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Generalized Hausdorff Matrices as Bounded Operators on l^p *

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1. Introduction

For $p \geqq 1$ let l^p be the normed linear space of all complex sequences $x=\{x_n\}$ with norm

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

Let $B(l^p)$ be the normed linear space of all bounded linear operators on l^p into l^p, so that a matrix $A=(a_{nk}) \in B(l^p)$ if and only if, for every $x \in l^p$, $y_n=(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$ is defined for $n=0,1,\ldots$ and $y=\{y_n\} \in l^p$. The norm $\|A\|_p$ of a matrix $A \in B(l^p)$ is given by

$$\|A\|_p = \sup_{\|x\|_p \leqq 1} \|Ax\|_p.$$

Weighted mean matrices. Let $a=\{a_n\}$ be a sequence of positive numbers and let $A_n = \sum_{k=0}^{n} a_k$. The *weighted mean* matrix $M_a=(c_{nk})$ is defined by

$$c_{nk} = \frac{a_k}{A_n} \quad \text{for } 0 \leqq k \leqq n; \qquad c_{nk}=0 \quad \text{for } k>n.$$

The following theorem is due to Cartlidge [3].

Theorem A. *If* $p \geqq 1$, $p>c>0$ *and*

$$\frac{A_{n+1}}{a_{n+1}} \leqq c + \frac{A_n}{a_n} \quad \textit{for } n=s, s+1, \ldots,$$

then $M_a \in B(l^p)$ *and, when* $s=0$, $\|M_a\|_p \leqq \dfrac{p}{p-c}$.

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The primary object of this paper is to extend Theorem A to generalized Hausdorff matrices.

Generalized Hausdorff Matrices. Suppose in all that follows that $\lambda=\{\lambda_n\}$ is a sequence of real numbers with $\lambda_0 \geqq 0$, $\lambda_n>0$ for $n\geqq 1$, and that α is a function of bounded variation on $[0,1]$. For $0\leqq k\leqq n$, let

$$\lambda_{nk}(t)=-\lambda_{k+1}\dots\lambda_n\frac{1}{2\pi i}\int_C\frac{t^z\,dz}{(\lambda_k-z)\dots(\lambda_n-z)},\qquad 0<t\leqq 1,$$
$$\lambda_{nk}(0)=\lambda_{nk}(0+),$$

C being a positively sensed closed Jordan contour enclosing $\lambda_k, \lambda_{k+1}, \dots, \lambda_n$. We observe the convention that products such as $\lambda_{k+1}\dots\lambda_n=1$ when $k=n$. Let

$$\lambda_{nk}=\int_0^1\lambda_{nk}(t)\,d\alpha(t)\quad\text{for } 0\leqq k\leqq n,\qquad \lambda_{nk}=0\quad\text{for } k>n, \tag{2}$$

and denote the triangular matrix (λ_{nk}) by $H(\lambda,\alpha)$. This is called a *generalized Hausdorff matrix* (see [2]). We shall prove the following theorem.

Theorem 1. *If* $p\geqq 1$, $c>0$ *and*

$$\lambda_{n+1}\leqq c+\lambda_n\quad \textit{for } n=s, s+1, \dots, \tag{3}$$

and if $\int_0^1 t^{-c/p}\,|d\alpha(t)|<\infty$, *then*

$$H(\lambda,\alpha)\in B(l^p)\quad \textit{and}\quad \|H(\lambda,\alpha)\|_p\leqq \mu^{1/p}\int_0^1 t^{-c/p}\,|d\alpha(t)|$$

where

$$\mu=\begin{cases}1 & \textit{when } s=0\\ \max\limits_{0\leqq k\leqq n\leqq s}\dfrac{\lambda_{k+1}\dots\lambda_n}{(\lambda_k+c)\dots(\lambda_{n-1}+c)} & \textit{when } s\geqq 1.\end{cases}$$

Hardy [4] established this theorem for ordinary Hausdorff matrices, i.e., $\lambda_n=n$, and showed that in this case, if α is non-decreasing, then $\|H(\lambda,\alpha)\|_p=\int_0^1 t^{-1/p}\,d\alpha(t)$. Jakimovski, Rhoades and Tzimbalario [5] extended Hardy's results to the case $\lambda_n=n+a$, $a>0$.

A "generalized weighted Hausdorff" matrix $W=(w_{nk})$ is defined by

$$w_{00}=\lambda_{00},\qquad w_{nk}=\lambda_{nk}(\lambda_k/\lambda_n)^{1/p}\quad \text{for } n\geqq 1;$$

and $\hat{W}$ is defined to be the matrix $(|w_{nk}|)$. Borwein and Jakimovski [2] proved that *if* $p\geqq 1$,

$$0\leqq\lambda_0<\lambda_1<\dots<\lambda_n,\qquad \lambda_n\to\infty,\qquad \sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty,$$

and if (2) *holds with a normalized,* i.e., $\alpha(0)=0$ *and* $2\alpha(t)=\alpha(t+)+\alpha(t-)$ *for* $0<t<1$, *then* $W, \hat{W}\in B(l^p)$, $\|W\|_p \leqq \|\hat{W}\|_p$ *and*

$$\int_0^1 |d\alpha(t)| - |\alpha(0+)| \leqq \|\hat{W}\|_p \leqq \int_0^1 |d\alpha(t)|.$$

Let

$$D_0=(1+\lambda_0)\,d_0=1, \qquad D_n=\left(1+\frac{1}{\lambda_1}\right)\cdots\left(1+\frac{1}{\lambda_n}\right)=(1+\lambda_n)\,d_n \quad \text{for } n\geqq 1. \tag{4}$$

Then

$$D_n=\lambda_{n+1}d_{n+1}=\frac{\lambda_0}{1+\lambda_0}+\sum_{k=0}^{n} d_k \quad \text{for } n\geqq 0. \tag{5}$$

It is known (see [2]) that

$$0\leqq \lambda_{nj}(t)\leqq \sum_{k=0}^{n}\lambda_{nk}(t)\leqq 1 \quad \text{for } 0\leqq t\leqq 1,\ 0\leqq j\leqq n, \tag{6}$$

$$\int_0^1 \lambda_{nk}(t)\,dt=\frac{d_k}{D_n} \quad \text{for } 0\leqq k\leqq n. \tag{7}$$

When $\alpha(t)=t$ and $\lambda_0=0$, $H(\lambda,\alpha)$ reduces to the weighted mean matrix M_d with $d=\{d_n\}$ given by (4). Conversely if $d=\{d_n\}$ is a sequence of positive numbers with $d_0=1$, then (4) yields a sequence $\lambda=\{\lambda_n\}$ such that $H(\lambda,\alpha)$ becomes M_d when $\alpha(t)=t$. These observations together with (7) show that Theorem A is a special case of Theorem 1.

2. Preliminary Results

Lemma 1. *Let* $\lambda_{00}^*=\lambda_{00}$, $\lambda_{nk}^*=\lambda_{nk}\lambda_k/\lambda_n$ *for* $n\geqq 1$. *Then, for* $m\geqq n\geqq 0$,

$$\sum_{k=n}^{m}\lambda_{mk}=\sum_{k=n}^{m}\lambda_{kn}^*. \tag{8}$$

Proof. It follows easily from (1) and (2) that, for $m\geqq k\geqq 0$,

$$\lambda_{m+1,k}-\lambda_{mk}=(\lambda_{m+1,k}\lambda_k-\lambda_{m+1,k+1}\lambda_{k+1})/\lambda_{m+1}.$$

We proceed by induction on m. Clearly (8) holds for $m=n$. Assume (8) holds for some $m\geqq n$. Then

$$\begin{aligned}\sum_{k=n}^{m+1}\lambda_{m+1,k}-\sum_{k=n}^{m+1}\lambda_{kn}^* &= \sum_{k=n}^{m}(\lambda_{m+1,k}-\lambda_{mk})+\lambda_{m+1,m+1}-\lambda_{m+1,n}^*\\ &=\frac{1}{\lambda_{m+1}}\sum_{k=n}^{m}(\lambda_{m+1,k}\lambda_k-\lambda_{m+1,k+1}\lambda_{k+1})+\lambda_{m+1,m+1}-\lambda_{m+1,n}^*\\ &=\lambda_{m+1,n}\lambda_n/\lambda_{m+1}-\lambda_{m+1,m+1}+\lambda_{m+1,m+1}-\lambda_{m+1,n}^*\\ &=0.\end{aligned}$$

Thus (8) holds with $m+1$ in place of m. This completes the proof.

Lemma 2. *Let* $\lambda_{00}^*(t)=\lambda_{00}(t)$, $\lambda_{nk}^*(t)=\lambda_{nk}(t)\,\lambda_k/\lambda_n$ *for* $n\geqq 1$. *Then*

$$\sum_{k=n}^{\infty}\lambda_{kn}^*(t)\leqq 1 \qquad \textit{for}\ \ 0\leqq t\leqq 1,\ n\geqq 0.$$

Proof. By Lemma 1 and (6), we have that, for $m\geqq n\geqq 0$, $0\leqq t\leqq 1$,

$$\sum_{k=n}^{m}\lambda_{kn}^*(t)=\sum_{k=n}^{m}\lambda_{mk}(t)\leqq 1.$$

The desired result follows.

3. Proof of Theorem 1

Let $0<t\leqq 1$, and let

$$w_n=w_n(t)=\sum_{k=0}^{n}\lambda_{nk}(t)\,x_k \tag{9}$$

where $x=\{x_n\}\in l^p$. Then, by Hölder's inequality and (6),

$$|w_n|^p\leqq\sum_{k=0}^{n}\lambda_{nk}(t)\,|x_k|^p\left(\sum_{k=0}^{n}\lambda_{nk}(t)\right)^{p-1}\leqq\sum_{k=0}^{n}\lambda_{nk}(t)\,|x_k|^p$$

and so

$$\sum_{n=0}^{\infty}|w_n|^p\leqq\sum_{k=0}^{\infty}|x_k|^p\sum_{n=k}^{\infty}\lambda_{nk}(t). \tag{10}$$

Let $\tilde{\lambda}_n=\lambda_n+c$ and define $\tilde{\lambda}_{nk}(t)$ by (1) with $\{\tilde{\lambda}_n\}$ in place of $\{\lambda_n\}$. Since $\lambda_{k+1}\ldots\lambda_n\leqq\mu\,\tilde{\lambda}_k\ldots\tilde{\lambda}_{n-1}$ for $0\leqq k\leqq n$ by (3), it follows from (1) that

$$\lambda_{nk}(t)\,t^c=\mu\,\tilde{\lambda}_{nk}(t)\,\tilde{\lambda}_k/\tilde{\lambda}_n \qquad \text{for}\ \ n\geqq k.$$

Hence, by Lemma 2,

$$\sum_{n=k}^{\infty}\lambda_{nk}(t)\,t^c\leqq\mu,$$

and so, by (10),

$$\sum_{n=0}^{\infty}|w_n|^p\leqq\mu\,t^{-c}\sum_{k=0}^{\infty}|x_k|^p. \tag{11}$$

Now let

$$y_n=\sum_{k=0}^{n}\lambda_{nk}\,x_k.$$

Then, by (2) and (9),

$$y_n=\int_0^1 w_n(t)\,d\alpha(t). \tag{12}$$

It follows from (11) and (12), by a form of Minkowski's inequality, that

$$\left(\sum_{n=0}^{\infty}|y_n|^p\right)^{1/p} \leqq \int_0^1 \left(\sum_{n=0}^{\infty}|w_n|^p\right)^{1/p} |d\alpha(t)|$$
$$\leqq \mu^{1/p}\int_0^1 t^{-c/p}\,|d\alpha(t)| \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p},$$

i.e., $\|y\|_p \leqq \|x\|_p \mu^{1/p}\int_0^1 t^{-c/p}\,|d\alpha(t)|$.

This completes the proof of Theorem 1.

4. A Subsidiary Theorem

Theorem 2. *If* $p>1$, $d_{n+1}\geqq d_n$ *for* $n\geqq s$, *and* $\int_0^1 t^{-1/p}\,|d\alpha(t)|<\infty$, *then* $H(\lambda,\alpha)\in B(l^p)$.

Proof. By (4) and (5), we have that, for $n\geqq s$,

$$\lambda_{n+1}-\lambda_n=\frac{D_{n+1}}{d_{n+1}}-\frac{D_n}{d_n}=D_n\left(\frac{1}{d_{n+1}}-\frac{1}{d_n}\right)+1\leqq 1.$$

The desired result is now an immediate consequence of Theorem 1. Cartlidge [3] proved the special case $\alpha(t)=t$ (i.e., $H(\lambda,\alpha)=M_d$) of Theorem 2.

References

1. Borwein, D., Jakimovski, A.: Matrix operators on l^p, Rocky Mountain J. Math. **9**, 463–477 (1979)
2. Borwein, D., Jakimovski, A.: Generalization of the Hausdorff moment problem. Canad. J. Math. **33**, 946–960 (1981)
3. Cartlidge, J.M.: Weighted mean matrices as operators on l^p. Ph.D. thesis, Indiana University 1978
4. Hardy, G.H.: An inequality for Hausdorff means. J. London Math. Soc. **18**, 46–50 (1943)
5. Jakimovski, A., Rhoades, B.E., Tzimbalario, J.: Hausdorff matrices as bounded operators over l^p. Math. Z. **138**, 173–181 (1974)

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