*Acta Math. Hung.*
**45**(1—2) (1985), 151—158.

# CONDITIONS FOR INCLUSION BETWEEN NÖRLUND SUMMABILITY METHODS

D. BORWEIN and B. THORPE (London, Canada)[1]

## 1. Introduction

Let $p=\{p_n\}_{n\geqq 0}$ denote a sequence of complex numbers, let $P_n=\sum_{k=0}^{n} p_k$ and let $p(z)=\sum_{n=0}^{\infty} p_n z^n$. A sequence $\{s_n\}_{n\geqq 0}$ is Nörlund summable $(N,p)$ to $l$ if $P_n\neq 0$ for $n\geqq 0$ and $\lim_{n\to\infty} \sum_{\nu=0}^{n} p_{n-\nu}s_\nu/P_n=l$. We use the same notation with other letters in place of $p$, $P$. It is well known that necessary and sufficient conditions for $(N,p)$ to be regular (i.e., finite limit preserving) are

$$\text{(a)}\quad \sum_{\nu=0}^{n} |p_\nu| = O(|P_n|) \quad\text{and}\quad \text{(b)}\quad p_n=o(P_n),$$

cf. Theorem 16 of [2] where Hardy considers the special case $p_n\geqq 0$ so that (a) is automatically satisfied. In this paper we make a contribution to the solution of an open problem raised by Theorem 19 of [2] and mentioned explicitly on page 91 of [2]. In particular, we consider the question whether the condition $\sum_{\nu=0}^{n} |k_\nu|=O(|Q_n|)$ alone is necessary and sufficient for $(N,p)$ to imply $(N,q)$ when $P_n=O(1)$, $|Q_n|\to\infty$, both $(N,p)$ and $(N,q)$ are regular, the sequence $\{k_n\}_{n\geqq 0}$ being obtained from the generating function $k(z)=q(z)/p(z)$. We can solve the problem completely for $p(z)$ a polynomial, and for a wide class of functions $p(z)$ with algebraic and logarithmic singularities on $|z|=1$, but the general case leads to delicate questions that escape our analysis.

## 2. The main problem

In Theorem 19 of [2], under the hypotheses that $(N,p)$ and $(N,q)$ are both regular, Hardy shows that the two conditions

$$\text{(A)}\qquad \sum_{\nu=0}^{n} |k_{n-\nu}P_\nu| = O(|Q_n|),$$

$$\text{(B)}\qquad k_n=o(Q_n),$$

[1] Supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant A-2983.

are necessary and sufficient for $(N, p)$ to imply $(N, q)$[2]. Following his argument (for the case $p_n \geqq 0$, $q_n \geqq 0$) it is not difficult to verify that (B) may be omitted in the cases (i) $|P_n| \to \infty$, (ii) $P_n = O(1)$ and $Q_n = O(1)$. In the remaining case, $P_n = O(1)$ and $|Q_n| \to \infty$, it is natural to conjecture that (A) alone is necessary and sufficient for $(N, p)$ to imply $(N, q)$. To deal with this problem we consider regular Nörlund methods $(N, p)$ with $P_n = O(1)$. It is easy to see from the regularity conditions that this is equivalant to considering sequences $\{p_n\}$ with $\sum_{n=0}^{\infty} |p_n| < \infty$, $p(1) \neq 0$ and $P_n \neq 0$ for $n \geqq 0$.

Given $\sum_{n=0}^{\infty} |p_n| < \infty$, $p_0 \neq 0$ and $p(1) \neq 0$, the little Nörlund method $(Z, p)$ is defined as follows:

$$s_n \to l(Z, p) \quad \text{if} \quad \lim_{n \to \infty} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu = l p(1).$$

This method is regular, and equivalent to $(N, p)$ when $(N, p)$ is regular and $P_n = O(1)$. In this case (A) is equivalent to

$$\text{(C)} \qquad \sum_{\nu=0}^{n} |k_\nu| = O(|Q_n|)$$

provided $(N, q)$ is regular. A simple direct argument shows that, provided $(Z, p)$ is defined and $(N, q)$ is regular, (B) and (C) are necessary and sufficient for $(Z, p)$ to imply $(N, q)$.

In Section 3 we prove that the conjecture is true when $p(z)$ has no zeros on $|z| = 1$, and in Sections 4 and 5 we investigate what happens when $p(z)$ has zeros on $|z| = 1$ and when $(N, q)$ is the Cesàro method $(C, \alpha)$ respectively.

## 3. The case $p(z) \neq 0$ for $|z| = 1$

Before considering this case we show that (C) does imply that (B) holds in the (C, $\delta$) sense for every $\delta > 0$. In fact we prove slightly more.

THEOREM 1. *Suppose that $(Z, p)$ is defined, $(N, q)$ is regular and*

$$(1) \qquad k_n = O(|Q_n|).$$

*Then*

$$\frac{k_n}{Q_n} \to 0 \quad (Z, p).$$

PROOF. Consider the identity

$$\sum_{\nu=0}^{n} p_{n-\nu} \frac{k_\nu}{Q_\nu} = \sum_{\nu=0}^{n} p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}} = \frac{q_n}{Q_n} + \sum_{\nu=0}^{n} p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}} \left(1 - \frac{Q_{n-\nu}}{Q_n}\right).$$

[2] Since Hardy only considers Nörlund methods with $p_n \geqq 0$, $q_n \geqq 0$ his conditions have to be modified in the obvious way.

The first term on the right-hand side tends to 0 by the regularity of $(N, q)$. By the Weierstrass $M$-test, the series on the right-hand side is absolutely and uniformly convergent with respect to $n$ since

$$\left|p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}}\left(1-\frac{Q_{n-\nu}}{Q_n}\right)\right| \leqq M|p_\nu|\ ^3$$

by (1) and the regularity of $(N, q)$, and so the second term on the right-hand side tends to 0 (by taking the limit as $n\to\infty$ inside the sum). This completes the proof.

COROLLARY. *Under the hypotheses of Theorem* 1,

$$\frac{k_n}{Q_n} \to 0 \quad (C, \delta)$$

*for every* $\delta>0$.

PROOF. Let $t_n=\sum_{\nu=0}^{n} p_{n-\nu}s_\nu$ where $s_\nu=k_\nu/Q_\nu$. Then, by (1), $s(z)=\sum_{n=0}^{\infty} s_n z^n$ is analytic in $|z|<1$, and $(1-z)s(z)=(1-z)t(z)/p(z)\to 0$ as $z\to 1$ through real values in $|z|<1$, since $t_n\to 0$ and $p(1)\neq 0$. It follows that $s_n\to 0$ (Abel) and the result is now a consequence of Théorème VI' (sequence version) of [5] or Theorems 70 and 92 of [2].

We give an example to show that we cannot replace $\delta>0$ by $\delta=0$ in the corollary. Let $\{p_n\}$, $\{q_n\}$ be defined from the generating functions $p(z)=1+z$, $q(z)=(1-z^2)^{-1}$ so that $k(z)=[(1+z)(1-z^2)]^{-1}$. Then $Q(z)=(1-z)^{-1}q(z)$ and so $Q(-z)=k(z)$, i.e., $Q_n=(-1)^n k_n$. It is clear that the hypotheses of Theorem 1 hold, but that in this case $k_n/Q_n=(-1)^n\to 0$ $(C, \delta)$ for all $\delta>0$ whereas $k_n/Q_n\not\to 0$ as $n\to\infty$. We remark that this example does not satisfy (C) and so is not a counterexample to the conjecture.

If $p(z)$ has no zeros on $|z|=1$, we can use Theorem 1 together with the following tauberian result to establish the conjecture in this case.

THEOREM 2. *Let $(Z, p)$ be defined. Then $(Z, p)$ sums no bounded divergent sequence if and only if $p(z)\neq 0$ for $|z|=1$.*

PROOF. For the sufficiency of the condition we first observe that $p(z)$ has only a finite number of zeros in $|z|<1$ (otherwise they would accumulate on the boundary). Let these be at the points $z=z_i$ with multiplicity $\lambda_i$ $(i=1, 2, \dots, l)$. Then, by Theorem 1 of [7], we have that $s_n\to 0$ $(Z, p)$ if and only if $s_n=t_n+\sum_{i=1}^{l} f_i(n)z_i^{-n}$ where $\{t_n\}$ converges to 0 and $f_i(n)$ is a polynomial in $n$ of degree $(\lambda_i-1)$. By Lemma 2 of [8], $\{\sum_{i=1}^{l} f_i(n)z_i^{-n}\}_{n\geqq 0}$ is unbounded unless $f_i(n)\equiv 0$ $(i=1, 2, \dots, l)$. Hence the only sequences summable $(Z, p)$ are convergent or unbounded.

To prove the necessity of the condition suppose $p(\beta)=0$, $|\beta|=1$, $\beta\neq 1$. Since we are assuming $\sum_{n=0}^{\infty}|p_n|<\infty$, $p(z)=\sum_{n=0}^{\infty} p_n z^n$ converges for $|z|\leqq 1$ and so $p(\beta)=$

[3] We use $M$ to denote a positive constant, independent of the variables, that may be different at each occurrence.

$=\sum_{n=0}^{\infty} p_n\beta^n=0$. It is now easy to see that the bounded divergent sequence $\{\beta^{-n}\}$ is summable to 0 $(Z, p)$, and the result follows.

COROLLARY. *Suppose that* $(Z, p)$ *is defined,* $p(z)\neq 0$ *for* $|z|=1$, $(N, q)$ *is regular and* (C) *holds. Then* $(Z, p)$ *implies* $(N, q)$.

PROOF. By the remarks at the end of Section 2 it is sufficient to show that (B) holds. Since (C) implies that (1) holds, Theorem 1 gives that the bounded sequence $\{k_n/Q_n\}$ is summable $(Z, p)$ to 0, and Theorem 2 shows that it must converge to 0, i.e. (B) must hold.

## 4. The case where $p(z)$ may have zeros on $|z|=1$

A summability method based on a regular, normal (i.e., lower triangular with non-zero diagonal) sequence to sequence matrix $A=(a_{nk})$ is said to be perfect if $\sum_{n=\nu}^{\infty}\alpha_n a_{n\nu}=0$ $(\nu=0, 1, \ldots)$ together with $\sum_{n=0}^{\infty}|\alpha_n|<\infty$ implies $\alpha_n=0$ $(n=0, 1, \ldots)$. See [4] and [9] for some basic properties. For the methods $(N, p)$ and $(Z, p)$ we have $a_{n\nu}$ equal to $p_{n-\nu}/P_n$ and $p_{n-\nu}$ respectively. It is clear that neither $(N, p)$ nor $(Z, p)$ is perfect if $p(z)$ has a zero in $|z|<1$ (since, if $p(w)=0$ with $0<|w|<1$, then $\alpha_n=P_n w^n$ is a non-zero term of an absolutely convergent series that satisfies the conditions for perfectness of $(N, p)$, and likewise with $\alpha_n=w^n$ for $(Z, p)$). This observation also settles an undecided question mentioned on page 707 of [4]. Hill asks whether the Nörlund method $(N, p)$ with generating function $p(z)=(1+az)(1-z)^{-2}$ is perfect for $a>1$. Since $p(z)$ has a zero at $z=-1/a$ which is in $|z|<1$, $(N, p)$ cannot be perfect.

THEOREM 3. *Suppose that* $(Z, p)$ *is perfect,* $(N, q)$ *is regular and* (C) *holds. Then* $(Z, p)$ *implies* $(N, q)$.

PROOF. This follows directly from Theorem II. 8 of [9] with $(Z, p)=A$, $(N, q)=B$, and the observation that (C) is necessary and sufficient for every sequence summable to 0 $(Z, p)$ to be bounded $(N, q)$.

The remainder of this section is devoted to finding examples of perfect $(Z, p)$ methods. We introduce the notation $\{c_n\}$ for the coefficients of the generating function $c(z)=1/p(z)$. It follows from Theorem 8 of [4] that when $(Z, p)$ is defined then $c_n=O(1)$ is a sufficient condition for it to be perfect.

LEMMA 1. *If* $p(z)=\left(1-\frac{z}{\beta}\right)^{\lambda}$ *where* $\beta\neq 1$, $|\beta|=1$, $\lambda>0$, *then* $(Z, p)$ *is perfect.*

PROOF. We have $p_n=A_n^{-\lambda-1}\beta^{-n}$ where $A_n^{-\lambda-1}=\binom{n-\lambda-1}{n}$ is defined from the relation

$$(1-z)^{\lambda} = \sum_{n=0}^{\infty} A_n^{-\lambda-1}z^n, \tag{2}$$

so that $\sum_{n=0}^{\infty}|p_n|<\infty$, $p_0=1$ and $p(1)\neq 0$. Suppose that $\sum_{n=0}^{\infty}|\alpha_n|<\infty$ and $\sum_{n=\nu}^{\infty}\alpha_n p_{n-\nu}=0$

$(\nu=0, 1, \ldots)$. This can be written as

$$\sum_{n=\nu}^{\infty} \alpha_n A_{n-\nu}^{-\lambda-1} \beta^{\nu-n} = \beta^{\nu} \sum_{n=\nu}^{\infty} A_{n-\nu}^{-\lambda-1}(\alpha_n \beta^{-n}) = 0,$$

and using the notation for fractional differences (see [1]) this is equivalent to

$$\Delta^{\lambda}(\alpha_\nu \beta^{-\nu}) = 0 \quad (\nu = 0, 1, \ldots).$$

If $\lambda \in \mathbf{N}$, then an inductive argument (as on page 706 of [4]) shows that $\alpha_\nu=0$ $(\nu=0, 1, \ldots)$. If $\lambda \in (N, N+1)$ for $N \in \mathbf{N}$, then

$$\Delta^{N+1-\lambda}\big(\Delta^{\lambda}(\alpha_\nu \beta^{-\nu})\big) = \Delta^{N+1}(\alpha_\nu \beta^{-\nu}) = 0$$

by the absolute convergence of the double series involved, and so the result follows from the integer case. Thus $(Z, p)$ is perfect.

The following lemma is a special case of Theorem 5 of [4].

LEMMA 2. *If $(Z, m)$, $(Z, l)$ are perfect and $p(z)=m(z)l(z)$, then $(Z, p)$ is perfect.*

LEMMA 3. *If $\sum_{n=0}^{\infty} |r_n| < \infty$ and $r(z) \neq 0$ for $|z| \leqq 1$, then $(Z, r)$ is perfect.*

PROOF. By the Wiener—Levy theorem (page 246 of [12]), $1/r(z) = \sum_{n=0}^{\infty} t_n z^n$ where $\sum_{n=0}^{\infty} |t_n| < \infty$. Suppose $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\sum_{n=s}^{\infty} \alpha_n r_{n-s} = 0$ $(s=0, 1, \ldots)$. Then, for $\nu \geqq 0$,

$$0 = \sum_{s=\nu}^{\infty} t_{s-\nu} \sum_{n=s}^{\infty} \alpha_n r_{n-s} = \sum_{n=\nu}^{\infty} \alpha_n \sum_{s=\nu}^{n} r_{n-s} t_{s-\nu} = \alpha_\nu,$$

the interchange of order of summation being legitimate because the double series involved is absolutely convergent. Hence $(Z, r)$ is perfect.

As an immediate consequence of Lemmas 1 and 2 we see that, if $(Z, r)$ is perfect and $p(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{\beta_i}\right)^{\lambda_i} r(z)$ where $\beta_i \neq 1$, $|\beta_i| = 1$, $\lambda_i > 0$ $(i=0, 1, \ldots, n)$, then $(Z, p)$ is perfect. Thus Theorem 3 holds for such a $(Z, p)$ method.

LEMMA 4. *If $p(z) = \left(1 - \frac{z}{\beta}\right)^{\lambda} \left(-\frac{\beta}{z} \log\left(1 - \frac{z}{\beta}\right)\right)^{\mu}$ where $\beta \neq 1$, $|\beta| = 1$, $0 < \lambda < 1$ and $\mu \in \mathbf{R}$, then $(Z, p)$ is perfect.*

PROOF. If $\mu=0$, this is a case of Lemma 1. Suppose $\mu \neq 0$. Then we have

$$p_n \sim M n^{-\lambda-1} (\log n)^{\mu} \beta^{-n}$$

by page 93 of [6]. (Although Littlewood gives this formula only for $\lambda < 0$ we can establish the result in our case by using backward induction and the differential equation on page 93 of [6].) Hence $\sum_{n=0}^{\infty} |p_n| < \infty$, $p_0 = 1$ and $p(1) \neq 0$. Moreover, $c(z) =$

$=1/p(z)=\left(1-\frac{z}{\beta}\right)^{-\lambda}\left(-\frac{\beta}{z}\log\left(1-\frac{z}{\beta}\right)\right)^{-\mu}$, so that again by Littlewood's result

$$c_n \sim Mn^{\lambda-1}(\log n)^{-\mu}\beta^{-n}.$$

Hence $c_n=O(1)$, and so $(Z, p)$ is perfect by Theorem 8 of [4].

By using Lemma 2, we see that if $p(z)$ is any finite product of functions of the form of those in Lemmas 1 and 4, then $(Z, p)$ is perfect and Theorem 3 holds for such a $(Z, p)$ method. In view of the results above, it would be of interest to know whether every $(Z, p)$ method with $p(z)$ having no zeros inside the unit circle is perfect. A likely candidate for a counterexample can be obtained by considering generalized Laguerre polynomials. Let

$$p(z) = \left(1-\frac{z}{\lambda}\right)^{-\alpha-1} \exp\left(\frac{-z}{\lambda-z}\right) \quad \text{for} \quad \lambda \neq 1, \quad |\lambda| = 1, \quad \alpha \in \mathbf{R},$$

so that

$$p_n\lambda^n = L_n^{\alpha}(1) \sim Mn^{(\alpha/2)-(1/4)} \cos(2\sqrt{n}+\theta)$$

by (8.22.1) of [10]; where $\theta$ is a constant depending only on $\alpha$. Thus, if $\alpha<-3/2$, then $\sum_{n=0}^{\infty}|p_n|<\infty$, $p_0=1$ and $p(1)\neq 0$. However, in this case (8.22.3) of [10] gives

$$c_n\lambda^n = L_n^{-\alpha-2}(-1) \sim Mn^{-(\alpha/2)-(5/4)} \exp(2\sqrt{n}),$$

and this leads us to suspect that $(Z, p)$ need not be perfect but we are unable to prove it.

THEOREM 4. *Suppose that $(Z, r)$ is perfect and that*

$$p(z) = \prod_{j=1}^{m}\left(1-\frac{z}{\alpha_j}\right)^{\nu_j} \prod_{i=1}^{n}\left(1-\frac{z}{\beta_i}\right)^{\lambda_i}\left(-\frac{\beta_i}{z}\log\left(1-\frac{z}{\beta_i}\right)\right)^{\mu_i} r(z) \quad \textit{where} \quad |\alpha_j|<1, \quad \nu_j \in \mathbf{N}$$

$(j=1, 2, \ldots, m)$, $\beta_i\neq 1$, $|\beta_i|=1$, $\lambda_i>0$, $\mu_i\in\mathbf{R}$ $(i=1, 2, \ldots, n)$. *Suppose that $(N, q)$ is regular and that* (C) *holds. Then $(Z, p)$ implies $(N, q)$.*

Note that, by Lemma 3, sufficient conditions for $(Z, r)$ to be perfect are that $\sum_{n=0}^{\infty}|r_n|<\infty$ and that $r(z)\neq 0$ for $|z|\leqq 1$.

PROOF OF THEOREM 4. Let $s(z)=\prod_{j=1}^{m}\left(1-\frac{z}{\alpha_j}\right)^{\nu_j}$ and $t(z)=p(z)/s(z)$. Then $k(z)s(z)t(z)=q(z)$. Define $l(z)=k(z)s(z)$ so that $l(z)t(z)=q(z)$. By Lemmas 1, 2 and 4, $(Z, t)$ is perfect and

$$\sum_{\nu=0}^{n}|l_\nu| = \sum_{\nu=0}^{n}\left|\sum_{\mu=0}^{\nu}k_{\nu-\mu}s_\mu\right| \leqq \sum_{\mu=0}^{n}|s_\mu|\sum_{\nu=\mu}^{n}|k_{\nu-\mu}| = O(|Q_n|)$$

by (C). Thus, by Theorem 3, $(Z, t)$ implies $(N, q)$. Similarly, using the corollary to Theorem 2 in place of Theorem 3, we get that $(Z, s)$ implies $(N, q)$. Since $p(z)= =s(z)t(z)$, by Corollary 3 of [7], we see that $w_n\to 0\ (Z, p)$ if and only if $w_n=a_n+b_n$ where $a_n\to 0\ (Z, s)$ and $b_n\to 0\ (Z, t)$. Hence, by the above, it is easy to see that $(Z, p)$ implies $(N, q)$.

## 5. The case $(N, q)=(C, \alpha)$

Although we cannot settle the general case with an arbitrary regular $(N, q)$ method, consideration of the special case when $(N, q)$ is the Cesàro method $(C, \alpha)$ leads to some interesting questions on the summability of the power series $\sum_{n=0}^{\infty} c_n z^n$ on its circle of convergence. The Cesàro method $(C, \alpha)$ for $\alpha>-1$ is the Nörlund method $(N, q)$ with $q_n=A_n^{\alpha-1}$ where this is defined by (2). For $(N, q)$ to be regular and $Q_n\to\infty$ we have to consider $\alpha>0$. In this case $k(z)=(1-z)^{-\alpha}/p(z)=(1-z)^{-\alpha}c(z)$ so that $k_n=C_n^{\alpha-1}$ where we use the notation for Cesàro sums (see for example, page 96 of [2] with $c_n$ replacing $a_n$). For the question under consideration, Hardy's Theorem 19 becomes: if $(N,p)$ is regular, $P_n=O(1)$ and $\alpha>0$, then the conditions

(3) $$\sum_{\nu=0}^{n} |C_\nu^{\alpha-1}| = O(n^\alpha),$$

(4) $$C_\nu^{\alpha-1} = o(n^\alpha),$$

are necessary and sufficient for $(N,p)$ to imply $(C, \alpha)$ (where $p(z)c(z)=1$). The problem is to show that (4) follows from (3) and the other hypotheses.

THEOREM 5. *If $(N,p)$ is regular, $P_n=O(1)$, $\alpha>0$, then (3) is sufficient for $(N,p)$ to imply $(C, \alpha+\delta)$ for every $\delta>0$.*

PROOF. By the corollary to Theorem 1, $C_n^{\alpha-1}/A_n^\alpha\to 0$ $(C, \delta)$, i.e., $c_n\to 0$ $(C, \delta)\times\times(C, \alpha)$, the iterated Cesàro method, and by page 23 of [5] or Ch. 11 of [2] this is equivalent to $c_n\to 0$ $(C, \alpha+\delta)$, i.e., (4) with $\alpha$ replaced by $(\alpha+\delta)$. Also, (3) implies that (3) holds with $\alpha$ replaced by $(\alpha+\delta)$, since (3) is exactly the condition for the series $\sum_{n=0}^{\infty} c_n$ to be strongly bounded $[C, \alpha]_1$ (see page 488 of [11]). Hence, by Hardy's result, $(N,p)$ implies $(C, \alpha+\delta)$.

We are unable to decide whether we can take $\delta=0$ in Theorem 5. It is clear that (3) alone does not imply (4) (consider $C_n^{\alpha-1}=n^\alpha$ if $n=2^s$ $(s=0, 1, \ldots)$ and 0 otherwise) but we have been unable to construct an example with the $c_n$'s satisfying the further hypotheses that $c(z)p(z)=1$, $(N,p)$ regular and $P_n=O(1)$. We can, however, make the following simplification.

THEOREM 6. *If $(N,p)$ is regular, $P_n=O(1)$, $\alpha>0$, then (3) and*

(5) $$c_n = o(n^\alpha)$$

*are necessary and sufficient for $(N,p)$ to imply $(C, \alpha)$.*

PROOF. By the remarks before Theorem 5 it is enough to show that, under the other hypotheses of the theorem, (4) is equivalent so (5). Now (4) says that $c_n\to 0$ $(C, \alpha)$, and so by the limitation theorem for $(C, \alpha)$ (Theorem 46 of [2]) (5) must hold. Conversely, by the convergence of $\sum_{n=0}^{\infty} |p_n|$ and the regularity of $(N,p)$, we see that $p(z)$ is continuous at $z=1$ and $p(z)\to p(1)\neq 0$ as $z\to 1$ in any manner from within the unit circle. Also, (3) implies that $\sum_{n=0}^{\infty} c_n z^n$ is convergent for $|z|<1$ and, by the continuity of $\sum_{n=0}^{\infty} c_n z^n = 1/p(z)$ at $z=1$, we have that $\sum_{n=0}^{\infty} c_n z^n \to 1/p(1)$ as $z\to 1$ in any manner from within the unit circle. Hence, by a result of Dienes (cf. Théorème XXVI of [5] or Theorem 9.23 of [12]), (5) implies that $\sum_{n=0}^{\infty} c_n$ is summable $(C, \alpha)$. By the remarks at the bottom of page 102 of [2], $c_n \to 0$ $(C, \alpha)$, i.e., (4) holds, and this proves the result.

If we only require an implication from $(N, p)$ to Cesàro summability of some positive order then we have a more complete result, cf. [3].

THEOREM 7. *Suppose that $(N, p)$ is regular and $P_n = O(1)$. In order that $(N, p)$ should imply Cesàro summability of some positive order it is necessary and sufficient that $c_n = O(n^\gamma)$ for some $\gamma > 0$.*

PROOF. To show that the condition is necessary, suppose $(N, p)$ implies $(C, \alpha)$ for $\alpha > 0$. Then $\sum_{n=0}^{\infty} c_n = 1/p(1)$ $(N, p)$ and so $\sum_{n=0}^{\infty} c_n = 1/p(1)$ $(C, \alpha)$. Hence, by the limitation theorem for $(C, \alpha)$, $c_n = o(n^\alpha)$ and so the condition holds.

For the sufficiency part, $c_n = O(n^\gamma)$ implies that $\sum_{n=0}^{\infty} c_n z^n$ is convergent for $|z|<1$ and that $c_n = o(n^\delta)$ for $\delta > \gamma$. Hence, by Dienes' theorem, as in the proof of Theorem 6, $\sum_{n=0}^{\infty} c_n = 1/p(1)$ $(C, \delta)$. Thus, $c_n = o(n^\delta)$ and, by II of [11], $\sum_{n=0}^{\infty} c_n = 1/p(1)$ $[C, \delta+1]_1$, and so (3) and (5) hold with $\alpha$ replaced by $\delta + 1$. Therefore, by Theorem 6, $(N, p)$ implies $(C, \delta+1)$.

## References

[1] A. F. Andersen, Comparison theorems in the theory of Cesàro summability, *Proc, Lond. Math. Soc.*, **27** (1928), 39—71.
[2] G. H. Hardy, *Divergent Series* (Oxford, 1949).
[3] G. H. Hardy and J. E. Littlewood, Solution of the Cesàro summability problem for power series and Fourier series, *Math. Zeit.*, **19** (1924), 67—96.
[4] J. D. Hill, On perfect methods of summability, *Duke Math. J.*, **3** (1937), 702—714.
[5] E. Kogbetliantz, Sommation des séries et integrales divergentes par les moyennes arithmétiques et typiques, *Mémorial des sciences mathématiques*, No. 51 (Paris, 1931).
[6] J. E. Littlewood, *Lectures on the Theory of Functions* (Oxford, 1943).
[7] W. Miesner, The convergence fields of Nörlund means, *Proc. Lond. Math. Soc.*, **15** (1965), 495—507.
[8] G. Petersen, A note on divergent series, *Canad. J. Math.*, **4** (1952), 445—454.
[9] A. Peyerimhoff, *Lectures on Summability*, Springer Lecture Notes, No. 107 (Berlin).
[10] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publication, Vol. **23** (New York, 1939).
[11] C. E. Winn, On strong summability for any positive order, *Math. Zeit.*, **37** (1933), 481—492.
[12] A. Zygmund, *Trigonometric Series* (Cambridge, 1959).

*(Received January 3, 1983)*

THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO
CANADA N6A 5B7