Solutions to Selected Exercises in Chapter 5

Exercises from Section 5.1

5.1.1. (a) Suppose $A \subset X$ is not empty. Observe that $(A^\circ)_\circ$ is a closed balance convex set containing A so it contains the closed balanced convex hull of A. Now suppose $x \in (A^\circ)_\circ$ but is not in the closed balanced hull of A. By the basic separation theorem (4.1.12), choose $\phi \in X^*$ such that $\langle \phi, x\rangle > 1 > \sup_A \phi$. Then $\phi \in A^\circ$, but this implies $x \notin (A^\circ)_\circ$ which is a contradiction.
(b) This is similar to (a). Suppose $B \subset X^*$ is not empty. Observe that $(B_\circ)^\circ$ is a weak*-closed balance convex set containing B so it contains the weak*-closed balanced convex hull of B. Suppose $\phi \in (B_\circ)^\circ$ but is not in the weak*-closed balanced hull of B. By the weak*-separation theorem (4.1.12), choose $x \in X$ such that $\langle \phi, x\rangle > 1 > \sup_B x$. Then $x \in B_\circ$, but this implies $\phi \notin (B_\circ)^\circ$ which is a contradiction. $\square$

5.1.2. (a) $\Rightarrow$ (c): Suppose (c) is not true. Choose a supporting functional $\phi \in S_{X^*}$ of x. Then there exist $\phi_n \in B_{X^*}$ such that $\phi_n(x) \to 1$, and $\epsilon > 0$ and $h \in B_{X^*}$ such that $(\phi - \phi_n)(h) \geq \epsilon$. Now choose $t_n \to 0^+$ such that $1 - \phi_n(x) \leq \frac{t_n\epsilon}{2}$. Then

$$\begin{aligned}\|x + t_n h\| + \|x - t_n h\| - 2\|x\| &\geq \phi_n(x + t_n h) + \phi(x - t_n h) - 2\|x\| \\ &\geq \phi_n(x) + \phi(x) + t_n(\phi_n - \phi)(h) - 2 \\ &\geq t_n\epsilon + \phi_n(x) - 1 \geq \frac{t_n\epsilon}{2}.\end{aligned}$$

Consequently, $\|\cdot\|$ is not Gâteaux differentiable at x by Proposition 5.1.3(b).
(c) $\Rightarrow$ (b): Let $\Lambda_n, \phi_n, x_n, y_n$ be as in (b). Choose a supporting functional $\phi \in S_{X^*}$ with $\phi(x) = 1$. Now $\phi_n(x) \to 1$ and $\Lambda_n(x) \to 1$ since $x_n \to x$ and $y_n \to x$. According to (c) $\phi_n \to_{w^*} \phi$ and $\Lambda_n \to_{w^*} \phi$ and so $(\phi_n - \Lambda_n) \to_{w^*} 0$ as desired.
(b) $\Rightarrow$ (a): Suppose $\|\cdot\|$ is not Gâteaux differentiable at x. Then for $\phi \in S_{X^*}$ with $\phi(x) = 1$, there exist $t_n \to 0^+$, $h \in S_X$ and $\epsilon > 0$ such that

$$\|x + t_n h\| - \|x\| - \phi(t_n h) > \epsilon t_n$$

for all n. Choose $\phi_n \in S_{X^*}$ so that $\phi_n(x + t_n h) = \|x + t_n h\|$. The previous inequality implies $\phi_n(t_n h) - \phi(t_n h) > \epsilon t_n$ and so $\phi_n \not\to \phi$ which shows (b) is not true. $\square$

5.1.3. This follows from Šmulian's theorems (4.2.10) and (4.2.11) with the observation $\phi \in S_{X^*}$, $\phi \in \partial_\epsilon \|\cdot\|$ if and only if $\phi(x) \geq 1 - \epsilon$. $\square$

5.1.4. Suppose $|||x|||_1 + |||y|||_1 = |||x + y|||_1 = 2$ where $|||x|||_1 = |||y|||_1 = 1$. Then $\|Tx\|_Y + \|Ty\|_Y = \|Tx + Ty\|_Y$. Because $\|\cdot\|_Y$ is strictly convex and T is one-to-one, Fact 5.1.9 implies $Tx = \lambda Ty$ for some $\lambda > 0$. Then $x = \lambda y$ and because $|||x|||_1 = |||y|||_1 = 1$, this implies $x = y$, and so $|||\cdot|||_1$ is strictly convex as desired. $\square$

5.1.5. Suppose $\|\cdot\|$ is a dual norm on X^* that is Fréchet differentiable at $\phi \in S_{X^*}$. Choose $x^{**} \in S_{X^{**}}$ such that $\langle x^{**}, \phi\rangle = 1$. Now choose $x_n \in B_X$ so that $\phi(x_n) \to 1$. By Smulian's theorem (5.1.4), $x_n \to x^{**}$ so $x^{**} \in S_X$. Now if $x \in S_X$ is such that $\phi(x) = 1$, then $x = x^{**}$ and the statement follows.

Conversely, suppose $\phi \in S_{X^*}$, and $x \in S_X$ are such that $\phi(x) = 1$ and $\|x_n - x\| \to 0$ whenever $x_n \in B_X$ are such that $\phi(x_n) \to 1$. Suppose $\|\cdot\|$ is not Fréchet differentiable at ϕ. Then there are $h_n \in S_{X^*}$, $t_n \to 0^+$ and $\epsilon > 0$ such that

$$\|\phi + t_n h_n\| + \|\phi - t_n h_n\| - 2 > \epsilon t_n$$

for all n. Choose $x_n \in S_X$ so that $\langle \phi + t_n h_n, x_n \rangle > \|x + t_n h_n\| - t_n\epsilon/3$ and $y_n \in S_X$ so that $\langle \phi - t_n h_n, x_n \rangle > \|x - t_n h_n\| - t_n\epsilon/3$. Then $\phi(x_n) \to 1$ and $\phi(y_n) \to 1$, but $\langle x_n - y_n, t_n h_n \rangle > \epsilon t_n/3$ and so $x_n \not\to x$ and $y_n \not\to x$ which is a contradiction. □

5.1.6. Let $x \in S_X$ and choose a supporting functional $f \in S_{X^*}$ so that $f(x) = 1$. Suppose $f_n \in B_{X^*}$ satisfies $f_n(x) \to 1$. Then, $\|f + f_n\| \geq (f + f_n)(x) \to 2$. Because $\|\cdot\|$ is locally uniformly convex, $\|f_n - f\| \to 0$. According to Smulian's theorem (5.1.4), $\|\cdot\|$ is Fréchet differentiable at x. □

5.1.7. (a) Suppose $x := (x_i)$ is such that $x_i \neq 0$ for all $i \in \mathbb{N}$. Then the unique supporting functional in $\ell_\infty(\mathbb{N})$ is $\Lambda := (\operatorname{sign} x_i)_{i=1}^\infty$. Thus $\|\cdot\|_1$ is Gâteaux differentiable at x by Corollary 5.1.7. Conversely, suppose $x := (x_i)$ and $x_{i_0} = 0$, then there are infinitely many support functionals in $\ell_\infty(\mathbb{N})$ because the i_0-th coordinate can be any number whose absolute value does not exceed 1.
(b) We need consider only the points $x = (x_i) \in S_{\ell_1}$ of Gâteaux differentiability, let $y^n \in S_{\ell_\infty}$ where $y_i^n = \operatorname{sign}(x_i)$, $i = 1, \ldots, n$, and $y_i = 0$ otherwise. Then $y^n(x) = 1 - \epsilon_n$ where $\epsilon_n = \sum_{i=n+1}^\infty |x_i|$. Now $y_n(x) \to 1$, but $\|y_n - y_{n+1}\|_\infty = 1$ and so (y_n) does not converge in norm. According to Šmulian's theorem (5.1.4), $\|\cdot\|_1$ is not Fréchet differentiable at x.
(c) Suppose $x = (x_\gamma)_{\gamma \in \Gamma}$. Then $x_{\gamma_0} = 0$ for some $\gamma_0 \in \Gamma$, and so there fails to be a unique supporting functional for x. □

5.1.10. Let $\{x_n\}_{n=1}^\infty \subset S_X$ be dense. For each n, choose $f_n \in S_{X^*}$ such that $f_n(x_n) = 1$ (Remark 4.1.16). Because the norm-attaining functionals are dense in S_{X^*} by the Bishop-Phelps theorem (4.3.4), it suffices to show that $\overline{\{f_n\}_{n=1}^\infty}$ contains the norm-attaining functionals in S_{X^*} (this will show S_{X^*} is separable). Now let $f \in S_X$ be a norm-attaining functional, say $f(x) = 1$. Choose $x_{n_k} \to x$. Now $f_{n_k}(x_{n_k}) = 1$, and so $f_{n_k}(x) \to 1$. According to Šmulian's theorem (5.1.4), $f_{n_k} \to f$. Thus, $\overline{\{f_n\}_{n=1}^\infty}$ contains the norm-attaining functionals as desired. □

5.1.15. (a) Suppose $\|\cdot\|$ is not uniformly convex. Then we choose $(x_n), (y_n) \subset B_X$ such that $\|x_n + y_n\| \to 2$ but $\|x_n - y_n\| \to 0$. Passing to a subsequence and using compactness, we have $x_{n_k} \to \bar{x}$, $y_{n_k} \to \bar{y}$, $\|\bar{x} + \bar{y}\| = 2$ and $\|\bar{x} - \bar{y}\| = 0$. Hence $\|\cdot\|$ is not strictly convex.
(b) Similarly, suppose $\|\cdot\|$ is not uniformly smooth. This means its derivative (if it exists) is not uniformly continuous on S_X, and hence not continuous on S_X. Thus $\|\cdot\|$ is not Gâteaux differentiable (since a differentiable convex function on a Euclidean space has continuous derivative).
(c) Suppose $x_0 \in S_X$ an exposed point of B_X. Let $\phi \in S_{X^*}$ be an exposing functional. Suppose $(x_n) \subset B_X$ and $\phi(x_n) \to 1$, and, again, by compactness, we may assume $x_n \to \bar{x} \in B_X$. Then $\phi(\bar{x}) = 1$ Thus $\bar{x} = x_0$. From this, we may deduce that x_0 is strongly exposed by ϕ. □

5.1.16. The respective cases follow by using characterizations in Fact 5.1.9(c), Fact 5.1.12(b) and Fact 5.1.17(b). We illustrate this in the uniformly convex case. Suppose $(x_n), (y_n)$ are bounded sequences such that

$$2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$$

Then Fact 5.1.8 implies

$$2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2 \to 0$$

and the uniform convexity of $\|\cdot\|_1$ as characterized in Fact 5.1.17(b) implies $\|x_n - y_n\|_1 \to 0$ and so $\|x_n - y_n\| \to 0$. Using Fact 5.1.17(b) we deduce $|\!|\!|\cdot|\!|\!|$ is uniformly convex. $\square$

5.1.26. Define $|\!|\!|\cdot|\!|\!|$ by

$$|\!|\!|x|\!|\!| := \max\left\{\frac{1}{2}\|x\|, |x_1|\right\} + \sqrt{\sum_{n=1}^{\infty} \frac{x_i^2}{2^i}}$$

The $|\!|\!|\cdot|\!|\!|$ is strictly convex by Proposition 5.1.10(a), and hence the dual norm is Gâteaux differentiable. Now $e_1/|\!|\!|e_1|\!|\!|$ is exposed by $\phi := e_1|\!|\!|e_1|\!|\!|$ but is not strongly exposed by ϕ since $|\!|\!|e_1 + e_n|\!|\!| \to |\!|\!|e_1|\!|\!|$, $\phi(e_1 + e_n) \to \phi(e_1)$ but $|\!|\!|(e_1 + e_n) - e_1|\!|\!| > 1/2$ for all n. The dual norm will not be Fréchet differentiable at ϕ, since ϕ does not strongly expose $e_1/|\!|\!|e_1|\!|\!|$. $\square$

Exercises from Section 5.2

5.2.1. From the definition it follows that a strongly exposed point of f is exposed. Conversely, suppose x_0 is an exposed point of f. Choose $\phi \in \partial f(x_0)$ so that $f - \phi$ attains its strict minimum at x_0. Now suppose x_0 is not a strongly exposed point of $f - \phi$. Then we can find a sequence $(x_n) \subset E$ so that $(f - \phi)(x_n) \to (f - \phi)(x_0)$ but $\|x_n - x_0\| \geq \epsilon > 0$ for all n. Let $0 < \lambda_n \leq 1$ be chosen so that $\lambda_n\|x_n - x_0\| = \epsilon$. Then set

$$y_n := x_0 + \lambda_n(x_n - x_0) = \lambda_n x_n + (1 - \lambda_n)x_0.$$

Using the convexity of $f - \phi$ we have

$$(f - \phi)(x_0) < (f - \phi)(y_n) \leq \lambda_n(f - \phi)(x_n) + (1 - \lambda_n)(f - \phi)(x_0) \to (f - \phi)(x_0).$$

Now (y_n) is a bounded sequence, so passing to a subsequence we have $y_{n_k} \to \bar{y}$ for some $\bar{y} \in E$ and $\bar{y} \neq x_0$. Using the lower semicontinuity of $f - \phi$, we obtain

$$(f - \phi)(\bar{y}) \leq \liminf_k (f - \phi)(y_{n_k}) = (f - \phi)(x_0)$$

which is a contradiction with the fact that $f - \phi$ attains its strict minimum at x_0. Thus x_0 is a strongly exposed point of $f - \phi$.

Consider the function $g(x, y) := x^2$ when $x > 0$, $g(0, 0) := 0$ and $g(x, y) := +\infty$ otherwise. Now let ϕ denote the 0 functional on $\mathbb{R}^2$. Then $g - \phi$ is convex and attains its strict minimum at 0. Therefore, g is exposed by ϕ at $(0, 0)$. However, $(g - \phi)(n^{-1}, 1) \to (g - \phi)(0, 0)$ and so g is not strongly exposed by ϕ at $(0, 0)$. $\square$

5.2.2. (a) Suppose f is Tikhonov well-posed with minimum at $\bar{x}$. This says $f - \phi$ attains its strong minimum at $\bar{x}$ where ϕ is the zero-functional. According to Exercise 5.2.6 f is coercive, and so by the Moreau-Rockafellar theorem (4.4.10), f^* is continuous at 0. Now $0 \in \partial f^(\bar{x})$ and so $\bar{x} \in \partial f^*(0)$ by Proposition 4.4.5(b). Suppose $x_n \in \partial_{\epsilon_n} f^*(0)$ where $\epsilon_n \to 0$, then $0 \in \partial_{\epsilon_n} f(x_n)$ by Proposition 4.4.5(b). Then $\|x_n - \bar{x}\| \to 0$ by the equivalence of (c) and (d) in Theorem 5.2.3. According to Šmulian's theorem (Exercise 4.2.10), f^* is Fréchet differentiable at 0 with $\nabla f^*(0) = \bar{x}$.

The converse was more generally shown in (e) $\Rightarrow$ (d) of Theorem 5.2.3 assuming f is lower semicontinuous. However, when f is also convex one doesn't need the more difficult Exercise 4.4.2. Indeed, suppose additionally f is lower semicontinuous and convex, and that f^* is Fréchet differentiable at 0 with $\bar{x} = \nabla f^*(0)$. According to Proposition 4.4.5(a), $0 \in \partial f(\bar{x})$. Now suppose $f(x_n) \leq f(\bar{x}) + \epsilon_n$ where $\epsilon_n \to 0^+$. Then by the equivalence of (c) and (d) in Theorem 5.2.3 $0 \in \partial_{\epsilon_n} f(x_n)$. Proposition 4.4.5(b) implies $x_n \in \partial_{\epsilon_n} f^*(0)$. Šmulian's theorem (4.2.10) then shows $x_n \to 0$ as desired.

(b) When f is lower semicontinuous and convex, part (a) shows $f - \phi_0$ attains its strong minimum at x_0 if and only if $(f - \phi_0)^*$ is Fréchet differentiable at 0 with derivative x_0 which occurs if and only if f^* is Fréchet differentiable at ϕ_0 with Fréchet derivative x_0. Thus (a) and (e) are equivalent in Theorem 5.2.3 when f is a proper lower semicontinuous convex function.

(c) This follows from Proposition 5.2.4(a) and expressing Tikhonov well-posedness in terms of strongly exposed points by equivalence of (a) and (c) in Theorem 5.2.3. $\square$

5.2.3. Let $\phi \in \partial f(\bar{x})$. Then Proposition 4.4.5(a) ensures that $\bar{x} \in \partial f^*(\phi)$. Because f^* is Fréchet differentiable at ϕ, this implies $\bar{x}$ is the Fréchet derivative $\nabla f^*(\phi)$. Now $x_n \to \bar{x}$ weakly implies $\phi(x_n) \to \phi(\bar{x})$ and $f(x_n) \to f(\bar{x})$ was given. Therefore,

$$(f - \phi)(x_n) \to (f - \phi)(\bar{x}).$$

According to Theorem 5.2.3, $\|x_n - \bar{x}\| \to 0$ as desired.

Certainly it was needed that $x_n \to \bar{x}$ weakly, otherwise we choose $f := \frac{1}{2}\|\cdot\|^2$ on ℓ_2. Then $f^* = f$, and f^* is Fréchet differentiable. However, $f(e_n) = f(e_1)$ for all n, but e_n does not converge weakly to e_1. $\square$

5.2.4. (a) Suppose x_0 exposes f^* at ϕ_0, then $x_0 \in \partial f^*(\phi_0)$ by Proposition 5.2.2. According to Proposition 4.4.5(a), $\phi_0 \in \partial f(x_0)$, and then the Fenchel–Young equality (Proposition 4.4.1) implies $f^*(\phi_0) - \langle x_0, \phi_0 \rangle = -f(x_0)$. The assumption in the exercise then implies $f^*(\phi_n) - \langle x_0, \phi_n \rangle \to -f(x_0)$. So let $\epsilon_n \to 0^+$ be chosen so that $f^*(\phi_n) - \langle x_0, \phi_n \rangle < -f(x_0) + \epsilon$ for each $n \in \mathbb{N}$. The defintion of f^* then ensures $\phi_n(x) - f(x) - \phi_n(x_0) \leq -f(x_0) + \epsilon_n$ for all $x \in X$. Thus $\phi_n \in \partial_{\epsilon_n} f(x_0)$.

For (b), consider the function $f(t) := t^2$ if $t \leq 1$, and $f(t) := 2t - 1$ if $t \geq 1$; see Figure 5.3. $\square$

5.2.5. Suppose $f^* : X^* \to (-\infty, +\infty]$ is a proper, weak*-lower semicontinuous convex function that is exposed at $\phi_0 \in X^*$ by $x_0 \in X$ and that

$$f^*(\phi_n) - \langle x_0, \phi_n \rangle \to f^*(\phi_0) - \langle x_0, \phi_0 \rangle. \tag{1}$$

(a) Let $(\phi_n)_{n=1}^\infty \subset X^*$ be bounded. Now suppose by way of contradiction $\phi_n \not\to_{w^*} \phi_0$. Because $(\phi_n)_{n=1}^\infty$ is bounded, it then has a weak*-convergent subnet (ϕ_{n_α}) that converges to $\bar{\phi} \neq \phi_0$. Now,

$$\begin{aligned} f^*(\phi_0) - \langle x_0, \phi_0 \rangle &= \limsup_n f^*(\phi_n) - \langle x_0, \phi_n \rangle \quad \text{[by (1)]} \\ &\geq \liminf_\alpha f^*(\phi_{n_\alpha}) - \langle x_0, \phi_{n_\alpha} \rangle \\ &\geq f^*(\bar{\phi}) - \langle x_0, \bar{\phi} \rangle \quad \text{[since } f^* \text{ is w*-lsc]}. \end{aligned}$$

This constradicts that $f^* - x_0$ attains its minimum uniquely at ϕ_0.

(b) An example where (ϕ_n) is bounded and f^* is Lipschitz with (1) holding is as follows. Define $f^* : \ell_1 \to \mathbb{R}$ where $f((x_i)) := \sum 2^{-i}|x_i|$. Then f^* is exposed at 0 by the zero functional in c_0

since f attains its minimum uniquely at 0, but $f^*(ne_n) - \langle 0, ne_n\rangle = \frac{n}{2^n}$ and $ne_n \not\to_{w^*} 0$ where (e_n) is the standard basis of ℓ_1.
(c) Suppose f is continuous at x_0, where f^* is the conjugate of f. According to Exercise 5.2.4(a), the condition (1) implies $\phi_n \in \partial_{\epsilon_n} f(x_0)$ where $\epsilon_n \to 0$. Hence it is easy to check that (ϕ_n) must be bounded, because f is continuous at x_0.
Further Notes. We will say an exposed point ϕ of the function $h : X^* \to (-\infty, +\infty]$ is *w^*-exposed* by x^{**} if $\phi_n \to_{w^*} \phi$ whenever

$$(h - x^{**})(\phi_n) \to (h - x^{**})(\phi).$$

Then one extend this and the previous exercise to show: *Suppose $f : X \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, $x_0 \in X$ and f^* is exposed by x_0 at $\phi_0 \in X^*$. Then f^* is w^*-exposed by x_0 at ϕ_0 if and only if f is continuous at x_0.* The 'if' portion follows directly from Exercise 5.2.4(a) and part (c) of this exercise. For the 'only if' implication, suppose f is not continuous at x_0. By replacing f^* with $f^* - x_0$ and then shifting f^* we may suppose $f^*(0) = 0$ is the strict minimum of f^* and f is not continuous at 0. By the Moreau-Rockafellar dual theorem, f^* is not coercive because f is not continuous at 0. Thus we choose x_n^* with $\|x_n^*\| \to \infty$ but $f^*(x_n^*) \le N$ for some $N > 0$ and all n. By passing to a subsequence as necessary, we may assume $\|x_n^*\| > n^2$. Let $\phi_n = \dfrac{1}{n}x_n^*$. By the convexity of f^* we obtain

$$f^*(\phi_n) = f^*\left(\frac{n-1}{n}0 + \frac{1}{n}x_n^*\right) \le \frac{n-1}{n}f^*(0) + \frac{1}{n}f^*(x_n^*) \le \frac{N}{n}.$$

Thus we obtain

$$(f^* - 0)(\phi_n) \to (f^* - 0)(0)$$

but (ϕ_n) does not converge weak* to 0, because it is unbounded (Uniform boundedness principle) since a pointwise convergent sequence is pointwise bounded. □

5.2.6. (a) There exists $\delta > 0$ such that $(f - \phi)(u) \ge (f - \phi)(x) + \delta$ whenever $\|u - x\| = 1$ for otherwise we would choose u_n such that $\|u_n - x\| = 1$ and $(f - \phi)(u_n) \to (f - \phi)(x)$ but then we obtain the contradiction $\|u_n - x\| \to 0$ because $f - \phi$ attains its strong minimum at x. Now suppose $\|u - x\| = \alpha$ with $\alpha \ge n$. The convexity of $(f - \phi)$ now implies

$$\frac{1}{\alpha}(f - \phi)(u) + \left(1 - \frac{1}{\alpha}\right)(f - \phi)(x) \ge (f - \phi)\left(x + \frac{1}{\alpha}(u - x)\right) \ge (f - \phi)(x) + \delta.$$

Then $\alpha^{-1}(f-\phi)(u) \ge \alpha^{-1}(f-\phi)(x)+\delta$ and so $(f-\phi)(u) \ge (f-\phi)(x)+n\delta$ whenever $\|x-u\| \ge n$ where $n \in \mathbb{N}$. Consequently, if $\|u\| \to \infty$, $\|u - x\| \to \infty$, and so $(f - \phi)(u) \to \infty$. This shows $f - \phi$ is coercive.
(b) By part (a), $f^* - x_0$ is coercive, and by the Moreau-Rockafellar theorem (4.4.11) we deduce f^{**} is continuous at x_0. Because f is lower semicontinuous, Proposition 4.4.2(a) ensures that $f^{**}|_X = f$, and the conclusion follows. □

5.2.7. This is a proof of the equivalence of (a) and (b) in Theorem 5.2.3.
(a) ⇒ (b): Suppose $(x_0, f(x_0))$ is strongly exposed by $(\phi_0, -1)$, and that $(\phi_0 - f)(x_n) \to (\phi_0 - f)(x_0)$. Then

$$(\phi_0, -1)(x_n, f(x_n)) \to (\phi_0, -1)(x_0, f(x_0))$$

and (a) implies $\|(x_n, f(x_n)) - (x_0, f(x_0))\| \to 0$ which implies $\|x_n - x_0\| \to 0$.

(b) $\Rightarrow$ (a): Suppose that $\phi_0 - f$ has a strong maximum at x_0. Then by Proposition 5.2.2, $(\phi_0, -1)$ exposes $\operatorname{epi} f$ at $(x_0, f(x_0))$. Now if $(x_n, t_n) \in \operatorname{epi} f$ and $(\phi_0, -1)(x_n, t_n) \to (\phi_0, -1)(x_0, f(x_0))$, then $(\phi_0, -1)(x_n, f(x_n)) \to (\phi_0, -1)(x_0, f(x_0))$ since $f(x_n) \leq t_n$ for all n. Therefore,

$$(\phi_0 - f)(x_n) \to (\phi_0 - f)(x_0). \tag{2}$$

Now, (b) implies that $\|x_n - x_0\| \to 0$. Therefore $\phi_0(x_n) \to \phi_0(x_0)$; this with (2) implies $f(x_n) \to f(x_0)$. Therefore, $\|(x_n, f(x_n)) - (x_0, f(x_0))\| \to 0$ as desired. □

5.2.8. To see that Theorem 5.2.3(c) does not generally imply Theorem 5.2.3(e) for proper functions, let $f(t) = \min\{|t|, 1\}$. Then $f - 0$ attains a strong minimum at 0, but $f^* = \delta_{\{0\}}$ is not Fréchet differentiable at 0. Let $g(t) = |t|$ if $t \neq 0$ and

To see that Theorem 5.2.3(e) does not generally imply Theorem 5.2.3(e) for functions that are not lower semicontinuous, let $g(t) = |t|$ if $t \neq 0$ and $g(0) = +\infty$ (or simply $g(0) > 0$ will do). Then $g^* = \delta_{[-1,1]}$ so $\nabla g^*(0) = 0$ as a Fréchet derivative, but $g - 0$ does not attain its strong minimum at 0, and in fact does not attain its infimum. □

5.2.8 (a) For any norm, 0 is the only exposed point of $f(x) = \|x\|$ and, in fact, f is strongly exposed at 0 by the 0 functional. For any $u \neq 0$, any functional ϕ that exposes f at u would satisfy $\|\phi\| = 1$, and $\phi(u) = \|u\|$. Then

$$(f - \phi)(tu) = 0 = (f - \phi)(u) \quad \text{ for any } t \geq 0.$$

Thus u cannot be an exposed point of f.

(b) Observe that $\phi \in S_{X^*}$ strongly exposes x_0 if and only if $\phi(x_0) = 1$ and $x_n \to x_0$ whenever $\|x_n\| \to 1$ and $\phi(x_n) \to 1$. We know from the duality mapping that $\Lambda \in \partial\|x\|^2$ if and only if $\Lambda := 2\|x\|\phi_x$ where $\phi_x \in S_X$ and $\phi_x(x) = \|x\|$. (An elementary check of this is as follows. Suppose $\Lambda = 2\|x\|\phi_x$. Then

$$\begin{aligned} \Lambda(y) - \Lambda(x) &= 2\|x\|\phi_x(y) - 2\|y\|\phi_x(x) = 2\|x\|(\phi_x(y) - \phi_x(x)) \\ &\leq 2\|x\|(\|y\| - \|x\|) \leq (\|y\| + \|x\|)(\|y\| - \|x\|) = \|y\|^2 - \|x\|^2. \end{aligned}$$

Thus $\Lambda \in \partial\|x\|^2$. Conversely, considering $g(t) := \|tx\|^2$ we have $g'(1) = 2\|x\|$ so $\|\Lambda\| = 2\|x\|$ when $\Lambda \in \partial\|x\|^2$; moreover, since $\Lambda(y) \leq \Lambda(x)$ whenever $\|y\|^2 = \|x\|^2$ it is clear Λ attains its norm at x when $\Lambda \in \partial\|x\|^2$.)

Then $\phi \in \partial f(x_0)$ if and only if $\|\phi/2\| = 1$ and $\phi(x_0)/2 = 1$, and $(\phi - f)(x_n) \to (\phi - f)(x_0)$ implies $\|x_n\| \to 1$. Thus $(\phi - f)(x_n) \to (\phi - f)(x_0)$ implies $\phi(x_n) \to \phi(x_0)$ and $\|x_n - x_0\| \to 0$. □

5.2.10. This provides of a proof of Proposition 5.2.4(b).

Suppose f^* is exposed at ϕ by $x \in X$. Then $x \in \partial f^*(\phi)$ and so Proposition 4.4.5(a) implies $\phi \in \partial f(x)$. Now Proposition 5.2.2 implies $\partial f(x) = \{\phi\}$ and so f is Gâteaux differentiable at x according to Corollary 4.2.5.

Conversely, suppose f is Gâteaux differentiable at x with $f'(x) = \phi$. Then $\partial f(x) = \{\phi\}$. Then $x \in \partial f^*(\phi)$ according to Proposition 4.4.5(a), and moreover, $x \notin \partial f^*(\Lambda)$ for $\Lambda \neq \phi$ (or else $\Lambda \in \partial f(x)$). Therefore, Proposition 5.2.2 implies that f^* is exposed by x at ϕ. □

Exercises from Section 5.3

5.3.1. Suppose that it is known $\|x_n - x_0\| \to 0$ whenever

$$\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f\left(\frac{x_n + x_0}{2}\right) \to 0 \tag{3}$$

and (x_n) is a bounded sequence in the domain of f. Suppose there exists an unbounded sequence $(x_n) \subset \operatorname{dom} f$ for which (3) holds, but $\|x_n - x_0\| \not\to 0$. By shifting f, we may assume $x_0 = 0$, and $f(0) = 0$. Further, by passing to a subsequence, we may assume $\|x_n\| \to \infty$, and in particular $\|x_n\| > 2$ for all n. Now let $t_n := \|x_n\|$, and let $u_n = \frac{2}{t_n}x_n$. Since $\|u_n - 0\| = 2$, we know

$$\frac{1}{2}f(0) + \frac{1}{2}f(u_n) - f\left(\frac{0+u_n}{2}\right) \not\to 0.$$

So we can find $\epsilon > 0$ such that

$$f\left(\frac{u_n}{2}\right) \le \frac{1}{2}f(0) + \frac{1}{2}f(u_n) - \epsilon \le \frac{1}{t_n}f(x_n) - \epsilon,$$

where we used $f(u_n) \le \frac{t_n-2}{t_n}f(0) + \frac{2}{t_n}f(x_n) = \frac{2}{t_n}f(x_n)$ for the last inequality. Now we compute

$$\begin{aligned}
f\left(\frac{0+x_n}{2}\right) &= f\left(\frac{t_n}{2t_n-2}\cdot\frac{1}{t_n}x_n + \frac{t_n-2}{2t_n-2}x_n\right)\\
&\le \frac{t_n}{2t_n-2}f\left(\frac{1}{t_n}x_n\right) + \frac{t_n-2}{2t_n-2}f(x_n)\\
&\le \frac{t_n}{2t_n-2}\left(\frac{1}{t_n}f(x_n)-\epsilon\right) + \frac{t_n-2}{2t_n-2}f(x_n)\\
&= \frac{1}{2}f(x_n) - \frac{t_n}{2t_n-2}\epsilon \le \frac{1}{2}f\left(\frac{0+x_n}{2}\right) - \frac{\epsilon}{2}.
\end{aligned}$$

This contradicts (3) and completes the proof. □

Exercises from Section 5.4

5.4.1. (a) First, $f'(t) = pt^{p-1}$ when $t \ge 0$, and $f'(t) = -p|t|^{p-1}$ when $p < 0$. If s, t both have the same sign, then Lemma 5.4.4 implies $|f'(t) - f'(s)| \le p|t-s|^{p-1}$. If $s < 0 < t$, then

$$|f'(t) - f'(s)| = p|t|^{p-1} + p|s|^{p-1} \le 2p|t-s|^{p-1}$$

and so f' is $(p-1)$-Hölder as desired. Now Exercise 5.4.13 implies f has modulus of smoothness of power type p. The statements on the moduli of convexity of $|t|^p$ for $p > 1$ now follow from Theorem 5.4.2. Note that an alternate approach to this and part (b) is given in Exercise 5.4.2 .

5.4.2. (a) Observe that g is convex because $g'' \ge 0$ on $[a, \infty)$. Let $a \le x < y$ and write $x = \bar{x} - h$, $y = \bar{x} + h$ where $\bar{x} = (x+y)/2$. By Taylor's theorem

$$\begin{aligned}
g(y) &= g(\bar{x}) + g'(\bar{x})(h) + \ldots + \frac{g^{(n)}(\bar{x})(h)}{n!}h^n + \frac{g^{(n+1)}(c)}{(n+1)!}h^{n+1}\\
&\ge g(\bar{x}) + g'(\bar{x})(h) + \frac{g^{(n)}(\bar{x})(h)}{n!}h^n.
\end{aligned}$$

By convexity

$$g(x) \ge g(\bar{x}) + g'(\bar{x})(-h).$$

Adding the previous two inequalities and dividing by 2 yields

$$\frac{1}{2}g(y) + \frac{1}{2}g(x) \ge g\left(\frac{x+y}{2}\right) + \frac{\alpha|x-y|^n}{n!2^n}.$$

Consequently, $\delta_g(\epsilon) \geq \dfrac{\alpha\epsilon^n}{n!2^n}$ as needed for (a). Notice that (b) follows from (a) because $g^{(k)}(t) = (\ln b)^k b^t \geq (\ln b)^k$ for all $t \geq 0$, and take some care for noninteger values of p.
To prove (c), we follow the argument and notation as in (a). Observe $\bar{x} \geq h$. Thus when $n \leq p < n-1$,

$$\begin{aligned} g^{(n)}(\bar{x}) &= (p-1)(p-2)\cdots(p-n+1)\bar{x}^{p-n} \\ &\geq (p-1)(p-2)\cdots(p-n+1)h^{p-n}. \end{aligned}$$

Proceeding as in (a), we conclude

$$g(y) \geq g(\bar{x}) + g'(\bar{x})(h) + (p-1)(p-2)\cdots(p-n+1)\frac{h^{p-n}h^n}{n!}$$

and then

$$\frac{1}{2}g(y) + \frac{1}{2}g(x) \geq g\left(\frac{x+y}{2}\right) + (p-1)(p-2)\cdots(p-n+1)\frac{h^p}{n!}$$

which provides the desired result. □

5.4.3. Suppose f has modulus of convexity of power type $p > 0$, say $\delta_f(\epsilon) \geq K\epsilon^p$ for all $\epsilon \geq 0$. Fix $\bar{x}, h \in X$, and $\phi \in \partial f(\bar{x})$. Then

$$\begin{aligned} 2K\|h\|^p &\leq f(\bar{x}+h) + f(\bar{x}) - 2f\left(\bar{x} + \frac{1}{2}h\right) \\ &= (f-\phi)(\bar{x}+h) + (f-\phi)(\bar{x}) - 2(f-\phi)\left(\bar{x} + \frac{1}{2}h\right). \end{aligned}$$

Because $(f-\phi)$ attains its minimum at $\bar{x}$, this implies

$$(f-\phi)(\bar{x}+h) \geq (f-\phi)(\bar{x}) + 2K\|h\|^p.$$

Rearranging, $f(\bar{x}+h) \geq f(\bar{x}) + \phi(h) + 2K\|h\|^p$, so the result holds with $C = 2K$.
Conversely, suppose $x, y \in X$ and let $\epsilon = \|x-y\|$. Let $\bar{x} = (x+y)/2$ and let h be such that $y = \bar{x}+h$, $x = \bar{x}-h$, and fix $\phi \in \partial f(\bar{x})$. Then

$$f(\bar{x}+h) \geq f(\bar{x}) + \phi(h) + C\|h\|^p \quad \text{and} \quad f(\bar{x}-h) \geq f(\bar{x}) + \phi(-h) + C\|h\|^p.$$

Adding these two inequalities and dividing by 2 yields

$$\frac{1}{2}f(y) + \frac{1}{2}f(x) \geq f\left(\frac{x+y}{2}\right) + C\left(\frac{\epsilon}{2}\right)^p.$$

It then follows that $\delta_f(\epsilon) \geq \frac{C}{2^p}\epsilon^p$ as desired.
The interested reader should see [445, Corollary 3.5.11] for several other conditions equivalent to moduli of power type. □

5.4.4. (a) Observe first

$$\begin{aligned} \delta_f(\epsilon) &= \inf\left\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \|x-y\| \geq \epsilon,\ x, y \in \operatorname{dom} f\right\} \\ &\leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) + \frac{1}{2}g(x) + \frac{1}{2}g(y) - g\left(\frac{x+y}{2}\right) \end{aligned}$$

for all $x, y \in \operatorname{dom} h$, $\|x-y\| \geq \epsilon$ since g is convex. From this, $\delta_h(\epsilon) \geq \delta_f(\epsilon)$ for $\epsilon \geq 0$.
(b) When $h = f \,\square\, g$, Lemma 4.4.15 shows $h^* = f^* + g^*$. Because h is proper, we know h^* is proper, and thus by (a), $\delta_{h^*} \geq \delta f^*$. Then Theorem 5.4.1(a) ensures $\rho_h \leq \rho_f$ as desired. □

5.4.5. (a) Suppose f is affine, then $f(t) = at + b$ for some $a, b \in \mathbb{R}$. Thus $f'' = 0$. On the other hand, suppose $f''(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. By replacing f with $-f$ as necessary, there is an open interval I containing x_0 and $\epsilon > 0$ so that $f'' > \epsilon$ on I. Then $|f'(t) - f'(s)| \geq \epsilon |t-s|$ for all $s, t \in I$, as $|s-t| \to 0^+$ this will contradict the α-Hölder condition.
(b) If $f'' = 0$ almost everywhere, then by the Fundamental theorem of calculus, f' is constant, and consequently, f is affine (observe f' is absolutely continuous because it satisfies a Hölder condition). In the case f'' is not 0 almost everywhere, by convexity we know $f'' \geq 0$ almost everywhere, and so we find some $\epsilon > 0$ so that $S := \{t : f'' \geq \epsilon\}$ has positive measure. Use that the *metric density* of S is 1 at almost every point of S (see [384, p. 141]) to fix $r_0 > 0$, and $x_0 \in S$ so that

$$\frac{\lambda(E \cap (x_0 - r_0, x_0 + r_0))}{2r} \geq \frac{1}{2} \quad \text{for all } 0 < r < r_0.$$

Then for $x_0 - r < s < t < x_0 + r_0$, the Fundamental theorem of calculus implies

$$f'(t) - f'(s) = \int_s^t f''(x)\,dx \geq \frac{\epsilon}{2}(t-s)$$

Hence f' does not satisfy and α-Hölder condition for $\alpha > 1$ on the interval $(x_0 - r_0, x_0 + r_0)$ which is a contradiction. □

5.4.6. Both (a) and (b) are straightforward from the definitions involved. (c) Use Exercise 5.4.5(b) and check that the connection between modulus of smoothness of power type and α-Hölder derivatives is valid for $\alpha \geq 1$ (see Exercise 5.4.13).
(d) Observe that power type duality is valid for $p > 1$. Suppose f is uniformly convex with modulus of convexity of power type p_0 where $p_0 < 2$. Let $h := f + |\cdot|^2$ on one dimension. Then h has modulus of convexity of power type p for any $p_0 < p \leq 2$. Indeed, choose $C_1 > 0$ and $C_2 > 0$ so that

$$\frac{1}{2}h(s) + \frac{1}{2}h(t) - h\left(\frac{s+t}{2}\right) \geq C_1|s-t|^{p_0}$$

and

$$\frac{1}{2}|t| + \frac{1}{2}|s| - \left|\frac{s+t}{2}\right| \geq C_2|s-t|^2$$

now separate the cases when $|s-t| \geq 1$ and $|s-t| \leq 1$. To show $\delta_h(\epsilon) \geq C\epsilon^p$ where $C := \min\{C_1, C_2\}$ and $p_0 \leq p \leq 2$. In particular, h has modulus of convexity of power type p for some (any) p between 1 and 2. By duality, deduce that h^* has modulus of smoothness of power type q for $q > 2$. Consequently h^* is affine, i.e. $h^*(t) = at + b$. Conclude that $\operatorname{dom} h = \{a\}$ is a singleton. This is true along any line, so $\operatorname{dom} f$ must be a singleton as desired.
An alternate proof for (d) in the case $0 < p < 1$ is as follows. As in Exercise 5.4.3 on can show that for a proper lower semicontinuous convex function of power type $p > 0$, there exists $C > 0$ such that

$$f(\bar{x} + h) \geq f(\bar{x}) + \phi(h) + C\|h\|^p \quad \text{whenever} \quad h \in X,\ \bar{x} \in \operatorname{dom}(\partial f),\ \phi \in \partial f(\bar{x}).$$

Because $\operatorname{dom}(\partial f) \neq \emptyset$, it follows from the convexity of f that $\operatorname{dom} f$ is a singleton whenever f has modulus of convexity of power type $p \in (0, 1)$.

For (e), let $|\!|\!|\cdot|\!|\!|$ on $\mathbb{R}^2$ be a norm that does not satisfy a modulus of convexity of power type p for any p (see [245]). Let $\|\cdot\|$ be the usual norm on $\mathbb{R}^2$. Check that $f := |\!|\!|\cdot|\!|\!|^2 + \max\{\|\cdot\|^2 - 2, 0\}$ is one such function. $\square$

5.4.7. (a) First we fix positive constants A, B corresponding to the respective moduli, and let $C > 0$ be as given. That is,

$$\delta_f(\epsilon) \geq A\epsilon^p \text{ for all } \epsilon > 0, \quad \delta_{\|\cdot\|}(\epsilon) \geq B\epsilon^p \text{ for all } 0 \leq \epsilon \leq 2, \text{ and } \quad f'_+(t) \geq Ct^{p-1} \text{ for all } t > 0.$$

Let $\epsilon > 0$ be fixed, and suppose $x, y \in X$ satisfy $\|x - y\| \geq \epsilon$. We may assume $\|y\| \leq \|x\|$. Suppose first, $\|y\| + \epsilon/2 \leq \|x\|$. Using the modulus of convexity of f we obtain

$$(4) \quad \frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - f\left(\left\|\frac{x+y}{2}\right\|\right) \geq \frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - f\left(\frac{\|x\| + \|y\|}{2}\right) \geq A\left(\frac{\epsilon}{2}\right)^p.$$

Thus for the remainder of the proof we will assume $\|y\| + \epsilon/2 > \|x\|$. Let $a := \|y\|$ and $\tilde{x} = x/\|x\|$, $\tilde{y} = y/\|y\|$. Then $\|y - a\tilde{x}\| > \epsilon/2$. Consequently, $\|\tilde{y} - \tilde{x}\| > \frac{\epsilon}{2a}$. Then the modulus of convexity implies $\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\| \leq 1 - B\left(\frac{\epsilon}{2a}\right)^p$ and thus

$$(5) \qquad \left\|\frac{x+y}{2}\right\| \leq a\left(\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\|\right) + \frac{\|x\| - a}{2} \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| - Ba\left(\frac{\epsilon}{2a}\right)^p.$$

We now consider the case, $Ba\left(\frac{\epsilon}{2a}\right)^p \geq a/2$. Recalling that $\|x\| + \|y\| \geq \|x - y\| \geq \epsilon$, we have $\|y\| \geq \epsilon/4$ since $\|y\| \geq \|x\| - \epsilon/2$. Because $a = \|y\|$, it follows that $a/2 \geq \epsilon/8$. Thus, letting $t_0 := (\|x\| + \|y\|)/2 - a/2$, we have $t_0 \geq a/2$ and the nondecreasing property of f ensures

$$f\left(\left\|\frac{x+y}{2}\right\|\right) \leq f(t_0).$$

Now we use this with the convexity of f to compute,

$$\begin{aligned}
\frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) \;&\geq\; f\left(\frac{\|x\| + \|y\|}{2}\right) \;\geq\; f(t_0) + f'_+(t_0)\cdot(a/2) \\
&\geq\; f(t_0) + f'_+(a/2)\cdot(a/2) \;\geq\; f(t_0) + f'_+(\epsilon/8)\cdot(\epsilon/8) \\
(6) \qquad &\geq\; f\left(\left\|\frac{x+y}{2}\right\|\right) + C\left(\frac{\epsilon}{8}\right)^p.
\end{aligned}$$

For our remaining case, we suppose $Ba\left(\frac{\epsilon}{2a}\right)^p \leq a/2$. Then the right hand side of (5) is at least $a/2$. Now use the fact $f'(t) \geq C(a/2)^{p-1}$ when $t \geq a/2$ to compute

$$\begin{aligned}
f\left(\left\|\frac{x+y}{2}\right\|\right) \;&\leq\; f\left(\frac{1}{2}\|x\| + \frac{1}{2}\|y\|\right) - Ba\left(\frac{\epsilon}{2a}\right)^p \cdot C\left(\frac{a}{2}\right)^{p-1} \\
(7) \qquad &\leq\; \frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - BC\left(\frac{\epsilon}{4}\right)^p.
\end{aligned}$$

Putting (4), (6) and (7) together we see that $f \circ \|\cdot\|$ has modulus of convexity of power type p as desired.

(b) In fact the following stronger statement is true: Suppose $f : [0, +\infty) \to [0, +\infty)$ is convex and increasing. Then $f \circ \|\cdot\|$ is uniformly convex if and only if

$$(8) \qquad \liminf_{t\to\infty} f'_+(t)\cdot\delta_{\|\cdot\|}\left(\frac{\epsilon}{t}\right)\cdot t > 0$$

for each $\epsilon > 0$, f is uniformly convex and $\|\cdot\|$ is uniformly convex. For details on this, see Theorem 2.1 of the paper found at

http://faculty.lasierra.edu/∼jvanderw/ConvexFunctions/Notes/cmb651v2.pdf

Further information related to other parts of this question can also be found in that note.

(c) Additionally, we will use the known moduli of ℓ_p norms (see [180]). That is, if $1 < p \leq 2$, $\|\cdot\|_p$ has modulus of convexity of power type 2. If $p > 2$, then $\|\cdot\|_p$ has modulus of convexity of power type p but not less, and trivially a norm with modulus of convexity of power type p also satisfies power type r when $r \geq p$. Also, according to Exercise 5.4.2, $t \mapsto |t|^p$ is uniformly convex on $[0, \infty)$ with modulus of convexity of power type p.

(i) Therefore, applying (a), we see that $f := \|\cdot\|_p^2$ is uniformly convex with modulus of convexity of power type 2 when $1 < p \leq 2$. Likewise, when $r > p$ we may apply $\|\cdot\|_p$ has modulus of convexity of power type r, so we may likewise apply (a) to verify $f := \|\cdot\|_p^r$ is uniformly convex with modulus of convexity of power type r.

(ii) Example 5.3.11 ensures that $f := \|\cdot\|^p$ for $p > 1$ is uniformly convex on bounded sets when $\|\cdot\|$ is uniformly convex. When $p \geq 2$, as in (i) $\|\cdot\|_p$ has modulus of convexity of power type $r \geq p$, thus we may apply (a) to deduce $f := \|\cdot\|_p^r$ uniformly convex with modulus of convexity of power type r. When $r \geq p \geq 2$, we may apply the condition in (a) to see that $f := \|\cdot\|_p^r$ is uniformly convex with modulus of convexit of power type r.

(iii) Use (a) for this part as well. □

5.4.10. (a) Suppose f has modulus of convexity of power type $p > 1$, that is $\delta_f(\epsilon) \geq C\epsilon^p$ for some $C > 0$ and all $\epsilon > 0$. According to Theorem 5.4.1(b), we have $\rho_{f^*}(\tau) = \sup\left\{\tau\frac{\epsilon}{2} - \delta_f(\epsilon) : \epsilon \geq 0\right\}$ for all $\tau \geq 0$. Therefore, $\rho_{f^*}(\tau) \leq \sup\left\{\tau\frac{\epsilon}{2} - C\epsilon^p : \epsilon \geq 0\right\}$. The supremum occurs when $\epsilon = \left(\frac{\tau}{2pC}\right)^{\frac{1}{p-1}}$, and so $\rho_{f^*}(\tau) \leq \frac{1}{2(2pC)^{\frac{1}{p-1}}}\tau^{\frac{p}{p-1}}$ as needed.

Conversely, suppose $\rho_{f^*}(\tau) \leq C\tau^{\frac{p}{p-1}}$. It follows from Theorem 5.4.1(b), that $\tau\frac{\epsilon}{2} - C\tau^{\frac{p}{p-1}} \leq \delta_f(\epsilon)$ for $\epsilon \geq 0$ and $\tau \geq 0$. For fixed $\epsilon \geq 0$, the supremum on the left hand side occurs when $\tau = \left(\frac{(p-1)\epsilon}{2pC}\right)^{p-1}$ and thus $\delta_f(\epsilon) \geq K\epsilon^p$ where

$$K = \left(\frac{(p-1)\epsilon}{2pC}\right)^{p-1}\left[\frac{1}{2} - \frac{p-1}{2p}\right] > 0 \text{ because } p > 1, C > 0.$$

This proves (a).

(b) Suppose f^* has modulus of convexity of power type p. By part (a), f^{**} has modulus of smoothness of power type q, and hence so does $f = f^{**}|_X$. Conversely, suppose f has modulus of smoothness of power type q. Proceeding as in the previous paragraph, but using Theorem 5.4.1(a) we obtain that f^* has modulus of convexity of power type p. □