

**THE NONLOCAL NATURE OF THE SUMMABILITY
FOURIER SERIES BY CERTAIN ABSOLUTE RIESZ METHODS**

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(Communicated by Andrew M. Bruckner)

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THE NONLOCAL NATURE OF THE SUMMABILITY OF FOURIER SERIES BY CERTAIN ABSOLUTE RIESZ METHODS

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ABSTRACT. It is proved that for a large class of sequences $\{\lambda_n\}$ the summability at a point of a Fourier series $\sum A_n(t)$ by the absolute Riesz method $|R, \lambda_n, 1|$ is not a local property of the generating function. It is also proved, inter alia, that, for every $\varepsilon > 0$, the $|R, \lambda_n, 1|$ summability of the factored series $\sum A_n(t)\lambda_n^{-\varepsilon}$ at any point is always a local property of the generating function.

1. INTRODUCTION

Suppose throughout that, for $n = 1, 2, \dots$,

$$\mu_n > 0, \quad \lambda_n := \mu_1 + \mu_2 + \dots + \mu_n \rightarrow \infty,$$

and $s_n := a_1 + a_2 + \dots + a_n$. The series $\sum a_n$ is said to be summable by the absolute Riesz method $|R, \lambda_n, 1|$ if

$$c(w) := \frac{1}{w} \sum_{\lambda_n < w} (w - \lambda_n) a_n$$

is of bounded variation over (λ_1, ∞) , and it is said to be summable by the absolute weighted mean method $|M, \mu_n|$ if the sequence of means $\{t_n\}$ defined by

$$t_n := \frac{1}{\lambda_n} \sum_{\nu=1}^n \mu_\nu s_\nu$$

is of bounded variation, that is if

$$\sum_{n=1}^{\infty} |\Delta t_n| < \infty,$$

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where $\Delta t_n := t_n - t_{n+1}$. It is well known, and easily verified, that these two methods are equivalent.

Let

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} A_n(t) := \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)$$

be the Fourier series generated by a periodic function F with period 2π which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of F (i.e. depends only on the behaviour of F in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of F . On the other hand, Bosanquet and Kestleman [3] showed that the summability $|C, 1|$ ($= |M, 1|$) of the Fourier series at any point is not a local property of F , and Mohanty [7] subsequently showed that this is also the case with summability $|R, \lambda_n, 1|$ when $\lambda_n := \ell_k(n)$ for n sufficiently large, where

$$\ell_0(x) := x \quad \text{and} \quad \ell_k(x) := \log(\ell_{k-1}(x))$$

for $k = 1, 2, \dots$ and x sufficiently large. Mohanty also showed that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum_{n=2}^{\infty} A_n(t) / \log n$$

at any point is a local property of F , whereas the $|C, 1|$ summability of this series is not. Matsumoto [5] improved the first of these results by showing that the $|R, \log n, 1|$ summability of the series

$$\sum_{n=3}^{\infty} A_n(t) (\log \log n)^{-p}, \quad p > 1,$$

at any point is a local property of F , and Bhatt [1] went a step further by showing that the factor $(\log \log n)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n \log n$ where $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent. Mishra [6] proved that if $\{\gamma_n\}$ is as above, and if

$$\lambda_n = O(n\mu_n) \quad \text{and} \quad \lambda_n \Delta \mu_n = O(\mu_n \mu_{n+1}),$$

then the summability $|M, \mu_n|$ of the series

$$\sum_{n=1}^{\infty} A_n(t) \gamma_n \frac{\lambda_n}{n\mu_n}$$

at any point is a local property of F . This does not directly generalize any of the above-mentioned results involving $|R, \log n, 1|$ summability since the order relations are not satisfied by $\mu_n := 1/n$. Bor [2] recently showed that $|M, \mu_n|$ in Mishra's result can be replaced by a more general summability method $|M, \mu_n|_k$. The object of this paper is to prove the following two theorems which include most of the above-mentioned results as special cases.

Theorem 1. Suppose that a is a positive integer, and that f is a positive, unbounded function with an absolutely continuous positive derivative on $[e^a, \infty)$ such that, on this interval,

$$(1) \quad \frac{xf'(x)}{f(x)} \text{ decreases to } 0$$

and

$$(2) \quad xf''(x) = O(f'(x)).$$

Suppose also that

$$(3) \quad \lambda_n := f(e^n) \quad \text{for } n \geq a,$$

and that $0 < \alpha < \beta < 2\pi$. Then there is a function F , Lebesgue integrable over (α, β) and zero in the remainder of $(0, 2\pi)$, whose Fourier series is not summable $|R, \lambda_n, 1|$ at $t = 0$.

This shows that, subject to the hypotheses of the theorem, the summability $|R, \lambda_n, 1|$ of a Fourier series at any point is not a local property of its generating function. Since the hypotheses are satisfied by $f(x) := \ell_k(x)$ for $k = 1, 2, \dots$, Bosanquet and Kestleman's result, and also Mohanty's result, on the nonlocal nature of the summability of a Fourier series by certain absolute methods are special cases of Theorem 1.

Theorem 2. Suppose that the sequence $\{c_n\}$ is such that

$$(4) \quad \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty$$

and

$$(5) \quad \sum_{n=1}^{\infty} |\Delta c_n| < \infty.$$

Then the summability $|R, \lambda_n, 1|$ of the factored Fourier series

$$\sum_{n=1}^{\infty} A_n(t) c_n$$

at any point is a local property of the generating function F .

This theorem generalizes Bhatt's above-mentioned result, since it is known (see [1] for references) that if $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent, then

$$\gamma_n \geq \gamma_{n+1} \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \log n \Delta \gamma_n < \infty,$$

and so (4) and (5) are satisfied by $\mu_n := 1/n$, $c_n := \gamma_n \log n$. Since, by Dini's theorem, $\sum \mu_n \lambda_n^{-1-\varepsilon}$ is convergent whenever $\varepsilon > 0$, we have the following corollary of Theorem 2.

where the function h is as in the lemma. For $0 < t < 2\pi$, $t \neq \pi$, we have that

$$\begin{aligned} \sum_{n=a}^{\infty} h(\lambda_n) |\cos nt| &\geq \sum_{n=a}^{\infty} h(\lambda_n) \cos^2 nt \\ &\geq \frac{1}{2} \sum_{n=a}^{\infty} h(\lambda_n) - \frac{1}{2} \left| \sum_{n=a}^{\infty} h(\lambda_n) \cos 2nt \right| = \infty, \end{aligned}$$

by Lemma 1, the final sum being convergent because the sequence $\{h(\lambda_n)\}$ decreases to 0. The required result now follows from a theorem due to Bosanquet and Kestleman [3, Theorem 1]. \square

Proof of Theorem 2. Since the convergence of the Fourier series at a point is a local property of its generating function F , Theorem 2 follows immediately from Lemma 2. \square

Remark (added November 9, 1990). After this paper was accepted for publication I found out that Theorem 2 is in fact a special case of Theorem 3 in S. Baron's paper, *Local property of absolute summability of a Fourier series and the conjugate series*, Tartu Riikl. Ül. Toimetised Vih. 253 (1970), 212–228. My proof, however, is somewhat simpler and more direct than Baron's, partly because he deals with more general summability methods.

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