

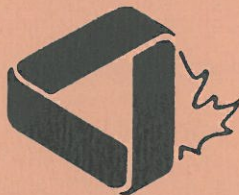
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BOREL-TYPE AND RIESZ SUMMABILITY METHODS

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ABSTRACT. It is proved that the summability of a series by the Borel-type summability method (B, α, β) together with a certain Tauberian condition implies its summability by the Riesz method $(R, \log(n+1), p)$.

1. Introduction. Suppose throughout that $\alpha > 0$, $\alpha N + \beta > 0$ with N a non-negative integer, $p \geq 0$, and $s_n := a_0 + a_1 + \dots + a_n$. The Borel-type summability method (B, α, β) and the Riesz method $(R, \log(n+1), p)$ are defined as follows:

$$s_n \rightarrow s(B, \alpha, \beta) \text{ if } \alpha e^{-x} \sum_{n=N}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \rightarrow s \text{ as } x \rightarrow \infty;$$

$$s_n \rightarrow s(R, \log(n+1), p) \text{ if } \sum_{\log(n+1) < w} \left(1 - \frac{\log(n+1)}{w}\right)^p a_n \rightarrow s \text{ as } w \rightarrow \infty.$$

Both methods are regular, and $(B, 1, 1)$ is the standard Borel exponential method B .

Let

$$L_n := \sum_{r=0}^n \frac{1}{r+1},$$

$$t_n := t_n^{(1)} := \frac{1}{L_n} \sum_{r=0}^n \frac{s_r}{r+1},$$

and, for $k = 2, 3, \dots$,

$$t_n^{(k)} := \frac{1}{L_n} \sum_{r=0}^n \frac{t_r^{(k-1)}}{r+1}.$$

The k -times iterated weighted mean method $(M, 1/(n+1), k)$ is defined by:

$$s_n \rightarrow s(M, 1/(n+1), k) \text{ if } t_n^{(k)} \rightarrow s \text{ as } n \rightarrow \infty.$$

The object of this paper is to prove the following Tauberian theorem.

THEOREM. Suppose that $s_n \rightarrow s(B, \alpha, \beta)$ and

$$(1) \quad s_n = O((n^{1/2} \log n)^p),$$

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where p is a positive integer. Then $s_n \rightarrow s(R, \log(n+1), p)$.

The case $\alpha = \beta = 1$ of the theorem was recently established by Kwee [8]. Our proof owes much to his. The present theorem is more general than Kwee's result since it is known ([1, Result (I)] and [2, Lemma 4]) that

if $s_n \rightarrow s(B, \alpha, \beta)$ and $\alpha > \gamma > 0$, then $s_n \rightarrow s(B, \gamma, \delta)$ provided

$$\sum_{n=N}^{\infty} s_n \frac{z^n}{\Gamma(\gamma n + \delta)}$$

is an entire function of z for N sufficiently large.

The proviso is certainly satisfied when (1) holds.

2. Preliminary results.

LEMMA 1 [1, RESULT (II)]. If $s_n \rightarrow s(B, \alpha, \beta)$ and $\delta > \beta$, then $s_n \rightarrow s(B, \alpha, \delta)$.

LEMMA 2 [4, THEOREM 1]. If $s_n \rightarrow s(B, \alpha, \beta)$ and $s_n - s_{n-1} = O(n^{-1/2})$, then $s_n \rightarrow s$.

This is a special case of a general Tauberian theorem [5, Theorem 1].

LEMMA 3. For k a positive integer, $s_n \rightarrow s(M, 1/(n+1), k)$ if and only if $s_n \rightarrow s(R, \log(n+1), k)$.

This is due to Kwee [8, Lemma 4] who deduced the equivalence from Kuttner's result [7] that the methods $(M, 1/(n+1), k)$ and (R, L_n, k) are equivalent.

LEMMA 4 [3, LEMMA 2]. Let

$$c_n(x) := \alpha e^{-x} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},$$

and let $h_n := n - \frac{x}{\alpha}$, $\frac{1}{2} < \xi < \frac{2}{3}$, and $0 < \eta < 2\xi - 1$. Then, as $x \rightarrow \infty$,

$$(i) \sum_{n=N}^{\infty} c_n(x) \rightarrow 1;$$

$$(ii) \sum_{|h_n| > x^\xi} c_n(x) = O(e^{-x^\eta});$$

$$(iii) c_n(x) = \frac{\alpha}{\sqrt{2\pi x}} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \{1 + O(x^{3\xi-2})\} \text{ when } |h_n| \leq x^\xi.$$

LEMMA 5. Suppose that k is a positive integer, and that $s_n \rightarrow s(B, \alpha, \beta)$. Then $t_n^{(k)} \rightarrow s(B, \alpha, \beta)$.

PROOF. Since $\{1/L_n\}$ is totally monotone there is a non-decreasing function χ on $[0, 1]$ [6, Theorem 207] such that

$$(2) \quad \frac{1}{L_n} = \int_0^1 t^n d\chi(t);$$

moreover, since $1/L_n \rightarrow 0$, we must have $\chi(1) = \chi(1-)$.

Suppose as we may without loss of generality that $s = 0$ and, in view of Lemma 1, that $\beta \geq \max(1, \alpha)$. Let $x > 0$ and

$$\psi(x) := \sum_{n=0}^{\infty} s_n \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

Then

$$(3) \quad \psi(x^\alpha) = o(x^{1-\beta} e^x) \text{ as } x \rightarrow \infty.$$

We first prove that

$$(4) \quad B(x) := e^{-x} \sum_{n=0}^{\infty} t_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

We have

$$\phi(x) := \sum_{n=0}^{\infty} \frac{s_n}{n+1} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} = x^{\beta - \alpha - 1} \int_0^{x^\alpha} \psi(t) dt = \alpha x^{\beta - \alpha - 1} \int_0^x \psi(t^\alpha) t^{\alpha - 1} dt.$$

Hence, by (3),

$$(5) \quad \begin{aligned} \phi(x) &= O(x^{\beta - \alpha - 1}) + O(x^{\beta - \alpha - 1}) \int_1^{x/2} t^{\alpha - \beta} e^t dt + x^{\beta - \alpha - 1} \int_{x/2}^x o(t^{\alpha - \beta} e^t) dt \\ &= O(x^{\beta - \alpha - 1}) + O(x^{\beta - \alpha - 1} e^{x/2}) + o(x^{\beta - \alpha - 1} (x/2)^{\alpha - \beta} e^x) \\ &= o\left(\frac{e^x}{x}\right) \text{ as } x \rightarrow \infty. \end{aligned}$$

Next, by (2),

$$\begin{aligned} B(x) &= e^{-x} \sum_{n=0}^{\infty} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \int_0^1 t^{\alpha n} d\chi(t^{\alpha n}) \sum_{r=0}^n \frac{s_{n-r}}{n-r+1} \\ &= x^{\beta - 1} e^{-x} \int_0^1 d\chi(t^\alpha) \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} \frac{(xt)^{\alpha n}}{\Gamma(\alpha n + \beta)} \frac{s_{n-r}}{n-r+1} \\ &= x^{\beta - 1} e^{-x} \int_0^1 d\chi(t^\alpha) \sum_{r=0}^{\infty} (xt)^{1-\beta} \sum_{n=0}^{\infty} \frac{(xt)^{\alpha n + \alpha r + \beta - 1}}{\Gamma(\alpha n + \alpha r + \beta)} \frac{s_n}{n+1}, \end{aligned}$$

the inversions in the order of operations, here and subsequently, being justified by absolute convergence. Since

$$\frac{(xt)^{\alpha n + \alpha r + \beta - 1}}{\Gamma(\alpha n + \alpha r + \beta)} = \frac{1}{\Gamma(\alpha r) \Gamma(\alpha n + \beta)} \int_0^{xt} (xt - u)^{\alpha r - 1} u^{\alpha n + \beta - 1} du$$

when $r > 0$, it follows that

$$\begin{aligned} B(x) &= x^{\beta - 1} e^{-x} \int_0^1 d\chi(t^\alpha) (xt)^{1-\beta} \left(\phi(xt) + \int_0^{xt} \phi(u) du \sum_{r=1}^{\infty} \frac{(xt - u)^{\alpha r - 1}}{\Gamma(\alpha r)} \right) \\ &= x^{\beta - 1} e^{-x} \int_0^1 d\chi(t^\alpha) (xt)^{1-\beta} \left(\phi(xt) + \int_0^{xt} E(xt - u) \phi(u) du \right), \end{aligned}$$

where

$$(6) \quad E(x) := \sum_{r=1}^{\infty} \frac{x^{\alpha r-1}}{\Gamma(\alpha r)} \sim \frac{e^x}{\alpha} \text{ as } x \rightarrow \infty,$$

by Lemma 4(i). Hence

$$(7) \quad \begin{aligned} B(x) &= x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha)(xt)^{1-\beta} \left(\phi(xt) + x \int_0^t E(xt-xu)\phi(xu) du \right) \\ &= x^{\beta-1} e^{-x} \int_0^1 (xt)^{1-\beta} \phi(xt) d\chi(t^\alpha) \\ &\quad + x^\beta e^{-x} \int_0^1 (xt)^{1-\beta} \phi(xt) dt \int_t^1 (u/t)^{1-\beta} E(xu-xt) d\chi(u^\alpha). \end{aligned}$$

Now let

$$F(x) := \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)},$$

so that $F(x)$ is value of $\phi(x)$ when $s_n \equiv 1$. Then

$$F(x) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{\alpha n + \beta}{n+1} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta+1)} \sim \frac{e^x}{x} \text{ as } x \rightarrow \infty,$$

by the regularity of $(B, \alpha, \beta + 1)$. Hence, by (5),

$$\phi(x) = o(F(x)) \text{ as } x \rightarrow \infty$$

and so, given $\epsilon > 0$, there is an $x_0 > 0$ such that

$$|\phi(x)| \leq \epsilon F(x) \text{ for } x \geq x_0.$$

Further, replacing $\phi(xt)$ by $F(xt)$ in (7) yields a $B(x)$ which tends to $1/\alpha$ as $x \rightarrow \infty$ and, since $\beta \geq 1$, to 0 as $x \rightarrow 0+$, and hence this $B(x)$ is dominated by a constant M for all $x > 0$. Thus the contribution to (7) of the parts of the integrals over the range $x_0/x \leq t \leq 1$ is in modulus less than ϵM for all $x > 0$. Since ϵ can be taken arbitrarily small, in order to establish (4) it suffices to show that the contribution to (7) of the parts of the integrals over the range $0 < t < x_0/x$ tends to 0 as $x \rightarrow \infty$.

Since $v^{1-\beta} \phi(v)$ is bounded for $0 < v \leq x_0$, it follows that

$$x^{\beta-1} e^{-x} \int_0^{x_0/x} (xt)^{1-\beta} \phi(xt) d\chi(t^\alpha) = O\left(x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha)\right) = o(1) \text{ as } x \rightarrow \infty.$$

Further, by (6) and because $\beta \geq 1$,

$$\begin{aligned} &x^\beta e^{-x} \int_0^{x_0/x} (xt)^{1-\beta} \phi(xt) dt \int_t^1 (u/t)^{1-\beta} E(xu-xt) d\chi(u^\alpha) \\ &= x^\beta e^{-x} O\left(\int_0^{x_0/x} dt \int_t^{x_0/x} e^{xu-xt} d\chi(u^\alpha) + \int_0^{x_0/x} dt \int_{x_0/x}^1 (u/t)^{1-\beta} e^{xu-xt} d\chi(u^\alpha)\right) \\ &= x^\beta e^{-x} O\left(e^{x_0} \int_0^{x_0/x} d\chi(u^\alpha) \int_0^u dt + \int_{x_0/x}^1 u^{1-\beta} e^{xu} d\chi(u^\alpha) \int_0^{x_0/x} t^{\beta-1} e^{-xt} dt\right) \\ &= O(x^{\beta-1} e^{-x}) + O\left(e^{-x} \int_{x_0/x}^1 u^{1-\beta} e^{xu} d\chi(u^\alpha) \int_0^{x_0} t^{\beta-1} e^{-t} dt\right) \\ &= o(1) + O\left(\int_{x_0/x}^{1/2} u^{1-\beta} e^{xu-x} d\chi(u^\alpha) + \int_{1/2}^1 u^{1-\beta} e^{xu-x} d\chi(u^\alpha)\right) \\ &= o(1) + O(x^{\beta-1} e^{-x/2}) + o(1) = o(1) \text{ as } x \rightarrow \infty, \end{aligned}$$

the final integral tending to 0 by the Lebesgue-Stieltjes theorem on dominated convergence since, for $1/2 \leq u < 1$, $u^{1-\beta} > u^{1-\beta} e^{xu-x} \rightarrow 0$ as $x \rightarrow \infty$, and $\chi(u^\alpha) \rightarrow \chi(1)$ as $u \rightarrow 1-$.

This establishes the case $k = 1$ of the lemma. Applying this case $k - 1$ times, we obtain the required result. ■

LEMMA 6. Suppose that $s_n \rightarrow s(B, \alpha, \beta)$ and that (1) holds with p a positive integer. Then

$$t_n^{(k)} = O((n^{1/2} \log n)^{p-k}) \text{ for } k = 1, 2, \dots, p.$$

PROOF. Assume again that $s = 0$. Let $x > 0$, $\frac{1}{2} < \xi < \frac{2}{3}$, $0 < \eta < 2\xi - 1$,

$$h_n =: n - \frac{x}{\alpha}, \quad m := \left[\frac{x}{\alpha} \right], \quad \text{and } B_n := \sum_{r=0}^n \frac{s_r}{r+1}.$$

Then

$$(8) \quad L_n \sim \log n \text{ and } B_n = O((n^{1/2} \log n)^p \log n)$$

and, by Lemma 5,

$$(9) \quad T(x) := e^{-x} \sum_{n=0}^{\infty} t_n \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = o(1) \text{ as } x \rightarrow \infty.$$

Write

$$(10) \quad \begin{aligned} T(x) &= e^{-x} \sum_{n=0}^{\infty} (B_n - B_m) \frac{x^{\alpha n+\beta-1}}{L_n \Gamma(\alpha n+\beta)} + e^{-x} B_m \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta-1}}{L_n \Gamma(\alpha n+\beta)} \\ &=: T_1(x) + T_2(x), \end{aligned}$$

$$(11) \quad T_1(x) = e^{-x} \left(\sum_{h_n < -x^\xi} + \sum_{|h_n| \leq x^\xi} + \sum_{h_n > x^\xi} \right) =: S_1(x) + S_2(x) + S_3(x).$$

By (8) and Lemma 4(ii), as $x \rightarrow \infty$,

$$(12) \quad S_1(x) = O\left(m^{p/2} (\log m)^{p+1} e^{-x} \sum_{h_n < -x^\xi} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right) = O(e^{-x^\eta}),$$

and

$$(13) \quad \begin{aligned} S_3(x) &= O\left(e^{-x} \sum_{h_n > x^\xi} (n^{1/2} \log n)^p \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right) \\ &= O\left(e^{-x} \sum_{h_n > x^\xi} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta-p)}\right) = O(e^{-x^\eta}). \end{aligned}$$

By (8) and Lemma 4(iii), as $x \rightarrow \infty$,

$$\begin{aligned}
 S_2(x) &= O\left(e^{-x} \sum_{|h_n| \leq x^\epsilon} (|h_n| + 1)x^{(p-2)/2}(\log x)^p \frac{x^{\alpha n + \beta - 1}}{L_n \Gamma(\alpha n + \beta)}\right) \\
 &= O\left(e^{-x} x^{(p-2)/2}(\log x)^{p-1} \sum_{|h_n| \leq x^\epsilon} (|h_n| + 1) \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}\right) \\
 (14) \quad &= O\left(x^{(p-2)/2}(\log x)^{p-1} \sum_{|h_n| \leq x^\epsilon} (|h_n| + 1) \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right) \\
 &= O\left(x^{(p-3)/2}(\log x)^{p-1} \int_{-\infty}^{\infty} (|t| + 1) \exp\left(-\frac{\alpha^2 t^2}{2x}\right) dt\right) \\
 &= O\left(x^{(p-1)/2}(\log x)^{p-1}\right) + O\left(x^{(p-2)/2}(\log x)^{p-1}\right) \\
 &= O\left(x^{1/2} \log x^{p-1}\right).
 \end{aligned}$$

It follows from (10), (11), (12), (13) and (14) that

$$(15) \quad T_1(x) = O\left(x^{1/2} \log x^{p-1}\right) \text{ as } x \rightarrow \infty.$$

Next,

$$(16) \quad T_2(x) = e^{-x} B_m \left(\sum_{h_n < -x^\epsilon} + \sum_{|h_n| \leq x^\epsilon} + \sum_{h_n > x^\epsilon} \right) =: V_1(x) + V_2(x) + V_3(x).$$

By (8) and Lemma 4(ii), as $x \rightarrow \infty$,

$$\begin{aligned}
 (17) \quad V_1(x) + V_3(x) &= e^{-x} t_m O\left(\sum_{h_n < -x^\epsilon} \frac{x^{\alpha n + \beta - 1} \log x}{\Gamma(\alpha n + \beta)} + \sum_{h_n > x^\epsilon} \frac{x^{\alpha n + \beta - 1} \log x}{\Gamma(\alpha n + \beta)}\right) \\
 &= O(t_m e^{-x^\eta}).
 \end{aligned}$$

Finally, as $x \rightarrow \infty$,

$$(18) \quad V_2(x) = t_m e^{-x} \sum_{h_n > x^\epsilon} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \frac{L_m}{L_n} = t_m \left(\frac{1}{\alpha} + o(1)\right),$$

since L_m/L_n in the above sum lies between $L_m/L_{\lfloor x/\alpha + x^\epsilon \rfloor}$ and $L_m/L_{\lfloor x/\alpha - x^\epsilon \rfloor}$ each of which tends to 1 as $x \rightarrow \infty$ and, by Lemma 4(i) and (ii),

$$\lim_{x \rightarrow \infty} e^{-x} \sum_{h_n > x^\epsilon} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} = \lim_{x \rightarrow \infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} = \frac{1}{\alpha}.$$

It follows from (16), (17) and (18) that

$$T_2(x) = t_m \left(\frac{1}{\alpha} + o(1)\right) \text{ as } x \rightarrow \infty,$$

and hence from (9) and (15) that

$$t_m = O\left(x^{1/2} \log x^{p-1}\right) \text{ as } x \rightarrow \infty.$$

Taking $x = \alpha n$, we get

$$t_n = t_n^{(1)} = O\left(n^{1/2} \log n^{p-1}\right).$$

When $p \geq 2$ we can replace t_n by $t_n^{(2)}$ in (9) and argue as above to obtain

$$t_n^{(2)} = O\left(n^{1/2} \log n^{p-2}\right).$$

The proof can now be completed by induction in the obvious way. ■

3. Proof of the theorem. By Lemma 6,

$$t_n^{(p)} - t_{n-1}^{(p)} = \frac{t_n^{p-1}}{(n+1)L_{n-1}} - \frac{1}{(n+1)L_n L_{n-1}} \sum_{r=0}^n \frac{t_r^{(p-1)}}{r+1} = O(n^{-1/2}).$$

Hence, by Lemma 5 and Lemma 2,

$$t_n^{(p)} \rightarrow s \text{ as } n \rightarrow \infty,$$

and so, by Lemma 3,

$$s_n \rightarrow s(R, \log(n+1), p). \quad \blacksquare$$

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