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On Relations between Weighted Mean and Power Series Methods of Summability

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1. INTRODUCTION

Suppose throughout that $\{p_n\}$ is a sequence of non-negative numbers with $p_0 > 0$, that

$$P_n := \sum_{k=0}^{n} p_k \to \infty,$$

and that

$$p(x) := \sum_{n=0}^{\infty} p_n x^n < \infty \qquad \text{for} \quad 0 < x < 1.$$

Let $\{s_n\}$ be a sequence of real numbers.

The weighted mean summability method M_p and the power series method J_p are defined as follows:

$s_n \to s(M_p)$ (and $\{s_n\}$ is said to be M_p-convergent) if

$$\frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \to s;$$

$s_n \to s(J_p)$ (and $\{s_n\}$ is said to be J_p-convergent) if $\sum_{n=0}^{\infty} p_n s_n x^n$ is convergent for $0 < x < 1$ and

$$\frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n \to s \qquad \text{as} \quad x \to 1-.$$

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It is known that both methods are regular (see [5, pp. 57, 80]), and (see [6]) that $s_n \to s(M_p)$ implies $s_n \to s(J_p)$. The following Tauberian theorem concerning the reverse implication is also known [3].

THEOREM T. *If $s_n \to s(J_p)$ and $s_n > -H$ for $n = 0, 1, ...,$ where H is a constant, and if*

$$\lim_{x \to 1-} \frac{p(x^m)}{p(x)} = \lambda_m > 0 \qquad \textit{for} \quad m = 2 \quad \textit{and} \quad m = 3, \tag{1}$$

then $s_n \to s(M_p)$.

It follows from Theorem 1.8 in [9] that the integers 2, 3 in (1) can be replaced by any pair of positive numbers $a, b \neq 1$ such that $\log_a b$ is irrational. It was proved in [3] that

$$\lim_{x \to 1-} \frac{p(x^2)}{p(x)} = \lambda \tag{2}$$

alone does not imply (1) when $0 < \lambda < 1$, though (1) and (2) are equivalent when $\lambda = 1$. In answer to a question raised in [3] we shall show that Theorem T does not remain valid when (1) is replaced by (2) with $0 \leqslant \lambda < 1$.

In Section 3 we construct, for each $\lambda \in (0, 1)$, a function $p(x)$ which satisfies (2) and a sequence of positive numbers $\{s_n\}$ which is J_p-convergent but not M_p-convergent.

In Section 5 we show that if $p_n := e^{g(n)}$, where $g(x)$ is a logarithmico-exponential function (see [4]) such that $g'(x) \to 0$ and $xg'(x) \to \infty$ as $x \to \infty$, then $p(x)$ satisfies (2) with $\lambda = 0$ (and consequently $\lim_{x \to 1-} (p(x^t)/p(x)) = 0$ for all $t \geqslant 2$), p_n increases faster than any power of n, and (cf. Lemma 2 in Section 2) $P_{n+1} \sim P_n \to \infty$, but the conditions $s_n \geqslant 0$ and $s_n \to s(J_p)$ do not imply that $s_n \to s(M_p)$. This result is a consequence of the fact that different limitation theorems hold for the two summability methods. The limitation theorem for the weighted mean method is well known. A limitation theorem for the power series method is derived in Section 4, while in Section 5 the asymptotic behaviour of the limitation order is determined for non-negative J_p-convergent sequences for the function p in question. The key to this analysis is Theorem A1, which deals with the asymptotic behaviour of certain Laplace transforms. Proofs of the asymptotic results are relatively straightforward and have been omitted.

2. Preliminary Results

Lemma 1. *Suppose* $\lim_{n\to\infty} (P_n/P_{mn}) = \lambda$, *where* m *is a positive integer. Then*

$$\lim_{x\to 1-} \frac{p(x^m)}{p(x)} = \lambda$$

provided either (i) $\lambda = 0$ *or* (ii) $P_{n+1} \sim P_n$.

Proof. Case (i). Let $P(x) := \sum_{n=0}^{\infty} P_n x^n$. Define a sequence $\{s_n\}$ by setting $s_n := P_k/P_n$ when $n = mk$, $k = 0, 1, ...$; $s_n := 0$ otherwise. Then $s_n \to 0$ and so

$$\frac{1}{P(x)} \sum_{n=0}^{\infty} P_n s_n x^n = \frac{P(x^m)}{P(x)} \to 0 \qquad \text{as} \quad x \to 1-.$$

Since $p(x) = (1-x)\, P(x)$ for $0 < x < 1$, it follows that

$$\frac{p(x^m)}{p(x)} = \frac{(1-x^m)\, P(x^m)}{(1-x)\, P(x)} \to 0 \qquad \text{as} \quad x \to 1-.$$

This completes the proof of Case (i). Case (ii) has been proved in essence in [2]. ∎

Lemma 2. *If* $p_n > 0$ *for* $n = 0, 1, ...$ *and the sequence* $\{P_{n+1}/P_n\}$ *is not convergent to* 1, *then the sequence* $\{p_{n+1}/p_n\}$ *is* J_p*-convergent to* 1 *but not* M_p*-convergent.*

Proof. Let $s_n := p_{n+1}/p_n$. Then

$$\frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n = \frac{p(x) - p_0}{xp(x)} \to 1 \qquad \text{as} \quad x \to 1-,$$

i.e., $s_n \to 1(J_p)$. On the other hand,

$$\frac{1}{P_n} \sum_{k=0}^{n} p_k s_k = \frac{P_{n+1} - P_0}{P_n}$$

does not converge to 1. Hence $\{s_n\}$ is not M_p-convergent, since $s_n \to s(M_p)$ implies $s_n \to s(J_p)$. ∎

3. Construction

For each $\lambda \in (0, 1)$ we shall construct a function $p(x)$ satisfying (2) such that the sequence $\{p_{n+1}/p_n\}$ is J_p-convergent but not M_p-convergent.

Let $\mu := 1/\lambda > 1$. Define a sequence $\{Q_n\}$ recursively by setting $Q_0 := 0$, $Q_1 := 1$ and

$$\frac{Q_{n+1}}{Q_n} := \begin{cases} \mu & \text{when} \quad n+1 = 2^k,\ k = 1, 2, ..., \\ 1 & \text{otherwise.} \end{cases}$$

Let $Q(x) := \sum_{n=0}^{\infty} Q_n x^n$ and $q(x) := (1-x)\,Q(x)$. Suppose that

$$2^k \leqslant n < 2^{k+1}.$$

Then it is easily verified that $Q_n = \mu^k$ so that

$$R_n := \sum_{r=0}^{n} Q_r = \sum_{r=0}^{k-1} (2\mu)^r + (n+1-2^k)\mu^k$$

$$= (2\mu)^k \left(2^{-k}n - 1 + \frac{1}{2\mu - 1} + o(1)\right),$$

and, since $2^{k+1} \leqslant 2n < 2^{k+2}$,

$$R_{2n} = (2\mu)^{k+1} \left(2^{-k}n - 1 + \frac{1}{2\mu - 1} + o(1)\right).$$

Since

$$0 < \frac{1}{2\mu - 1} \leqslant 2^{-k}n - 1 + \frac{1}{2\mu - 1} < \frac{2\mu}{2\mu - 1},$$

it follows that

$$\frac{R_n}{R_{2n}} \to \frac{1}{2\mu} = \frac{\lambda}{2} \qquad \text{and} \qquad \frac{R_{n-1}}{R_n} = 1 - \frac{Q_n}{R_n} \to 1.$$

Also $R_n \to \infty$ and $0 < Q(x) < \infty$ for $0 < x < 1$. Hence, by Lemma 1(ii),

$$\frac{Q(x^2)}{Q(x)} \to \frac{\lambda}{2} \qquad \text{as} \quad x \to 1-,$$

and consequently

$$\frac{q(x^2)}{q(x)} = (1+x)\,\frac{Q(x^2)}{Q(x)} \to \lambda \qquad \text{as} \quad x \to 1-.$$

Now define $p(x) := q(x) + e^x$, and note that $P_n \geqslant Q_n \to \infty$. Then $p(x)$ satisfies (2). Further $p_n > 0$ for $n = 0, 1, ...,$ and $\{P_{n+1}/P_n\}$ is not convergent since $\{Q_{n+1}/Q_n\}$ is not convergent. Hence, by Lemma 2, the sequence $\{p_{n+1}/p_n\}$ is J_p-convergent but not M_p-convergent.

Remark 1. It is easy to show (with the aid of Lemma 1(i)) that, if in the above construction we replace μ by μ^k in the definition of Q_{n+1}/Q_n, we obtain a function $p(x)$ satisfying (2) with $\lambda = 0$ for which $p_n > 0$ and $\{P_{n+1}/P_n\}$ is unbounded, so that $\{p_{n+1}/p_n\}$ is J_p-convergent but not M_p-convergent. However, the case for which (2) is satisfied with $\lambda = 0$ while $P_{n+1} \sim P_n$ is dealt with in Section 5.

4. Limitation Theorems

The following result is well known (see [5, p. 57] or [8, Theorem II.3]).

THEOREM L1. *If* $s_n \to 0(M_p)$, *then* $p_n s_n = o(P_n)$.

Next, we derive a limitation theorem for the J_p-method. We shall use the notation

$$\Delta_n := \inf_{0<t<1} p(t)t^{-n} \qquad \text{for} \quad n = 1, 2, \ldots.$$

LEMMA 3. *The sequence* $\{\Delta_n\}$ *has the following properties*:

(i) $\Delta_n \geqslant P_n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \Delta_n x^n$ *has radius of convergence* 1;

(iii) $\Delta_n = p(t_n)t_n^{-n}$ *for some* $t_n \in (0, 1)$ *such that*

$$t_m^{n-m} \leqslant \Delta_m/\Delta_n \leqslant t_n^{n-m} \qquad \textit{for} \quad m, n = 1, 2, \ldots;$$

(iv) *the sequences* $\{\Delta_n\}$, $\{\Delta_n/\Delta_{n+1}\}$, *and* $\{t_n\}$ *are non-decreasing with*

$$\lim_{n\to\infty} \Delta_n/\Delta_{n+1} = \lim_{n\to\infty} t_n = 1.$$

The proof of this lemma is straightforward.

THEOREM L2. (i) *If* $s_n \geqslant 0$ *for* $n = 0, 1, \ldots$ *and* $s_n \to 0(J_p)$, *then*

$$p_n s_n = o(\Delta_n).$$

(ii) *If* $\{\lambda_n\}$ *is any sequence of positive numbers converging to* 0, *then there is a sequence* $\{s_n\}$ *of non-negative numbers such that* $s_n \to 0(J_p)$ *and* $\{p_n s_n/\lambda_n \Delta_n\}$ *is unbounded.*

Proof. (i) The hypotheses imply that

$$0 \leqslant \frac{p_n s_n}{\Delta_n} = \frac{p_n s_n t_n^n}{p(t_n)} \leqslant \frac{1}{p(t_n)} \sum_{k=0}^{\infty} p_k s_k t_n^k \to 0,$$

since $t_n \to 1-$ by Lemma 3(iv).

(ii) Let $\{n_k\}$ be an increasing sequence of positive integers such that $p_{n_k} > 0$ and $\sum_{k=0}^{\infty} \sqrt{\lambda_{n_k}} < \infty$, and define

$$s_n := \begin{cases} \sqrt{\lambda_n}\, \Delta_n / p_n & \text{if} \quad n = n_k,\ k = 0, 1, ..., \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{p_n s_n / \lambda_n \Delta_n\}$ is unbounded. Further

$$\begin{aligned} 0 &\leqslant \limsup_{t \to 1-} \frac{1}{p(t)} \sum_{n=0}^{\infty} p_n s_n t^n \\ &\leqslant \lim_{t \to 1-} \frac{1}{p(t)} \sum_{k=0}^{N-1} p_{n_k} s_{n_k} + \sum_{k=N}^{\infty} \sqrt{\lambda_{n_k}} \\ &= \sum_{k=N}^{\infty} \sqrt{\lambda_{n_k}} \to 0 \qquad \text{as} \quad N \to \infty, \end{aligned}$$

and hence $s_n \to 0(J_p)$. ∎

Remark 2. It is readily shown that Theorem L2(ii) remains valid if J_p is replaced by M_p and Δ_n by P_n. The limitation conditions in both Theorems L1 and L2 are thus sharp.

5. Asymptotics

We suppose throughout this section that the function $g(x)$ is defined and continuous on $[0, \infty)$, and that it is a logarithmico-exponential function for sufficiently large x satisfying

$$g'(x) \to 0 \qquad \text{and} \qquad xg'(x) \to \infty \qquad \text{as} \quad x \to \infty. \tag{3}$$

We shall consider the J_p-method given by

$$p(x) := \sum_{n=0}^{\infty} p_n x^n \qquad \text{with} \quad p_n = e^{g(n)}.$$

Observe that $p_n \to \infty$ and $e^{g(x)} = o(e^{\varepsilon x})$ as $x \to \infty$ for all $\varepsilon > 0$, so that the power series for $p(x)$ has radius of convergence 1. Moreover, it follows from the properties of logarithmico-exponential functions (see [4]) that, for a sufficiently large positive x_0, $g'''(x)$ is continuous on $[x_0, \infty)$,

$$\left.\begin{aligned} &g'(x) > 0, g''(x) < 0, g'''(x) > 0, \frac{d}{dx}(x^2 g''(x)) < 0 \qquad \text{for} \quad x \geqslant x_0, \\ &g(x) \to \infty, g''(x) \to 0, x^2 g''(x) \to -\infty \qquad \text{as} \quad x \to \infty, \end{aligned}\right\} \tag{4}$$

and

$$g''(x)/g'^2(x) \to 0, \; xg'''(x)/g''(x) = O(1) \qquad \text{as} \quad x \to \infty. \tag{5}$$

THEOREM A1. *As* $x \to \infty$,

$$G(x) := \int_0^\infty e^{g(t) - tg'(x)}\, dt \sim \tilde{G}(x) := \sqrt{2\pi}\, e^{g(x) - xg'(x)}/\sqrt{-g''(x)}.$$

The theorem can be proved by considering

$$G_1(x) := \sqrt{-g''(x)} \int_0^\infty e^{g_1(t)}\, dt,$$

where

$$g_1(t) = g_1(t, x) := g(t) - g(x) - (t - x)\, g'(x),$$

and showing that $G_1(x) \to \int_{-\infty}^{\infty} e^{-\tau^2/2}\, d\tau = \sqrt{2\pi}$ as $x \to \infty$. This can done by means of what is often called Laplace's method (see [7, p. 80]).

Remark 3. Equivalent to Theorem A1 is the result that the Laplace transform $\int_0^\infty e^{g(t)} e^{-xt}\, dt \sim \tilde{G}(h(x))$ as $x \to 0+$, where h is the inverse function of g' on the interval $(0, g'(x_0)]$. Note that the function $g(h(x)) - xh(x)$, the exponent in the expression for $\tilde{G}(h(x))$, is the maximum of $g(t) - xt$ with respect to t and is frequently called the "complementary convex function" of g (see [1]).

The following two theorems can now be established without difficulty.

THEOREM A2. $\tilde{\Delta}_n := \inf_{x \geqslant x_0} \tilde{G}(x)\, e^{ng'(x)} \sqrt{2\pi}\, e^{g(n)}/\sqrt{-g''(n)}$. *Moreover,*

$$\tilde{\Delta}_n = \tilde{G}(x_n)\, e^{ng'(x_n)}, \qquad \text{where} \quad x_0 \leqslant x_n \leqslant n \quad \text{and} \quad x_n \sim n.$$

THEOREM A3. *The following asymptotic relations hold*:

$$\text{(i)} \quad P_n \sim \frac{p_n}{g'(n)}, \qquad P_{n+1} \sim P_n, \qquad \frac{P_{2n}}{P_n} \to \infty;$$

$$\text{(ii)} \quad \lim_{x \to 1-} \frac{p(x^2)}{p(x)} = 0;$$

$$\text{(iii)} \quad \frac{\Delta_n}{P_n} \sim \sqrt{2\pi}\, \frac{g'(n)}{\sqrt{-g''(n)}} \to \infty.$$

An immediate consequence of Theorem L2(ii) with $\lambda_n = P_n/\Delta_n$ and Theorems L1 and A3 is the following result concerning the function p considered in this section:

COROLLARY. *There is a sequence of non-negative numbers* $\{s_n\}$ *which is* J_p*-convergent to* 0 *but not* M_p*-convergent.*

We conclude by giving some examples of functions g satisfying the conditions of this section, together with the corresponding asymptotics of P_n, Δ_n, and Δ_n/P_n calculated by means of Theorem A3.

EXAMPLES.

(i) $g(x) := \log^2(x+1)$; $\quad P_n \sim \dfrac{n}{2\log n}\, e^{g(n)}$,

$$\Delta_n \sim \frac{\sqrt{\pi}\, n}{\sqrt{\log n}}\, e^{g(n)}, \qquad \frac{\Delta_n}{P_n} \sim 2\sqrt{\pi \log n}.$$

(ii) $g(x) := \sqrt{x}$; $\quad P_n \sim 2\sqrt{n}\, e^{\sqrt{n}}$, $\quad \Delta_n \sim 2\sqrt{2\pi}\, n^{3/4} e^{\sqrt{n}}$,

$$\frac{\Delta_n}{P_n} \sim \sqrt{2\pi}\, n^{1/4}.$$

(iii) $g(x) := \dfrac{x}{\log x}$; $\quad P_n \sim (\log n) e^{g(n)}$,

$$\Delta_n \sim \sqrt{2\pi n}\,(\log n) e^{g(n)}, \qquad \frac{\Delta_n}{P_n} \sim \sqrt{2\pi n}.$$

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