha-1}{\alpha-1}^p$$
$$= O(n^{p(\alpha-1)+1-p\alpha}) = O(n^{1-p}) \quad \text{as} \quad n\to\infty,$$

since $p(\alpha-1) > -1$. Therefore $\sigma_n(p)^{r/q} = O(n^{-r/q})$ as $n\to\infty$, and so condition (4) is satisfied.

It follows, by Theorem 1, that

$$T_n = \sum_{k=0}^{n} a_{nk}X_k \to 0 \text{ a.s.},$$

that is $X_n \to 0(C_\alpha)$ a.s.

(b) *The Euler matrix* E_α $(0<\alpha<1)$. This is the triangular matrix $\{a_{nk}\}$ given by

$$a_{nk} = \binom{n}{k}\alpha^k(1-\alpha)^{n-k} \quad \text{for} \quad 0\leqslant k\leqslant n; \quad a_{nk}=0 \quad \text{for} \quad k>n.$$

We shall prove the following result.

THEOREM 5. *Let $r>2q\geqslant 4$ where r is an even integer, and let $\{X_n\}$ satisfy conditions* (2) *and* (3). *Then $X_n\to 0(E_\alpha)$ a.s.*

Proof. It is known [9; p. 57] that $n^{1/2}a_{nk}\leqslant M_\alpha$ for $0\leqslant k\leqslant n$, where M_α is a positive number independent of k and n. Hence, for $\frac{1}{p}+\frac{1}{q}=1, n\geqslant 1$,

$$\sigma_n(p)^{p-1} = \sum_{k=0}^{n}|a_{nk}|^p \leqslant M_\alpha n^{-(p-1)/2}\sum_{k=0}^{n}a_{nk} = M_\alpha n^{-(p-1)/2},$$

and so $\sigma_n(p)^{r/q} = O(n^{-r/2q})$ as $n\to\infty$. Condition (4) is thus satisfied and consequently, by Theorem 1, $X_n\to 0(E_\alpha)$ a.s.

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