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# A Tauberian theorem concerning Dirichlet series

BY DAVID BORWEIN
*Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7*

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*Abstract*

It is shown that under certain general Tauberian conditions the asymptotic relationship

$$\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \sim s \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \quad \text{as} \quad x \to 0+$$

between two Dirichlet series implies the asymptotic relationship

$$\sum_{k=1}^{n} a_k s_k \sim s \sum_{k=1}^{n} a_k.$$

## 1. *Introduction*

Suppose throughout that $\lambda := \{\lambda_n\}$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geqslant 0$, and that $a := \{a_n\}$ is a sequence of non-negative numbers with $a_1 > 0$. Let

$$A_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}.$$

Suppose also that $A_n \to \infty$, and that the Dirichlet series $a(x)$ is convergent for all $x > 0$.

Let $\{s_n\}$ be a sequence of real numbers,

$$t_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k s_k \quad \text{and} \quad \sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}.$$

The weighted mean summability method $M_a$ and the Dirichlet series method $D_{\lambda,a}$ (see [**2**]) are defined as follows:

$$s_n \to s(M_a) \quad \text{if} \quad t_n \to s;$$

$$s_n \to s(D_{\lambda,a}) \quad \text{if} \quad \sigma(x) \to s \quad \text{as} \quad x \to 0+.$$

When $\lambda_n := n$ the method $D_{\lambda,a}$ reduces to the power series method $J_a$ (as defined in [**1**] for example). Since $A_n \to \infty$ both methods are regular (i.e. $s_n \to s$ implies $s_n \to s(M_a)$ and $s_n \to s(D_{\lambda,a})$), and also $s_n \to s(M_a)$ implies $s_n \to s(D_{\lambda,a})$ (see [**2**], theorem 1). The purpose of this paper is to prove the following Tauberian converse of the latter result:

THEOREM. *Suppose that*

$$\lambda_{n+1} \sim \lambda_n, \tag{1}$$

$$A_m/A_n \to 1 \quad \text{when} \quad \lambda_m/\lambda_n \to 1, \quad m > n \to \infty, \tag{2}$$

$$a_n \lambda_n s_n \geqslant -H(\lambda_{n+1}-\lambda_n) A_n, \tag{3}$$

*where H is a positive constant, and that* $s_n \to s(D_{\lambda,a})$. *Then* $s_n \to s(M_a)$.

A version of the theorem with (2) replaced by

$$a_n \lambda_n = o((\lambda_{n+1}-\lambda_n) A_n)$$

is known ([2], theorem 3). This latter result is a special case of the theorem since, as is easily shown, (2) is in fact a consequence of (1) and

$$a_n \lambda_n = O((\lambda_{n+1}-\lambda_n) A_n). \tag{4}$$

Thus the theorem also holds with (2) replaced by (4). The special case $\lambda_n := n$ of the theorem has been proved by Tietz ([4], theorem 1). His paper has references to previously proved special cases.

## 2. *Preliminary results*

LEMMA 1. *Suppose that (1) and (2) hold. Then*
(i) $a(x)/a(2x) = O(1)$ *as* $x \to 0+$, *and*
(ii) $a(1/\lambda_n) = O(A_n)$.

*Proof.* The lemma follows directly from results proved elsewhere ([3], lemmas 5(i), 3 and 4). ▮

LEMMA 2. *Suppose that (1), (2) and (3) hold, and that* $\sigma(x) = O(1)$ *as* $x \to 0+$. *Then* $t_n = O(1)$.

*Proof.* Arguing as in the first half of the proof of [2], theorem 3 with $c = 1/e$ but using lemma 1(i) (instead of $a(x)/a(2x) \to 1$ as $x \to 0+$), we deduce that

$$\frac{1}{a(1/\lambda_n)} \sum_{k=1}^{n} a_k s_k = O(1),$$

and hence, by Lemma 1(ii), that $t_n = O(1)$. ▮

LEMMA 3. *Suppose that (1), (2) and (3) hold, and that* $t_n = O(1)$. *Then*

$$\liminf (t_m - t_n) \geqslant 0 \quad \text{when} \quad \lambda_m/\lambda_n \to 1, m > n \to \infty. \tag{5}$$

*Proof.* Let $K = \sup_{n>0}|t_n|$. Since

$$t_n - t_{n-1} = (a_n/A_n)(s_n - t_{n-1}) \quad \text{for} \quad n > 1,$$

we have, by (3),

$$t_m - t_n = \sum_{k=n+1}^{m} \frac{a_k s_k}{A_k} - \sum_{k=n+1}^{m} \frac{a_k t_{k-1}}{A_k} \geqslant -H \sum_{k=n+1}^{m} \frac{\lambda_{k+1}-\lambda_k}{\lambda_k} - K\left(\frac{A_m}{A_n} - 1\right)$$

$$\geqslant -H\left(\frac{\lambda_{m+1}}{\lambda_{n+1}} - 1\right) - K\left(\frac{A_m}{A_n} - 1\right) \quad \text{for} \quad m > n > 0. \tag{6}$$

In view of (1) and (2), (5) follows from (6). ▮



## 3. *Proof of the theorem*

Let

$$b_n := A_n(\lambda_{n+1}-\lambda_n) \quad \text{and} \quad \tau(x) := \frac{1}{b(x)} \sum_{n=1}^{\infty} b_n t_n e^{-\lambda_n x}.$$

Then as in the proof of [2], theorem 3 (end of p. 522 to middle of p. 523) it follows from (1), (2), (3) and Lemma 2 that $\tau(x) \to s$ as $x \to 0+$, i.e. that

$$t_n \to s(D_{\lambda,b}). \tag{7}$$

Now let $A(x) := 0$ for $x < \lambda_1$,

$$A(x) := A_n \quad \text{for} \quad \lambda_n \leqslant x < \lambda_{n+1}, \quad n = 1, 2, \ldots,$$

and let $B(x) := \int_0^x A(v)\,dv$.

Then $B(\lambda_{n+1}) = B_n = \sum_{k=1}^{n} b_k$ and $B_n \geqslant A_1(\lambda_{n+1}-\lambda_1) \to \infty$. Further, it follows from (1) and (2), by [3], lemma 5(i) that $A(y)/A(x) \to 1$ when $y/x \to 1, y > x \to \infty$, which implies (see [3], lemma 3) that

$$A(y)/A(\tfrac{1}{2}y) = O(1) \quad \text{as} \quad y \to \infty. \tag{8}$$

Hence, for $y > x > \lambda_2$,

$$(B(y)-B(x))/B(y) = \frac{1}{B(y)} \int_x^y A(v)\,dv$$

$$\leqslant (yA(y)/B(y))(1 - x/y) \to 0 \quad \text{when} \quad y/x \to 1, y > x \to \infty,$$

since, by (8),

$$2B(y) \geqslant 2\int_{\frac{1}{2}y}^{y} A(v)\,dv \geqslant yA(\tfrac{1}{2}y) \geqslant KyA(y)$$

for some positive constant $K$. It follows that

$$B(y)/B(x) \to 1 \quad \text{when} \quad y/x \to 1, \quad y > x \to \infty,$$

and hence, in view of (1), that

$$B_m/B_n \to 1 \quad \text{when} \quad \lambda_m/\lambda_n \to 1, \quad m > n \to \infty. \tag{9}$$

Next, for $y > x > \lambda_2$,

$$(B(y)-B(x))/B(x) = \frac{1}{B(x)} \int_x^y A(v)\,dv$$

$$\geqslant (xA(x)/B(x))(y/x - 1) \geqslant y/x - 1,$$

which implies that

$$y/x \to 1 \quad \text{when} \quad B(y)/B(x) \to 1, \quad y > x \to \infty,$$

and hence that

$$\lambda_m/\lambda_n \to 1 \quad \text{when} \quad B_m/B_n \to 1, \quad m > n \to \infty. \tag{10}$$

By Lemmas 2 and 3, it follows from (10) that

$$\liminf (t_m - t_n) \geqslant 0 \quad \text{when} \quad B_m/B_n \to 1, \quad m > n \to \infty. \tag{11}$$

Finally, by a theorem proved elsewhere ([3], theorem 6), it follows from (1), (7), (9) and (11) that $t_n \to s$, i.e. $s_n \to s(M_a)$. ▮

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