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Tauberian theorems concerning Laplace transforms and Dirichlet series

By

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1. Introduction. Suppose throughout that all functions and sequences are real, that $A(x)$ is a non-decreasing, right-continuous and unbounded function on $[0, \infty)$ with $A(0) \geqq 0$, and that $s(x)$ is a locally bounded function, measurable on $(0, \infty)$ with respect to the Lebesgue-Stieltjes measure induced by $A(x)$. Let

$$a(x) := \int_0^\infty e^{-vx}\, dA(v), \qquad t(x) := \frac{1}{A(x)} \int_0^x s(v)\, dA(v)$$

and

$$\sigma(x) := \frac{1}{a(x)} \int_0^\infty s(v)\, e^{-vx}\, dA(v).$$

We suppose that the Laplace transform $a(x)$ is finite for all $x > 0$. The integrals are to be interpreted as follows: For $0 \leqq x < y$,

$$\int_x^y dA(v) = \int_{(x,y]} dA(v) = A(y+) - A(x+) = A(y) - A(x),$$

and

$$\int_x^y s(v)\, e^{-vx}\, dA(v) = \int_{(x,y]} s(v)\, e^{-vx}\, dA(v),$$

the integrals over $(x, y]$ being Lebesgue-Stieltjes integrals. Further

$$\int_x^\infty s(v)\, e^{-vx}\, dA(v) := \lim_{y\to\infty} \int_x^y s(v)\, e^{-vx}\, dA(v)$$

whenever the limit exists. It is easy to prove that $a(x) \to \infty$ as $x \to 0+$; and that if $s(x) \to s$ as $x \to \infty$, then $\sigma(x) \to s$ as $x \to 0+$. The primary object of this paper is to

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prove the following Tauberian converse of the latter result:

Theorem 1. *Suppose that*

(1) $$\frac{A(y)}{A(x)} \to 1 \quad when \quad \frac{y}{x} \to 1,\ y > x \to \infty,$$

(2) $$\liminf \{s(y) - s(x)\} \geqq 0 \quad when \quad \frac{A(y)}{A(x)} \to 1,\ y > x \to \infty,$$

and that $\sigma(x) \to s$ *as* $x \to 0+$. *Then* $s(x) \to s$ *as* $x \to \infty$.

In Sect. 4 we specialize this result to obtain a Tauberian theorem (Theorem 6) for the Dirichlet series summability method which generalizes a result due to Tietz [7, Satz 3.9] on the power series method. In Sect. 3 we prove the following ancillary (but independently interesting) Tauberian theorems:

Theorem 2. *Suppose that* (2) *holds, that*

(3) $$\frac{A(x+1)}{A(x)} \to 1 \ as\ x \to \infty,$$

(4) $$\frac{A(2x)}{A(x)} = O(1) \ as\ x \to \infty,$$

and that $\sigma(x) = O(1)$ *as* $x \to 0+$. *Then* $s(x) = O(1)$ *for* $x \geqq 0$.

Theorem 2 generalizes a result due to Tietz [7, Satz 3.6].

Theorem 3. *Suppose that* (1) *holds, that* $s(x) > -H$ *for* $x \geqq 0$, *where* H *is a constant, and that* $\sigma(x) \to s$ *as* $x \to 0+$. *Then* $t(x) \to s$ *as* $x \to \infty$.

In Sect. 4 we specialize Theorem 3 to obtain a Tauberian theorem (Theorem 7) for the Dirichlet series summability method which generalizes a result due to Tietz and Trautner [8, Korollar 4.2] on the power series method.

Theorem 4. *Suppose that* (1) *and* (2) *hold, and that* $\sigma(x) \to s$ *as* $x \to 0+$. *Then* $t(x) \to s$ *as* $x \to \infty$.

Since (1) implies (3) and, by Lemma 3 (below), also implies (4), Theorem 4 is an immediate consequence of Theorems 2 and 3.

Theorem 5. *Suppose that* (2) *and* (3) *hold, and that* $t(x) \to s$ *as* $x \to \infty$. *Then* $s(x) \to s$ *as* $x \to \infty$.

Since (1) implies (3), Theorem 1 follows from Theorems 4 and 5.
We proceed now to establish Theorems 2, 3 and 5.

2. Preliminary results.

Lemma 1. *Let*

$$K(u, v) := \frac{e^{-v/u}}{a(1/u)}.$$

Suppose that

(5) $\quad \phi(x)$ *is a non-decreasing function on* $[0, \infty)$ *such that*

$$\phi(x) \to \infty \quad \text{and} \quad \phi(x) - \phi(x-1) \to 0 \quad \text{as} \quad x \to \infty;$$

$$\int_0^x K(u, v)\, dA(v) \to 0 \quad \text{when} \quad \phi(u) - \phi(x) \to \infty,\ u > x \to \infty; \tag{6}$$

$$\int_x^\infty K(u, v)\, \{\phi(v) - \phi(x)\}\, dA(v) \to 0 \quad \text{when} \quad \phi(x) - \phi(u) \to \infty,\ x > u \to \infty; \tag{7}$$

and that there are positive constants α *and* β *such that*

$$s(y) - s(x) > -\alpha\{\phi(y) - \phi(x)\} - \beta \quad \text{for } y > x > 0. \tag{8}$$

Then $\sigma(x) = O(1)$ *as* $x \to 0+$ *implies* $s(x) = O(1)$ *for* $x \geqq 0$.

Note that $\int_0^\infty K(u, v)\, s(v)\, dA(v) = \sigma(1/u)$. The lemma is a variant of a result originally given by Vijayaraghavan [9] and can be proved along the lines of the proof of Theorem 238 in [4]. (See also the proofs of [3, Theorem 3] and [6, Lemma 1].)

Lemma 2. *Suppose that* (5) *holds and that*

$$\liminf\, \{s(y) - s(x)\} \geqq 0 \quad \text{when} \quad \phi(y) - \phi(x) \to 0,\ y > x \to \infty.$$

Then (8) *holds.*

P r o o f. (Cf. the proof of [3, Lemma 6].) By the second hypothesis, there are positive numbers c and δ such that $s(y) - s(x) > -1$ whenever $y > x \geqq c$ and $\phi(y) - \phi(x) < 2\delta$. Furthermore, by (5), c can be chosen so large that $\phi(x+) - \phi(x-) < \delta$ when $x \geqq c$.

Suppose first that $y > x \geqq c$. Define an increasing sequence $\{x_n\}$ such that $x_0 = x$ and $\delta \leqq \phi(x_n) - \phi(x_{n-1}) < 2\delta$ for $n = 1, 2, \ldots$. Since $\phi(x_n) \geqq \phi(x_0) + n\delta$ we have that $x_n \to \infty$. Hence there is a positive integer m for which $x_m \leqq y < x_{m+1}$. Therefore

$$s(y) - s(x) = \sum_{n=1}^{m} \{s(x_n) - s(x_{n-1})\} + s(y) - s(x_m) > -m - 1.$$

Since $m\delta \leqq \phi(x_m) - \phi(x_0) \leqq \phi(x) - \phi(y)$, it follows that

$$s(y) - s(x) > -\frac{1}{\delta}\{\phi(y) - \phi(x)\} - 1 \quad \text{when } y > x \geqq c.$$

If $c \geqq y > x > 0$, then, because $s(x)$ is locally bounded, there is a positive constant M such that $s(y) - s(x) > -M$. Finally, if $y > c > x > 0$, then

$$\begin{aligned} s(y) - s(x) &= s(y) - s(c) + s(c) - s(x) \\ &> -\frac{1}{\delta}\{\phi(y) - \phi(c)\} - 1 - M \geqq -\frac{1}{\delta}\{\phi(y) - \phi(x)\} - 1 - M. \end{aligned}$$

Consequently (8) holds with $\alpha = 1/\delta$ and $\beta = M + 1$. □

Lemma 3. *If* (1) *holds, then* (4) *holds.*

P r o o f. If (1) holds, then

$$\log \frac{1}{A(y)} - \log \frac{1}{A(x)} \to 0 \quad \text{when} \quad \frac{y}{x} \to 1,\ y > x \to \infty,$$

and it follows [4, p. 125] that there are positive constants H, x_0 such that

$$\log \frac{1}{A(2x)} - \log \frac{1}{A(x)} > -H \quad \text{for} \quad x > x_0.$$

This implies (4). □

Lemma 4. *Suppose that* (4) *holds. Then*

(i) $\quad \dfrac{a(x)}{a(2x)} = O(1)$ *as* $x \to 0+$, *and*

(ii) $\quad \dfrac{a(1/x)}{A(x)} = O(1)$ *as* $x \to \infty$.

P r o o f. Let $x > 0$, Then, for $x > \varepsilon > 0$, $y > 0$,

$$\{A(y) - A(0)\}\, e^{-yx} \leqq e^{-y\varepsilon} \int_0^y e^{-v(x-\varepsilon)}\, dA(v) \leqq e^{-y\varepsilon} a(x - \varepsilon) \to 0 \text{ as } y \to \infty.$$

Hence, as $y \to \infty$,

$$\begin{aligned} \int_0^y e^{-vx}\, dA(v) &= A(y)\, e^{-yx} - A(0) + x \int_0^y A(v)\, e^{-vx}\, dv \\ &\to x \int_0^\infty A(v)\, e^{-vx}\, dv - A(0). \end{aligned}$$

Thus

$$\begin{aligned} a(x) + A(0) &= x \int_0^\infty A(v)\, e^{-vx}\, dv = 2x \int_0^\infty A(2v)\, e^{-2vx}\, dv \\ &\leqq Hx \int_0^\infty A(v)\, e^{-2vx}\, dv = H\{a(2x) + A(0)\}, \end{aligned}$$

by (4), H being a positive constant. Since $a(2x) \to \infty$ as $x \to 0+$, (i) follows.

Next, by (i), there is a positive constant K such that $a(x) \leqq K a(2x)$ for all $x > 0$. Now choose $c > 0$ so that $b := K e^{-c/2} < 1$. Then

$$a(1/x) = \int_0^{cx} e^{-v/x}\, dA(v) + \int_{cx}^{\infty} e^{-v/2x} e^{-v/2x}\, dA(v)$$

$$\leqq A(cx) + e^{-c/2} \int_{cx}^{\infty} e^{-v/2x}\, dA(v) \leqq A(cx) + e^{-c/2} a(1/2x)$$

$$\leqq A(cx) + K e^{-c/2} a(1/x) = A(cx) + b\,a(1/x).$$

Hence, by (4),

$$\frac{a(1/x)}{A(x)} \leqq \frac{A(cx)}{A(x)} \cdot \frac{1}{1-b} = O(1) \quad \text{as } x \to \infty. \qquad \square$$

3. Proofs of Theorems 2, 3 and 5.

Proof of Theorem 2. (Cf. the proof of [7, Satz 3.6].) Let $x_0 \geqq 0$ be such that $A(x_0) \geqq e$ and take

$$\phi(x) := \log A(x) \text{ for } x \geqq x_0, \quad \phi(x) := 1 \text{ for } x < x_0.$$

Then ϕ satisfies (5), and

$$\text{(9)} \qquad \phi(u) - \phi(x) \to \infty \text{ implies } \frac{A(x)}{A(u)} \to 0.$$

Also, for $u > 0$,

$$e\,a(1/u) > \int_0^u e^{(u-v)/u}\, dA(v) \geqq A(u) - A(0),$$

and so, since $a(1/u) \to \infty$ as $u \to \infty$,

$$\text{(10)} \qquad \frac{A(u)}{a(1/u)} = O(1) \text{ as } u \to \infty.$$

Hence, for $u > x > x_0$,

$$\int_0^x K(u,v)\, dA(v) = \frac{1}{a(1/u)} \int_0^x e^{-v/u}\, dA(v)$$

$$\leqq \frac{A(x)}{a(1/u)} = \frac{A(x)}{A(u)} \frac{A(u)}{a(1/u)} \to 0$$

when

$$\phi(u) - \phi(x) \to \infty, \quad u > x \to \infty,$$

by (9) and (10). Thus (5) holds.

Next

$$\text{(11)} \qquad \phi(x) - \phi(u) \to \infty \text{ implies } \frac{A(x)}{A(u)} \to \infty.$$

It follows from (4) and (11) that

$$\text{(12)} \qquad \phi(x) - \phi(u) \to \infty \text{ implies } \frac{x}{u} \to \infty.$$

Suppose now that $\phi(x) - \phi(u) \to \infty$ $(x > u \to \infty)$. By (12), there is an $x_1 \geqq x_0$ such that $x > 2u$ for $x \geqq x_1$. Therefore, for $x \geqq x_1$,

$$\int_x^{\infty} K(u,v) \{\phi(v) - \phi(x)\}\, dA(v) = \frac{1}{a(1/u)} \int_x^{\infty} e^{-v/u} \log \frac{A(v)}{A(x)}\, dA(v)$$

$$\leqq \frac{1}{a(1/u) A(x)} \int_x^{\infty} e^{-v/u} \{A(v) - A(x)\}\, dA(v)$$

$$= \frac{1}{a(1/u) A(x)} \int_x^{\infty} e^{-v/u}\, dA(v) \int_x^{v} dA(t)$$

$$= \frac{1}{a(1/u) A(x)} \int_x^{\infty} e^{-t/u}\, dA(t) \int_t^{\infty} e^{-(v-t)/u}\, dA(v)$$

$$\leqq \frac{1}{a(1/u) A(x)} \int_x^{\infty} e^{-t/u}\, dA(t) \int_t^{\infty} e^{-(v-t)/x}\, dA(v)$$

$$\leqq \frac{a(1/x)}{a(1/u) A(x)} \int_x^{\infty} e^{-t/u} e^{t/x}\, dA(t)$$

$$= \frac{a(1/x)}{a(1/u) A(x)} \int_x^{\infty} e^{-t/2u} e^{-t(x-2u)/2xu}\, dA(t)$$

$$\leqq \frac{a(1/x)}{A(x)} \frac{a(1/2u)}{a(1/u)} e^{1-x/2u} \to 0$$

when

$$\phi(x) - \phi(u) \to \infty, \; x > u \to \infty,$$

by (12) and Lemmas 3 and 4. Therefore (7) holds; and, by Lemma 2, (2) implies (8).
The desired conclusion is now a consequence of Lemma 1. $\square$

Proof of Theorem 3. Suppose without loss of generality that $H = 0$, i.e., $s(x) > 0$ for $x \geqq 0$. Let

$$B(x) := \int_0^x s(v)\, dA(v).$$

Then, since $s(x)$ is locally bounded, $B(x)$ is non-decreasing and right-continuous on $[0, \infty)$. Further, for $x > 0$,

$$a(x)\sigma(x) = \int_0^\infty e^{-vx} s(v)\, dA(v) = \int_0^\infty e^{-vx}\, dB(v),$$

and so, since $\sigma(x) \to s$ as $x \to 0+$,

$$\int_0^\infty e^{-vx}\, dB(v) \sim s \int_0^\infty e^{-vx}\, dA(v) \quad \text{as } x \to 0+. \tag{13}$$

Since the function A satisfies (1) and $A(x) \to \infty$ as $x \to \infty$, it follows from (13), by a theorem due to Korenblum [5, Theorem 2], that

$$t(x) = \frac{B(x)}{A(x)} \to s \quad \text{as } x \to \infty. \quad \square$$

Proof of Theorem 5. Suppose without loss of generality that $s = 0$, i.e. $t(x) \to 0$ as $x \to \infty$. Let $\varepsilon > 0$. Then, by (2), there are positive numbers x_0, δ such that

$$s(y) - s(x) > -\varepsilon \quad \text{when} \quad \frac{A(y)}{A(x)} < 1 + 2\delta \quad \text{and } y > x > x_0.$$

Consequently if x, y satisfy these conditions

$$\{s(x) - \varepsilon\}\{A(y) - A(x)\} \leqq \int_x^y s(v)\, dA(v) = t(y)A(y) - t(x)A(x)$$
$$\leqq \{s(y) + \varepsilon\}\{A(y) - A(x)\},$$

and hence

$$s(x) - \varepsilon \leqq \frac{t(y)A(y) - t(x)A(x)}{A(y) - A(x)} \tag{14}$$
$$= t(y) + \frac{t(y) - t(x)}{\{A(y)/A(x)\} - 1} \leqq s(y) + \varepsilon.$$

Since $A(x) \to \infty$ as $x \to \infty$ and (3) holds, there is an $x_1 > x_0$ such that for every $x > x_1$ there is a $y > x$ satisfying

$$1 + \delta < \frac{A(y)}{A(x)} < 1 + 2\delta. \tag{15}$$

It follows on letting $x \to \infty$ in (14) that

$$\limsup_{x \to \infty} s(x) \leqq \varepsilon.$$

Likewise, there is a $y_1 > x_0$ such that for every $y > y_1$ there is an x satisfying $x_0 < x < y$ and (15). Hence, letting $y \to \infty$ in (14), we get

$$\liminf_{y \to \infty} s(y) \geqq -\varepsilon.$$

Therefore $s(x) \to 0$ as $x \to \infty$. $\square$

4. Specializations. Now suppose that $\lambda := \{\lambda_n\}$ is a strictly increasing unbounded sequence with $\lambda_1 > 0$, that $a := \{a_n\}$ is a sequence of non-negative numbers with $a_1 > 0$, and that $\{s_n\}$ is a sequence of real numbers. Let

$$A_n := \sum_{k=1}^n a_k \to \infty,$$

and let $A(x) := s(x) := 0$ for $x < \lambda_1$,

$$A(x) := A_n \quad \text{and} \quad s(x) := s_n \quad \text{for} \quad \lambda_n \leqq x < \lambda_{n+1}, \quad n = 1, 2, \ldots.$$

Then, for $x > 0$,

$$a(x) = \int_0^\infty e^{-vx}\, dA(v) = \sum_{n=1}^\infty a_n e^{-\lambda_n x},$$

$$\sigma(x) = \frac{1}{a(x)} \int_0^\infty s(v) e^{-vx}\, dA(v) = \frac{1}{a(x)} \sum_{n=1}^\infty a_n s_n e^{-\lambda_n x},$$

$$t(x) = \frac{1}{A(x)} \int_0^x s(v)\, dA(v);$$

and

$$A(\lambda_n) = A_n, \quad s(\lambda_n) = s_n, \quad t_n := \frac{1}{A_n} \sum_{k=1}^n a_k s_k = t(\lambda_n).$$

As before we assume that $a(x) < \infty$ for all $x > 0$, i.e., that the Dirichlet series $a(x)$ is convergent for all $x > 0$. The weighted mean summability method M_a and the Dirichlet series method $D_{\lambda, a}$ (see [2]) are defined as follows:

$$s_n \to s(M_a) \quad \text{if} \quad t_n \to s;$$

$$s_n \to s(D_{\lambda,a}) \quad \text{if} \quad \sigma(x) \to s \quad \text{as} \quad x \to 0+.$$

When $\lambda := n$ the method $D_{\lambda, a}$ reduces to the power series method J_a (as defined in [1] for example). Since $A_n \to \infty$ both methods are regular (i.e., $s_n \to s$ implies $s_n \to s(M_a)$ and

$s_n \to s(D_{\lambda,a})$). Theorem 1 specializes to the following Tauberian theorem, the case $\lambda_n := n$ of which has been proved by Tietz [7, Satz 3.9]:

Theorem 6. *Suppose that*

$$\lambda_{n+1} \sim \lambda_n, \tag{16}$$

$$\frac{A_m}{A_n} \to 1 \quad \textit{when} \quad \frac{\lambda_m}{\lambda_n} \to 1,\ m > n \to \infty, \tag{17}$$

$$\liminf (s_m - s_n) \geqq 0 \quad \textit{when} \quad \frac{A_m}{A_n} \to 1,\ m > n \to \infty, \tag{18}$$

and that $s_n \to s(D_{\lambda,a})$. *Then* $s_n \to s$.

In order to prove this theorem we require another lemma.

Lemma 5. *Suppose that* (16) *holds. Then*

(i) (1) *is equivalent to* (17);
(ii) (2) *implies* (18);
(iii) (18) *and*

$$A_{n+1} \sim A_n \tag{19}$$

imply (2).

Proof. Part (i). That (1) implies (17) is immediate. Suppose therefore that (17) holds. Assign $\varepsilon > 0$. Then there are positive numbers N, δ such that

$$\frac{A_m}{A_n} < 1 + \varepsilon \quad \text{when} \quad \frac{\lambda_m}{\lambda_n} < 1 + 2\delta \quad \text{and} \quad m \geqq n \geqq N.$$

Now choose a positive integer $M > N$ such that

$$\frac{\lambda_{n+1}}{\lambda_n} < \frac{1+2\delta}{1+\delta} \quad \text{for } n \geqq M.$$

Let $y > x > \lambda_M, \frac{y}{x} < 1 + \delta$. Then there are integers m, n such that

$$\lambda_{n+1} > x \geqq \lambda_n, \quad \lambda_{m+1} > y \geqq \lambda_m.$$

Hence $m \geqq n \geqq M$,

$$\frac{\lambda_m}{\lambda_n} < \frac{y}{x}\frac{\lambda_{n+1}}{\lambda_n} < (1+\delta)\frac{1+2\delta}{1+\delta} = 1 + 2\delta;$$

and therefore

$$\frac{A(y)}{A(x)} = \frac{A_m}{A_n} < 1 + \varepsilon.$$

Consequently (1) holds, and the proof of (i) is complete.

Part (ii). This is immediate.

Part (iii). Suppose that (17), (18) and (19) hold. Assign $\varepsilon > 0$. Then there are positive numbers N, δ such that

$$s_m - s_n > -\varepsilon \quad \text{when} \quad \frac{A_m}{A_n} < 1 + 2\delta \quad \text{and} \quad m \geqq n \geqq N.$$

Now choose a positive integer $M > N$ such that

$$\frac{A_{n+1}}{A_n} < \frac{1+2\delta}{1+\delta} \quad \text{for } n \geqq M.$$

Let $y > x > \lambda_M, \frac{A(y)}{A(x)} < 1 + \delta$. Then there are integers m, n such that

$$A_{n+1} > A(x) \geqq A_n, \quad A_{m+1} > A(y) \geqq A_m.$$

Hence $m \geqq n \geqq M$,

$$\frac{A_m}{A_n} < \frac{A(y)}{A(x)}\frac{A_{n+1}}{A_n} < (1+\delta)\frac{1+2\delta}{1+\delta} = 1 + 2\delta;$$

and therefore $s(y) - s(x) = s_m - s_n > -\varepsilon$. Thus (2) holds. □

Proof of Theorem 6. Since (16) and (17) imply (19), it follows, by Lemma 5, that (16), (17) and (18) imply (1) and (2). Theorem 6 is thus a consequence of Theorem 1. □

In view of Lemma 5 (i), we can also specialize Theorem 3 as follows:

Theorem 7. *If* (16) *and* (17) *holds,* $s_n > -H$ *for* $n = 1, 2, \ldots$ *where* H *is a constant, and* $s_n \to s(D_{\lambda,a})$, *then* $s_n \to s(M_a)$.

A similar theorem with a somewhat stronger hypothesis in place of (17) but without hypothesis (16) appears as Theorem 2 in [2]. The case $\lambda_n := n$ of Theorem 7 was proved by Tietz and Trautner [8, Korollar 4.2]. Theorems 6 and 7 evidently remain valid with $\lambda_1 = 0$.

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