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TAUBERIAN THEOREMS CONCERNING POWER SERIES WITH NON-NEGATIVE COEFFICIENTS

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1. Introduction

Suppose throughout that $\{a_n\}$ is a sequence of non-negative number, that

$$s_n := \sum_{k=0}^{n} a_k$$

and that

$$0 < f(x) := \sum_{k=0}^{\infty} a_k x^k < \infty \quad \text{for} \quad 0 < x < 1.$$

Hardy and Littlewood [4, Theorem 10] have proved the following theorem.

THEOREM H–L. *If*

$$f(x) \sim (1-x)^{-\rho} L(x) \quad \textit{as} \quad x \to 1-,$$

where $\rho \geqq 0$ *and* $L(1-\frac{1}{u})$ *is a logarithmico-exponential function such that*

$$u^{-\delta} \prec L\left(1-\frac{1}{u}\right) \prec u^{\delta},$$

then

$$s_n \sim \frac{n^{\rho}}{\Gamma(\rho+1)} L\left(1-\frac{1}{n}\right).$$

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See [3] for definitions and properties of logarithmico-exponential functions. Examples of logarithmico-exponential functions satisfying the above condition are given by

$$L\left(1-\frac{1}{u}\right) := (\log u)^{c_1}(\log\log u)^{c_2}\ldots,$$

where $c_1, c_2, \ldots$ are real numbers. Theorem H–L is Tauberian in nature in that it yields information about the asymptotic behavior of s_n from the asymptotic behavior of $f(x)$.

The primary object of this note is to supply a simple and straightforward proof of the following generalization of Theorem H–L.

THEOREM 1. (i) *Suppose*

$$\lim_{x\to 1-}\frac{f(x^m)}{f(x)} = \lambda_m > 0 \quad \textit{for} \quad m=2 \quad \textit{and} \quad m=3. \tag{1}$$

Then

$$f(x) = (1-x)^{-\rho}\phi(x)$$

where $\rho = -\log_2\lambda_2 \geqq 0$ *and, for all* $t \geqq 1$,

$$\lim_{x\to 1-}\frac{\phi(x^t)}{\phi(x)} = 1.$$

Moreover

$$s_n \sim \frac{n^\rho}{\Gamma(\rho+1)}\phi\left(1-\frac{1}{n}\right) = \frac{1}{\Gamma(\rho+1)}f\left(1-\frac{1}{n}\right)$$

and

$$s_{n+1}\sim s_n \quad \textit{and} \quad \lim_{n\to\infty}\frac{s_n}{s_{mn}} = \lambda_m > 0 \quad \textit{for} \quad m=2 \quad \textit{and} \quad m=3. \tag{2}$$

(ii) *Conversely,* (2) *implies* (1).

If follows from Theorem 1.8 in [5] that the integers 2, 3 in (1) can be replaced by any two positive numbers $p, q \neq 1$ such that $\log_q p$ is irrational. It was proved in [2] that

$$\lim_{x\to 1-}\frac{f(x^2)}{f(x)} = \lambda > 0 \tag{3}$$

alone does not imply (1) when $\lambda < 1$, though (1) and (3) are equivalent when $\lambda = 1$. Part (i) of Theorem 1 can be deduced from Karamata's Tauberian theorem and a known result about regularly varying functions (see Theorems 2.3 and 1.8 in [5]). We give an alternate proof which is more direct and more elementary, not involving, in particular, the extended continuity theorem for Laplace-Stieltjes transforms on which the proof of Karamata's theorem is based. Part (ii) of Theorem 1 is interesting in that it shows that (1) and (2) are in fact equivalent.

2. Preliminary results

THEOREM 2. *Suppose* $b_n \geqq 0$ *for* $n = 0, 1, \ldots,$

$$t_n := \sum_{k=0}^{n} b_k, \quad \textit{and} \quad g(x) := \sum_{n=0}^{\infty} b_n x^n < \infty \quad \textit{for} \quad 0<x<1.$$

If (1) *holds and* $\frac{g(x)}{f(x)} \to \lambda$ *as* $x \to 1-$, *then* $\frac{t_n}{s_n} \to \lambda$.

PROOF. The result is evidently true if $f(x)$ tends to a finite limit as $x \to 1-$. Suppose therefore that $f(x) \to \infty$ as $x \to 1-$.

Case (i) : $a_n > 0$ for $n = 0, 1, \ldots$. This case follows immediately from the theorem in [2].

Case (ii) : $a_n \geqq 0$ for $n = 0, 1, \ldots$. Let

$$f^\star(x) := f(x) + e^x, \quad g^\star(x) := g(x) + e^x$$

and define $a_n^\star$, $s_n^\star$, $b_n^\star$, $t_n^\star$ in the obvious way. Then $a_n^\star > 0$ for $n = 0, 1, \ldots,$ and, since $f(x) \to \infty$ as $x \to 1-$, (1) is satisfied with $f^\star$ in place of f. Further

$$\frac{g^\star(x)}{f^\star(x)} \to \lambda \quad \text{as} \quad x \to 1- \quad \text{if and only if} \quad \frac{g(x)}{f(x)} \to \lambda \quad \text{as} \quad x \to 1-,$$

and

$$\frac{t_n^\star}{s_n^\star} \to \lambda \quad \text{if and only if} \quad \frac{t_n}{s_n} \to \lambda.$$

Case (ii) now follows from Case (i). □

LEMMA 1. *If* (1) *holds, then, for* $m = 1, 2, \ldots$ *and* $\rho = -\log_2\lambda_2 \geqq 0$,

$$\lim_{x\to 1-}\frac{f(x^m)}{f(x)} = m^{-\rho}$$

and, for every $c \in (0,1)$,

$$\lim_{n\to\infty}\frac{s_n}{f(c^{1/n})} = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}.$$

PROOF. The result is evidently true with $\rho = 0$ if $f(x)$ tends to a finite limit as $x \to 1-$. Suppose therefore that $f(x) \to \infty$ as $x \to 1-$. It has been shown in [2] that this together with (1) implies the first conclusion. Further, when $\rho > 0$,

$$(m+1)^{-\rho} = \int_0^1 t^m d\chi(t) \quad \text{with} \quad \chi(t) := \frac{1}{\Gamma(\rho)} \int_0^t (-\log u)^{\rho-1} du.$$

It was proved in [1] that the above implies that, when $\rho = 0$,

$$\lim_{n\to\infty} \frac{s_n}{f(c^{1/n})} = 1,$$

and, when $\rho > 0$,

$$\lim_{n\to\infty} \frac{s_n}{f(c^{1/n})} = \int_c^1 t^{-1} d\chi(t) = \frac{1}{\Gamma(\rho)} \int_c^1 t^{-1}(-\log t)^{\rho-1} dt = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}. \ \square$$

The next lemma has been proved in essence in [1].

LEMMA 2. *If* $s_{n+1} \sim s_n$ *and* $\lim_{n\to\infty} \frac{s_n}{s_{mn}} = \lambda > 0$ *where* m *is a positive integer, then*

$$\lim_{x\to 1-} \frac{f(x^m)}{f(x)} = \lambda.$$

3. Proof of Theorem 1

(i) The first conclusion has been proved in [2]. To establish the asymptotic expression for s_n observe that, given $\gamma > 1$,

$$e^{-\gamma/n} < 1 - \frac{1}{n} < e^{-1/n}$$

for n sufficiently large. Hence for such n

$$\frac{s_n}{f(e^{-\gamma/n})} \geqq \frac{s_n}{f(1-1/n)} \geqq \frac{s_n}{f(e^{-1/n})}$$

and so, by Lemma 1,

$$\frac{\gamma^\rho}{\Gamma(\rho+1)} \geqq \limsup_{n\to\infty} \frac{s_n}{f(1-1/n)} \geqq \liminf_{n\to\infty} \frac{s_n}{f(1-1/n)} \geqq \frac{1}{\Gamma(\rho+1)}.$$

Since $\gamma^\rho \to 1$ as $\gamma \to 1-$, it follows that

$$\lim_{n\to\infty} \frac{s_n}{f(1-1/n)} = \frac{1}{\Gamma(\rho+1)},$$

i.e.,

$$s_n \sim \frac{n^\rho}{\Gamma(\rho+1)} \phi\left(1 - \frac{1}{n}\right) = \frac{1}{\Gamma(\rho+1)} f\left(1 - \frac{1}{n}\right).$$

To establish (2) we first observe that, by Lemma 1,

$$\lim_{n\to\infty} \frac{s_n}{s_{mn}} = \lim_{n\to\infty} \frac{s_n}{f(e^{-1/n})} \cdot \frac{f(e^{-1/mn})}{s_{mn}} \cdot \frac{f(e^{-1/n})}{f(e^{-1/nm})} = m^{-\rho}.$$

Next we suppose without loss of generality that $s_n \to \infty$. Then, by Theorem 2 with $b_n = a_{n+1}$, we see that

$$\frac{g(x)}{f(x)} = \frac{f(x) - a_0}{f(x)} \to 1 \text{ as } x \to 1-, \quad \text{and hence} \quad \frac{t_n}{s_n} = \frac{s_{n+1} - s_0}{s_n} \to 1$$

so that $s_{n+1} \sim s_n$.

(ii) This follows immediately from Lemma 2. □

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