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COMMUNICATION

ENUMERATION OF INJECTIVE PARTIAL TRANSFORMATIONS*

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1. Introduction

Let $\mathcal{R}_n$ denote the set of injective, partial self-maps on a set of n elements (this notation comes from [2]). If further, this set is linearly ordered, let $\mathcal{B}_n$ denote the subset of order decreasing injective partial self-maps. In this paper we compute the generating functions of $r_n = |\mathcal{R}_n|$ and $b_n = |\mathcal{B}_n|$. It turns out that these enumerative problems are closely related to Bell numbers, Sterling numbers of the second kind and Laguerre polynomials; known to many as some of the prettiest gems of combinatorial theory ([1] p. 67, 91, 116 and 232). It is initially very surprising that there should be such a direct correspondence between partitions and order decreasing injective partial maps.

The motivation for this problem originally came from [2], where it was made clear that we are (here) dealing with the most familiar interesting case in the recursive enumeration of generalized Bruhat cells on algebraic monoids. To be more specific, let $M = M_n(k)$ and let $\mathcal{R}_n$ be identified with the set of 0, 1 matrices with at most one nonzero entry in each row and column. Then there is exactly one element of $\mathcal{R}_n$ in each two-sided B-orbit on M (where B is the upper triangular group). $\mathcal{B}_n$ corresponds to the set of orbits of upper triangular matrices.

Notice that $\mathcal{R}_n$ is an inverse semigroup under composition of partial functions, usually referred to as the symmetric inverse semigroup on n letters. In fact, $\mathcal{R}_n$ plays the same role in inverse semigroup theory as does the symmetric group in group theory. It appears that there are many other infinite families of algebraic monoids which yield enumerative problems similar to the one solved here.

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2. Injective partial transformations

Let $r_n = |\mathcal{R}_n|$, the number of injective partial maps on the set of n elements. It is fairly easy to see that $r_n = \sum_{i=0}^{n} \binom{n}{i} n!/i!$ and this leads to one proof of the following proposition (see Remark 2.2). However, we first provide a direct enumerative proof.

Proposition 2.1.
(a) $r_0 = 1$, $r_1 = 2$.
(b) $r_n = 2nr_{n-1} - (n-1)^2 r_{n-2}$ for $n \geq 2$.

Proof. (a) is obvious, so consider (b). Let $\mathcal{R}_{ij}$ denote the set of injective, partial functions from a set of size i to a set of size j, and let $r_{ij} = |\mathcal{R}_{ij}|$. Note that $r_{ii} = r_i$. Let $n \geq 1$, and distinguish an element $x \in \underline{n+1}$, the set with $n+1$ elements. We claim that

$$r_{n+1} = r_n + (n+1)r_n + nr_{n-1,n}. \tag{1}$$

Indeed,

$$r_n = |\{f \in \mathcal{R}_{n+1} \mid x \notin \text{Domain}(f) \cup \text{Range}(f)\}|$$
$$(n+1)r_n = |\{f \in \mathcal{R}_{n+1} \mid x \in \text{Domain}(f)\}|, \text{ and}$$
$$nr_{n-1,n} = |\{f \in \mathcal{R}_{n+1} \mid x \in \text{Range}(f) \setminus \text{Domain}(f)\}|.$$

Similarly,

$$r_{n-1,n} = r_{n-1} + (n-1)r_{n-2,n-1} \tag{2}$$

since (choosing a distinguished element $y \in \underline{n}$) we have $r_{n-1} = |\{f \in \mathcal{R}_{n-1,n} \mid y \notin \text{Range}(f)\}|$ and $(n-1)r_{n-2,n-1} = |\{f \in \mathcal{R}_{n-1,n} \mid y \in \text{Range}(f)\}|$. So repeated application of (2) yields

$$nr_{n-1,n} = \sum_{i=1}^{n} \frac{n!}{(n-i)!} r_{n-i} = \sum_{i=0}^{n-1} \frac{n!}{i!} r_i. \tag{3}$$

Putting (1) and (3) together yields

$$r_{n+1} = (n+2)r_n + \sum_{i=0}^{n-1} \frac{n!}{i!} r_i \tag{4}$$

But then

$$r_{n+1} - (n+2)r_n = \sum_{i=0}^{n-1} \frac{n!}{i!} r_i$$
$$= nr_{n-1} + n \sum_{i=0}^{n-2} \frac{(n-1)!}{i!} r_i$$
$$= nr_{n-1} + n(r_n - (n+1)r_{n-1}).$$

The last equality follows from (4). Thus, $r_{n+1} = 2(n+1)r_n - n^2 r_{n-1}$. □

Remark 2.2. Alternately, we could let

$$c_n = \frac{1}{(n!)^2} r_n = \sum_{i=0}^{\infty} \frac{1}{(i!)^2 (n-i)!}.$$

Then

$$\sum_{n=0}^{\infty} c_n x^n = e^x \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},$$

so that

$$e^{-x} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

Then since

$$\frac{d}{dx}\left(x \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}\right)\right) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},$$

we obtain

$$e^{-x} \sum_{n=0}^{\infty} c_n x^n = \frac{d}{dx}\left(x \frac{d}{dx} e^{-x} \sum_{n=0}^{\infty} c_n x^n\right).$$

Expand this out, cancel the e^{-x} and equate coefficients to get the desired result.

We now compute the generating function of the r_n's by solving the appropriate differential equation.

Theorem 2.3. *Let $r(x) = \sum_{n=0}^{\infty} (r_n/n!)x^n$. Then $r(x)$ converges for $|x| < 1$ to the function $e^{x/(1-x)}/(1-x)$. Furthermore, $r(x)$ satisfies the differential equation*

$$\frac{r'(x)}{r(x)} = \frac{2-x}{(1-x)^2}.$$

Proof. For convergence we use the ratio test. If $a_n = r_{n+1}/(n+1)r_n$ then using 2.1(b), we get

$$a_{n-1} - a_n = \frac{n}{n+1}\left(\frac{a_{n-2} - a_{n-1}}{a_{n-1}a_{n-2}}\right) + \frac{1}{n(n+1)a_{n-2}},$$

whence it follows by induction that a_n is decreasing. But $a_n > 0$ for all n, so $a = \lim_{n\to\infty} a_n$ exists. By 2.1(b), we have $a = 2 - 1/a$ and so $a = 1$.

Again using 2.1(b), it follows that $(2-x)(d/dx)(xr(x)) = r'(x)$, and so $r'(x)/r(x) = (2-x)/(1-x)^2$. Thus, $r(x) = e^{x/(1-x)}/(1-x)$, since the initial conditions here are $r(0) = 1$ and $r'(0) = 2$. □

Remark. If ℓ_n denotes the nth Laguerre polynomial, then by the formula on page

116 of [1], $\sum_{n=0}^{\infty} \ell_n(1)/n! = e^{x/(1-x)}$. Thus, we obtain the curious formula

$$\frac{r_n}{n!} - \frac{r_{n-1}}{(n-1)!} = \ell_n(1).$$

3. Order decreasing, injective partial functions

For $n \geq 0$, let

$$\mathcal{B}_n = \{f \in \mathcal{R}_n \mid f(a) \leq a \text{ for all } a \in \text{Domain}(f)\}$$

and let $b_n = |\mathcal{B}_n|$. Also let $\mathcal{B}_{n,i} = \{f \in \mathcal{B}_n \mid |\text{Domain}(f)| = i\}$ and $b_{n,i} = |\mathcal{B}_{n,i}|$.

Proposition 3.1.
(a) $b_{n,0} = b_{n,n} = 1$ *for* $n \geq 0$.
(b) $b_0 = 1$ *and* $b_1 = 2$.
(c) $b_n = 2b_{n-1} + \sum_{i=0}^{n-2} (n-i-1)b_{n-1,i}$ *for* $n \geq 2$.
(d) $b_{n,i} = (n-i+1)b_{n-1,i-1} + b_{n-1,i}$ *for* $n > i > 0$.

Proof. (a) and (b) are obvious. So consider (c). Let $\underline{n}$ denote the linearly ordered set of n elements and let $x \in \underline{n}$ be the smallest element. Then

$$b_n = b_{n-1} + b_{n-1} + \gamma_{n-1} \tag{5}$$

where

$$\gamma_{n-1} = |\{f \in \mathcal{B}_n \mid f(y) = x \text{ for some } y \neq x\}|$$

and $b_{n-1} = |\{f \in \mathcal{B}_n \mid f(x) = x\}| = |\{f \in \mathcal{B}_n \mid x \notin \text{Range}(f)\}|$. Now by partitioning the set in the definition of γ_{n-1} by domain size, we see that

$$\gamma_{n-1} = \sum_{i=0}^{n-2} (n-i-1)b_{n-1,i}. \tag{6}$$

So (c) follows from (5) and (6).

For (d), simply notice that $b_{n,i} = b_{n-1,i-1} + b_{n-1,i} + (n-1)b_{n-1,i-1}$ since

$$b_{n-1,i-1} = |\{f \in \mathcal{B}_{n,i} \mid f(x) = x\}|,$$
$$b_{n-1,i} = |\{f \in \mathcal{B}_{n,i} \mid x \notin \text{Range}(f)\}, \text{ and}$$
$$(n-i)b_{n-1,i-1} = |(f \in \mathcal{B}_{n,i} \mid x = f(y) \text{ for some } y \neq x\}|. \quad \square$$

Proposition 3.2. *Define a sequence of polynomials $p_j(x)$ recursively, as follows:*

$$p_1(x) = x, \quad \text{and}$$
$$p_j(x) = (x+1)p_{j-1}(x) + p_{j-1}(x+1) \quad \text{for} \quad j \geq 2.$$

Then $b_n - 2b_{n-1} = \sum_{j=0}^{n-1} p_j(1)$.

Proof. Let $n \geq 2$ be given. We shall prove that

$$b_n - 2 = \sum_{j=1}^{i-1} p_j(1) + \sum_{k=1}^{n-i} p_i(k) b_{n-i,n-i-k} \tag{7}$$

for all $1 \leq i \leq n-1$. By 3.1(c), (7) holds for $i = 1$. Suppose now that (7) holds for some $1 \leq i \leq n-2$. Then by 3.1(d) we obtain

$$\begin{aligned} b_n - 2b_{n-1} &= \sum_{j=1}^{i-1} p_j(1) + p_i(n-1) \\ &\quad + \sum_{k=1}^{n-(i+1)} p_i(k)((k+1)b_{n-(i+1),n-(i+1)-k} + b_{n-(i+1),n-i-k}) \\ &= p_i(n-1) + \sum_{j=1}^{i-1} p_j(1) + p_i(1) - p_i(n-i) \\ &\quad + \sum_{k=1}^{n-(i+1)} ((k+1)p_i(k) + p_i(k+1))b_{n-(i+1),n-(i+1)-k} \\ &= \sum_{j=1}^{(i+1)-1} p_j(1) + \sum_{k=1}^{n-(i+1)} p_{i+1}(k) b_{n-(i+1),n-(i+1)-k} \end{aligned}$$

and so induction on i establishes (7) for all $1 \leq i \leq n-1$. When $i = n-1$, we obtain $b_n - 2b_{n-1} = \sum_{j=1}^{n-2} p_j(1) + p_{n-1}(1) = \sum_{j=1}^{n-1} p_j(1)$, as required. $\square$

Proposition 3.3. *Let* $\phi_x(t) = e^{e^t - 1} e^{t(x+1)}(e^t + x - 1)$ *and expand* ϕ_x *as*

$$\phi_x(t) = \sum_{n=1}^{\infty} \frac{q_n(x)}{(n-1)!} t^{n-1}$$

for appropriate functions q_n. *Then* $q_n(x) = p_n(x)$ *for all* $n \geq 1$.

Proof. By direct computation, $\phi_x'(t) = (x+1)\phi_x(t) + \phi_{x+1}(t)$. Hence,

$$\sum_{n=1}^{\infty} \frac{q_{n+1}(x)}{(n-1)!} t^{n-1} = (x+1) \sum_{n=1}^{\infty} \frac{q_n(x)}{(n-1)!} t^{n-1} + \sum_{n=1}^{\infty} \frac{q_n(x+1)}{(n-1)!} t^{n-1},$$

and so $q_1(x) = \phi_x(0) = x = p_1(x)$, and $q_{n+1}(x) = (x+1)q_n(x) + q_n(x+1)$. $\square$

Define $b(x) = \sum_{n=1}^{\infty} (b_n/(n+1)!)\, x^{n+1}$.

Theorem 3.4. $b(x) = e^{e^x - 1} - 1$.

Proof. From the definition of $p_n(x)$ for $n \geq 2$, we obtain that $p_n(0) = p_i(0) + \sum_{j=i}^{n-1} p_j(1)$ for each $1 \leq i \leq n-1$. For $i = 1$ we then obtain $p_n(0) = \sum_{j=1}^{n-1} p_j(1)$ and so by 3.2 we have

$$b_n = 2b_{n-1} + p_n(0) \text{ for } n \geq 2, \tag{8}$$

with $b_0 = 1$, $b_1 = 2$.

Let $c(x)=\sum_{n=0}^{\infty}(b_n/n!)x^n$. Then from 3.3 and (8), we get $c'(x)-2c(x)=e^x(e^x-1)e^{e^x-1}$. Solving this equation in the usual way we obtain $c(x)=e^{e^x+x-1}$. From this, our result follows easily. □

Note. By repeatedly differentiating $(1/e)\sum_{m=0}^{\infty}e^{mx}/m!-1$ we obtain the formula

$$b_n=\frac{1}{e}\sum_{m=1}^{\infty}\frac{m^{n+1}}{m!}$$

originally due to Kozniowski, in the context of Bell numbers (see 3.5 and 3.6).

Remark 3.5. Anyone familiar with partitions and Bell numbers may recognize $b(x)$ as the generating function $\sum_{n=0}^{\infty}(P_n/n!)x^n$ where p_n is the number of partitions of a set with n elements. Thus, perhaps unexpectedly, we get $p_{n+1}=b_n$.

We leave the reader with the following exercise: Find a canonical bijection between $\mathscr{B}_n$ and $\mathscr{P}(n+1)$, the set of partitions of a set of $n+1$ elements.

Remark 3.6. If we choose to evaluate b_n by means of the formula $b_n=\sum_{i=0}^{n}b_{n,i}$, it is convenient to define $\alpha_{n,i}=b_{n,n-i}$ for $0\leqslant i\leqslant n$. Then 3.1(a), (d) yields the recursive formulation

$$\alpha_{n,n}=\alpha_{n,0}=1 \quad \text{for} \quad n\geqslant 0, \quad \text{and}$$

$$\alpha_{n+1,i}=\alpha_{n,i-1}+(i+1)\alpha_{n,i} \quad \text{for} \quad 1\leqslant i\leqslant n$$

and, of course, we have

$$b_n=\sum_{i=0}^{n}\alpha_{n,i}.$$

Notice that the $\alpha_{n,i}$'s are the Sterling numbers of the second kind, which arise in the study of partitions (see 3.5). The following table records the first six cases.

$n \backslash j$	0	1	2	3	4	5	b_n
0	1						1
1	1	1					2
2	1	3	1				5
3	1	7	6	1			15
4	1	15	25	10	1		52
5	1	31	90	65	15	1	203

References

[1] M. Aigner, Combinatorial Theory, Grundlehren der Math. (Springer, 1979).

[2] L. Renner, Analogue of the Bruhat decomposition for algebraic monoids, J. Algebra 101 (1986) 303–338.