# Seeing Things by Walking on Real Numbers

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(Joint work with Francisco Aragón, David Bailey and Peter Borwein)

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http://carma.newcastle.edu.au/meetings/evims/

April 10, 2004: Destination Maitland: City of the Future

Revised 03-04 2004

# Contents:

One message is "Try drawing numbers"

1. Three movies
   - Three movies of numbers
2. Who we are
   - The current team
3. Randomness
   - What is Pi?
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   - Normality of Pi
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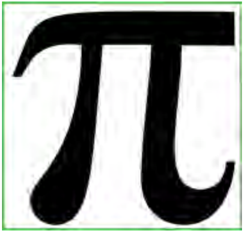
# A walk on 200 billion bits of Pi

Behind these three doors are movies of:

**A 'random' number** Pi A 'non-random' number



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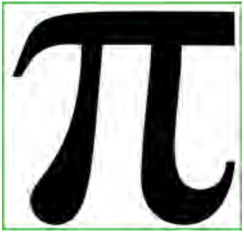

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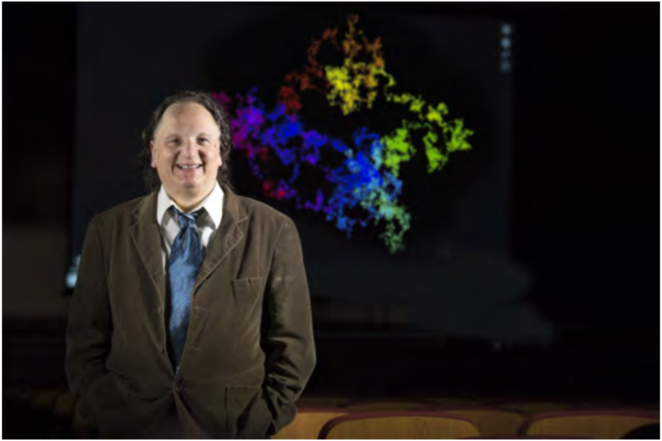
# Computer Assisted Research Mathematics and its Applications (CARMA and Me)



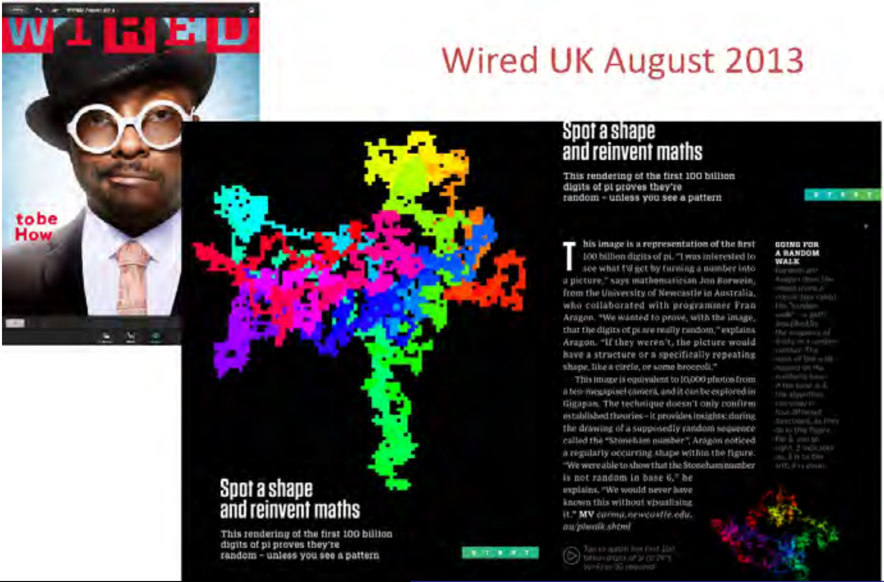
**MAA 3.14** article on Pi

http://www.carma.newcastle.edu.au/jon/pi-monthly.pdf

# My collaborators

# Outreach: images and animations led to high-level research which went viral

Wired UK August 2013



## Spot a shape and reinvent maths

This rendering of the first 100 billion digits of pi proves they're random – unless you see a pattern

This image is a representation of the first 100 billion digits of pi. "I was interested to see what I'd get by turning a number into a picture," says mathematician Jon Borwein, from the University of Newcastle in Australia, who collaborated with programmer Fran Aragon. "We wanted to prove, with the image, that the digits of pi are really random," explains Aragon. "If they weren't, the picture would have a structure or a specifically repeating shape, like a circle, or some broccoli."

This image is equivalent to 10,000 photos from a ten-megapixel camera, and it can be explored in Gigapan. The technique doesn't only confirm established theories – it provides insights: during the drawing of a supposedly random sequence called the "Stoneham number", Aragon noticed a regularly occurring shape within the figure. "We were able to show that the Stoneham number is not random in base 6," he explains. "We would never have known this without visualising it." MV carma.newcastle.edu.au/piwalk.shtml

GOING FOR A RANDOM WALK

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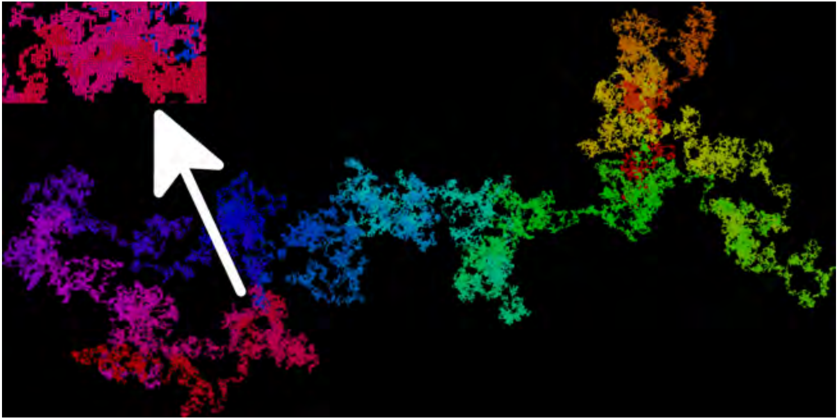

- 100 billion base four digits of $\pi = 3.14159...$ on Gigapan
- Really big pictures are often better than movies

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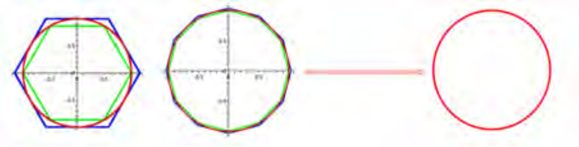

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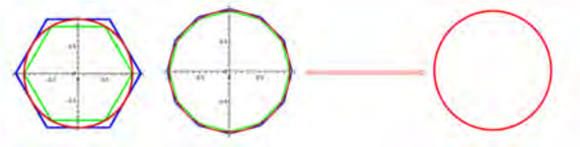

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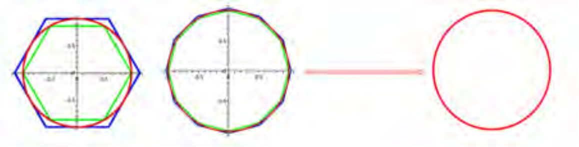
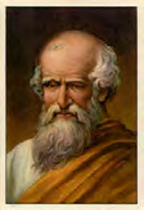


$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$ to obtain the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

# Where Greece was: Magna Graecia



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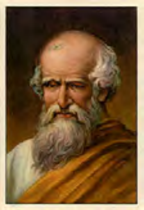

1. Syracuse
2. Troy
3. Byzantium
   Constantinople
4. Rhodes
   (Helios)
5. Hallicarnassus
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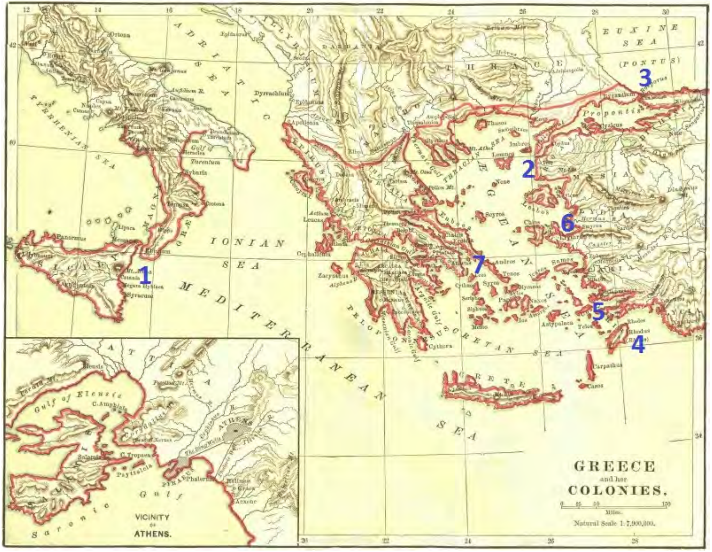
6. Ephesus
   (Artemis)
7. Athens (Zeus)



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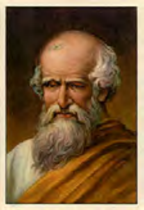

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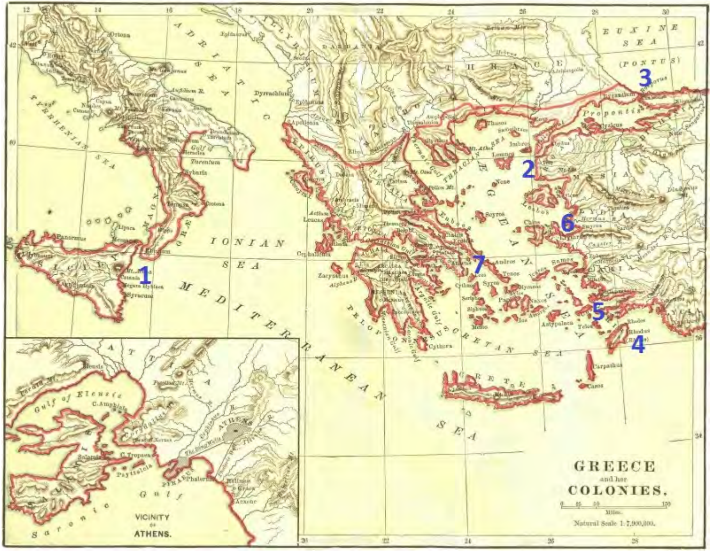


The others of the Seven Wonders of the Ancient World: Lighthouse of Alexandria, Pyramids of Giza, Gardens of Babylon

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# Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be "random":

$$\pi = 3.14159265358979323846264338327950288419716939937\ldots$$

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Are they really?

- **1949 ENIAC** (*Electronic Numerical Integrator and Calculator*) computed of $\pi$ to **2,037** decimals (in **70** hours)—proposed by polymath John von Neumann (**1903-1957**) to shed light on distribution of $\pi$ (and of $e$).

# Are the digits of $\pi$ random?

| Digit | Ocurrences |
|---|---|
| 0 | 99,993,942 |
| 1 | 99,997,334 |
| 2 | 100,002,410 |
| 3 | 99,986,911 |
| 4 | **100,011**,958 |
| 5 | **99,998**,885 |
| 6 | 100,010,387 |
| 7 | 99,996,061 |
| 8 | 100,001,839 |
| 9 | 100,000,273 |
| Total | **1,000,000,000** |

Table : Counts of first billion digits of $\pi$. Second half is ‘right’ for law of large numbers.

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  - Euler found the **292**
  - $e$ has a fine *continued* fraction
- There are infinitely many sevens in the decimal expansion of $\pi$
- There are infinitely many ones in the ternary expansion of $\pi$
- There are equally many zeroes and ones in the binary expansion of $\pi$
- Or pretty much anything else...

# What is "random"? A hard question

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It might be:

- Unpredictable (fair dice or coin-flips)?
- Without structure (noise)?
- Algorithmically random ($\pi$ is not)?
- Quantum random (radiation)?
- Incompressible (‘zip’ does not help)?

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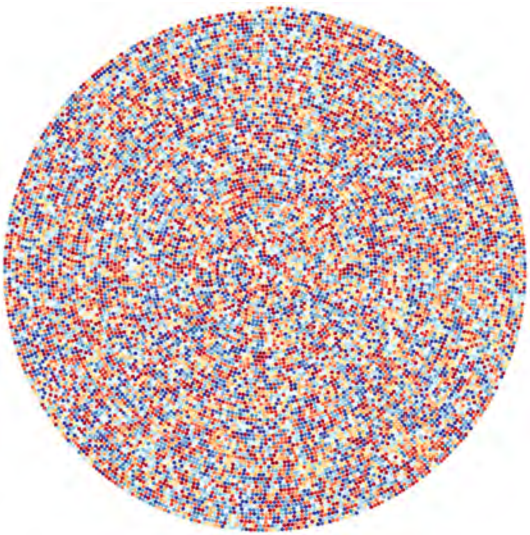
**b-normal:** All digits occur with the same probability in base $b$, say $b = 2, 4, 10$, or $16$.

# Randomness in Pi?

http://mkweb.bcgsc.ca/pi/art/



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# Is $\pi$ 10-normal?

| String | Occurrences | String | Occurrences | String | Occurrences |
|---|---|---|---|---|---|
| 0 | 99,993,942 | 00 | 10,004,524 | 000 | 1,000,897 |
| 1 | 99,997,334 | 01 | 9,998,250 | 001 | 1,000,758 |
| 2 | 100,002,410 | 02 | 9,999,222 | 002 | 1,000,447 |
| 3 | 99,986,911 | 03 | 10,000,290 | 003 | 1,001,566 |
| 4 | 100,011,958 | 04 | 10,000,613 | 004 | 1,000,741 |
| 5 | **99,998,885** | 05 | **10,002,048** | 005 | **1,002,881** |
| 6 | 100,010,387 | 06 | 9,995,451 | 006 | 999,294 |
| 7 | 99,996,061 | 07 | 9,993,703 | 007 | 998,919 |
| 8 | 100,001,839 | 08 | 10,000,565 | 008 | 999,962 |
| 9 | 100,000,273 | 09 | 9,999,276 | 009 | 999,059 |
| | | 10 | 9,997,289 | 010 | 998,884 |
| | | 11 | 9,997,964 | 011 | 1,001,188 |
| | | ⋮ | ⋮ | ⋮ | ⋮ |
| | | 99 | 10,003,709 | 099 | 999,201 |
| | | | | ⋮ | ⋮ |
| | | | | 999 | 1,000,905 |
| **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 |

Table : Counts for the first billion digits of $\pi$.

# Is $\pi$ 16-normal

That is, in Hex?

$\hookleftarrow$ Counts of first trillion hex digits

| | |
|---|---|
| 0 | 62499881108 |
| 1 | 62500212206 |
| 2 | 62499924780 |
| 3 | 62500188844 |
| 4 | 62499807368 |
| 5 | 62500007205 |
| 6 | 62499925426 |
| 7 | 62499878794 |
| 8 | 62500216752 |
| 9 | 62500120671 |
| A | 62500266095 |
| B | 62499955595 |
| C | 62500188610 |
| D | 62499613666 |
| E | 62499875079 |
| F | 62499937801 |
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- They are $353CB3F7F0C9ACCFA9AA215F2$

See www.karrels.org/pi/index.html

# Modern $\pi$ Calculation Records: and IBM Blue Gene/L at LBL

| Name | Year | Correct Digits |
|---|---|---|
| Miyoshi and Kanada | 1981 | 2,000,036 |
| Kanada-Yoshino-Tamura | 1982 | 16,777,206 |
| Gosper | 1985 | 17,526,200 |
| Bailey | Jan. 1986 | 29,360,111 |
| Kanada and Tamura | Sep. 1986 | 33,554,414 |
| Kanada and Tamura | Oct. 1986 | 67,108,839 |
| Kanada et. al | Jan. 1987 | 134,217,700 |
| Kanada and Tamura | Jan. 1988 | 201,326,551 |
| Chudnovskys | May 1989 | 480,000,000 |
| Kanada and Tamura | Jul. 1989 | 536,870,898 |
| Kanada and Tamura | Nov. 1989 | 1,073,741,799 |
| Chudnovskys | Aug. 1991 | 2,260,000,000 |
| Chudnovskys | May 1994 | 4,044,000,000 |
| Kanada and Takahashi | Oct. 1995 | 6,442,450,938 |
| Kanada and Takahashi | Jul. 1997 | 51,539,600,000 |
| Kanada and Takahashi | Sep. 1999 | 206,158,430,000 |
| Kanada-Ushiro-Kuroda | Dec. 2002 | 1,241,100,000,000 |
| Takahashi | Jan. 2009 | 1,649,000,000,000 |
| Takahashi | April 2009 | 2,576,980,377,524 |
| Bellard | Dec. 2009 | 2,699,999,990,000 |
| Kondo and Yee | Aug. 2010 | **5,000,000,000,000** |
| Kondo and Yee | Oct. 2011 | **10,000,000,000,000** |
| Kondo and Yee | Dec. 2013 | **12,100,000,000,000** |

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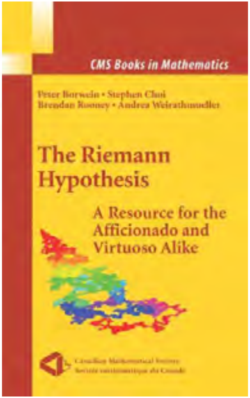
# One 1500-step ramble: a familiar picture

Liouville function



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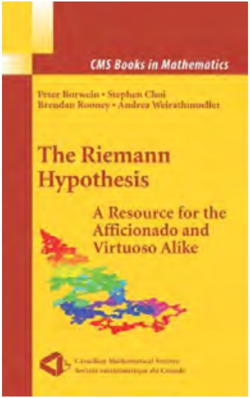


- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).

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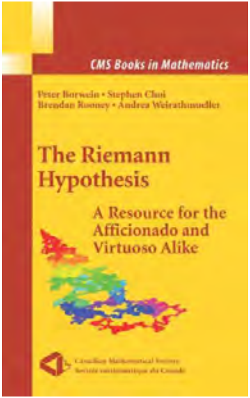
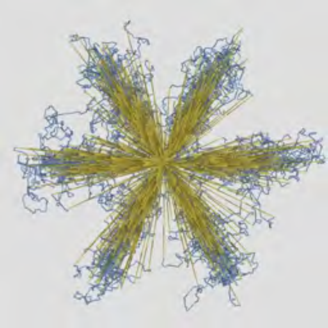



- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
- 1D or 2D *lattice*: probability one of returning to the origin.

# 1000 three-step rambles: a less familiar picture?

# Art meets science

AAAS & Bridges conference



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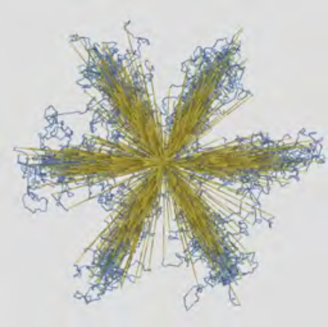

*A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds*.

(Nadia Whitehead 2014-03-25 16:15)

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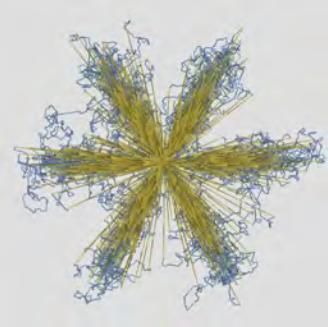
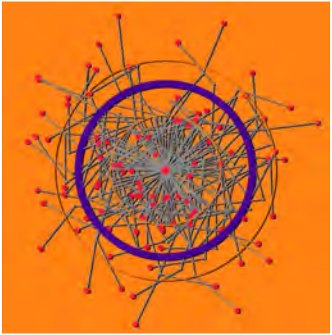


*A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds.*

(Nadia Whitehead 2014-03-25 16:15)

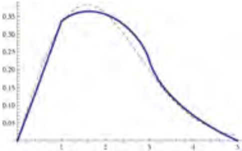
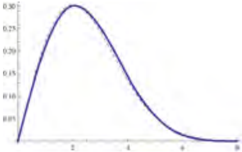
(JonFest 2011 Logo) *Three-step random walks. The (purple) expected distance travelled is 1.57459 ...*

*The closed form $W_3$ is given below.*



$$W_3 = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4}$$

# A Little History: From a vast literature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R: Rayleigh gave large $n$ estimates of density: $p_n(x) \sim \frac{2x}{n} e^{-x^2/n}$ (*Nature*, 1905) with $n = 5, 8$ shown above.

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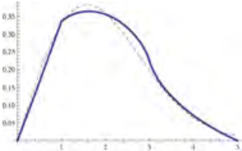

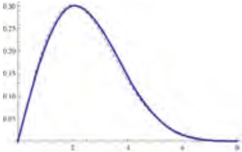

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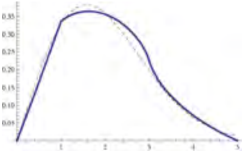
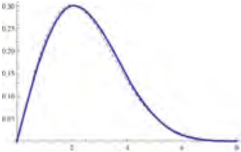


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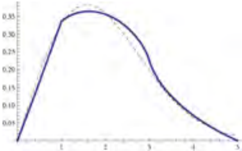

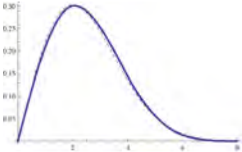

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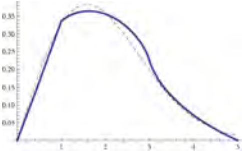

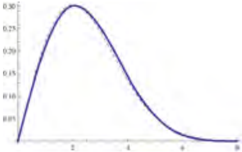

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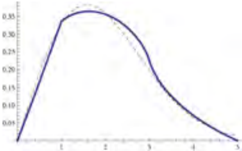
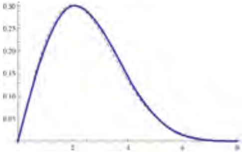


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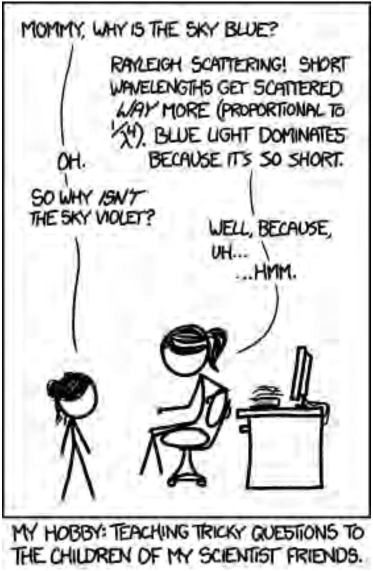
- **UNSW**: Donovan and Nuyens, WWII cryptography.
- appear in graph theory, quantum chemistry, in quantum physics as hexagonal and diamond lattice integers, etc ...

# Why is the sky blue?



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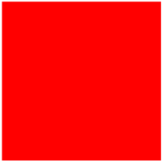
Pick a random number in $\{0,1,2,3\}$ and move according to $0=\rightarrow$, $1=\uparrow$, $2=\leftarrow$, $3=\downarrow$



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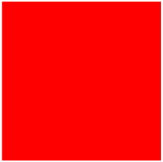

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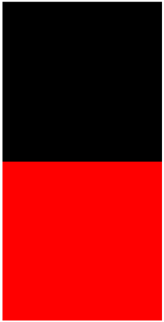

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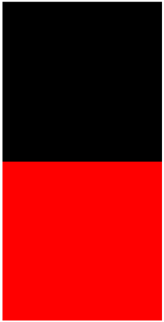

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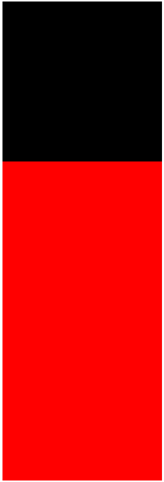

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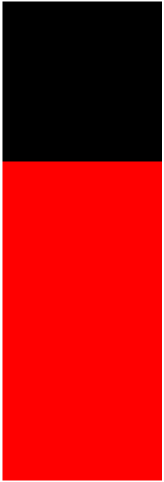

$$2 = \leftarrow$$

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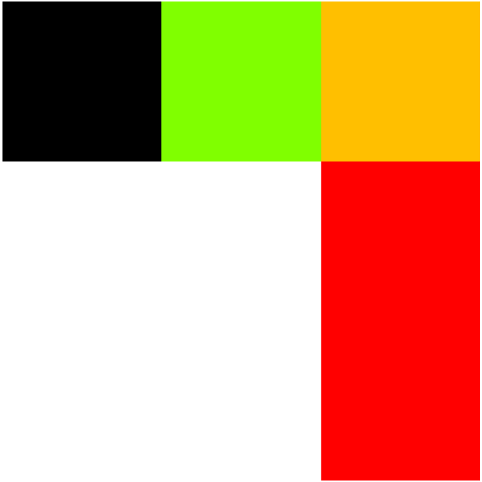

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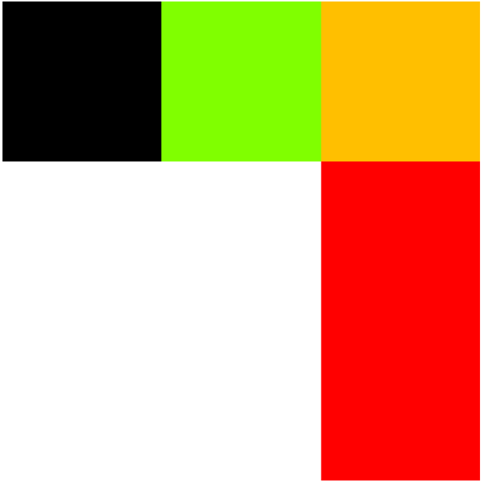

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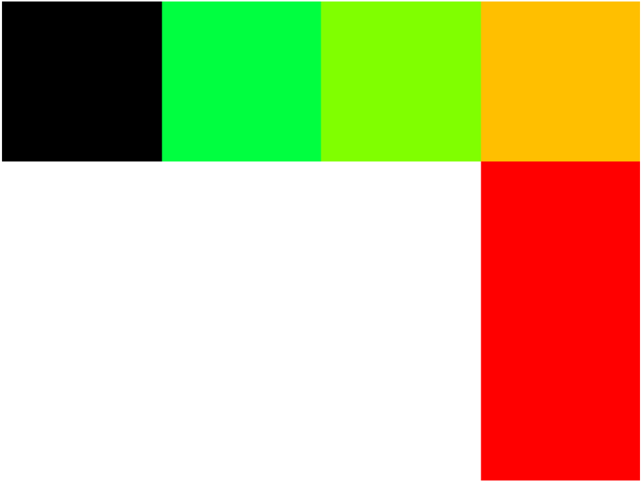

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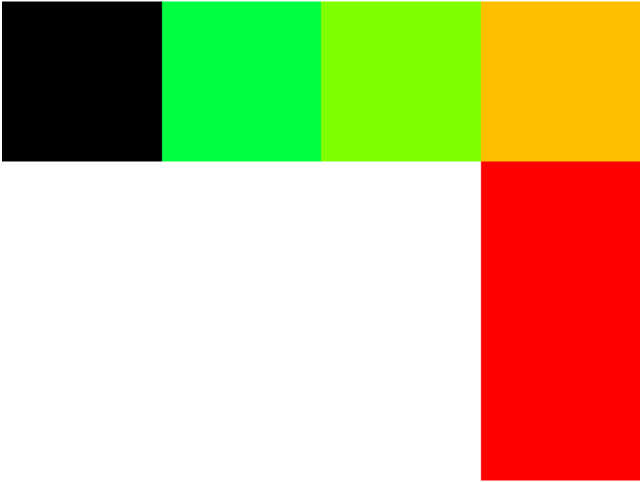

$$3 = \downarrow$$

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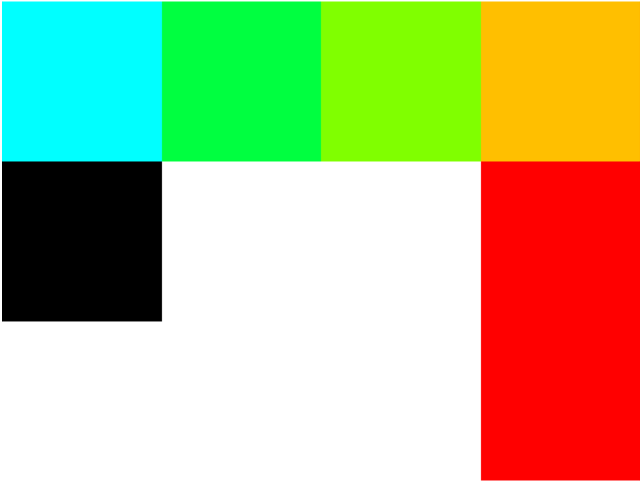

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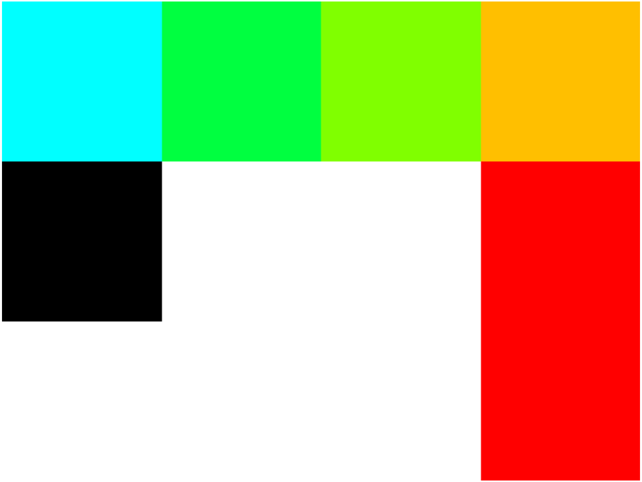

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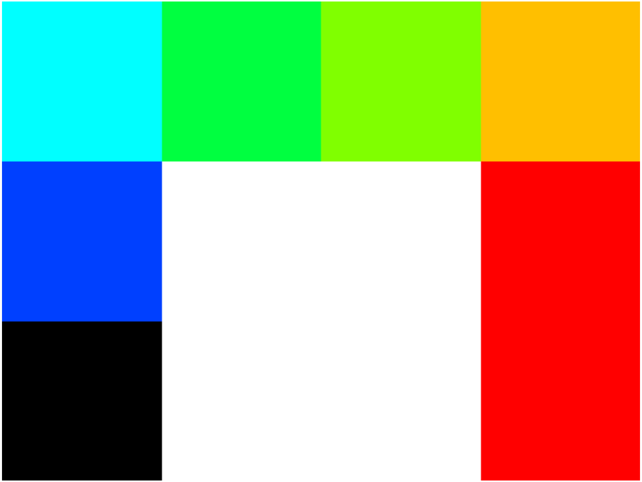

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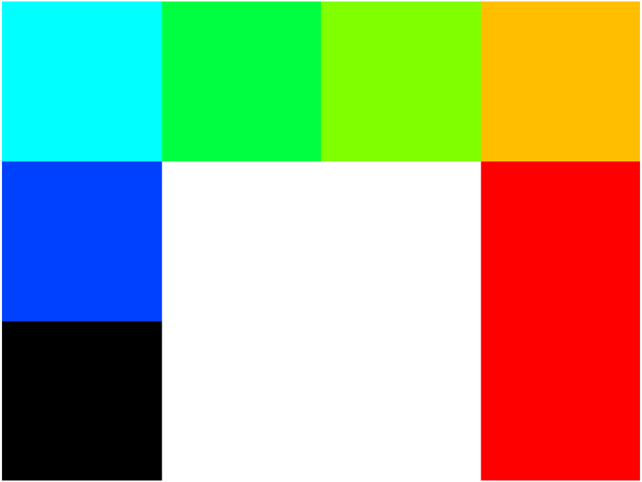

$$0=\rightarrow$$

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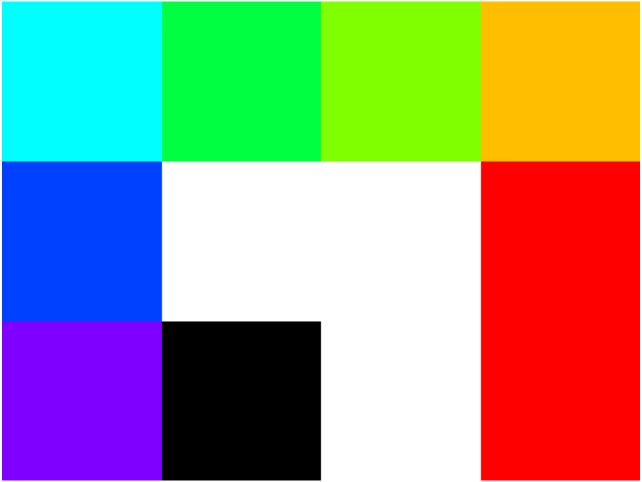

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# What is a random walk (base 4)?

Pick a random number in $\{0,1,2,3\}$ and move $0 = \rightarrow,\ 1 = \uparrow,\ 2 = \leftarrow,\ 3 = \downarrow$ ANIMATION

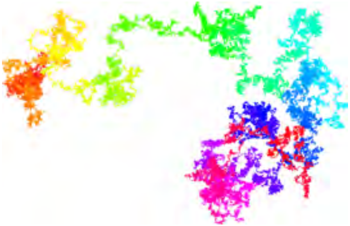

Figure : A million step base-4 pseudorandom walk. We use the spectrum to show when we visited each point (ROYGBIV and R).

# Random walks look similarish

Chaos theory (order in disorder)

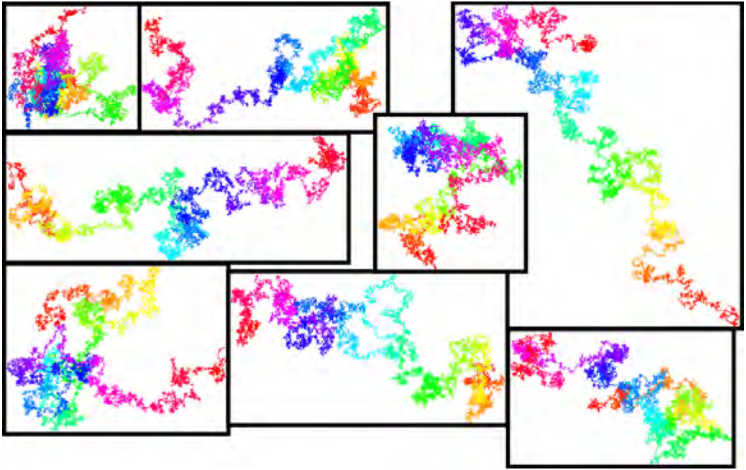

Figure : Eight different base-4 (pseudo)random[1] walks of one million steps.

[1] Python uses the *Mersenne Twister* as the core generator. It has a period of $2^{19937} - 1 \approx 10^{6002}$.

# Base-$b$ random walks: Our direction choice

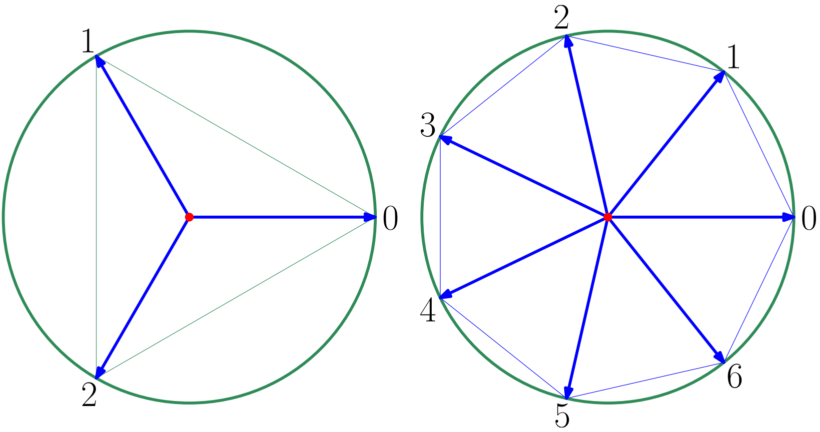


Figure : Directions for base-3 and base-7 random walks.

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# Two rational numbers

ANIMATION

The base-4 digit expansion of $Q1$ and $Q2$:

```
Q1=
0.22122101223212120012210122312100122210001123212312100012221000122 2
10001222100012221000012221000122201103010122010012010311033333333333
33333333333333330111111111111111111111111111111001000000000300300320032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232222300032222300322213300233212330232132321121121212223232 33
33303000001000323003230032220303203011033301110330110310111101133233 3
3232322321221211211121122322222122...
```

```
Q2=
0.22122101223212120012210122312100122210001123212312100012221000122 2
10001222100012221000012221000122201103010122010012010311033333333333
33333333333333330111111111111111111111111111111001000000000300300320032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232222300032222300322213300233212330232132321121121212223232 33
33303000001000323003230032220303203011033301110330110310111101100000 0
000000...
```

# Two rational numbers

ANIMATION

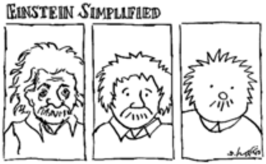


Figure : Self-referent walks on the rational numbers $Q1$ (top) and $Q2$ (bottom).

# Two more rationals

Hard to tell from their decimal expansions

The following relatively small rational numbers [G. Marsaglia, 2010]

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{1000000000061},$$

have base-10 periods with huge length of **1,750,000,000** digits and **1,000,000,000,060** digits, respectively.

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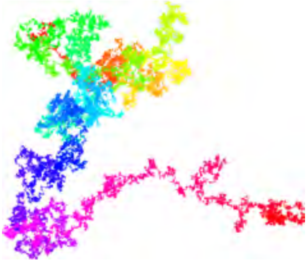
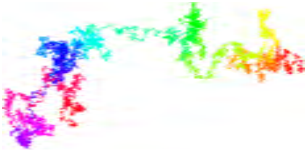


(a) $Q3$



(b) $Q4$

Figure : Walks on the first million base-10 digits of the rationals $Q3$ and $Q4$.

# Walks on the digits of numbers

ANIMATION

Figure : A walk on the first 10 million base-4 digits of $\pi$.

# Walks on the digits of numbers

Coloured by hits (more pink is more hits)

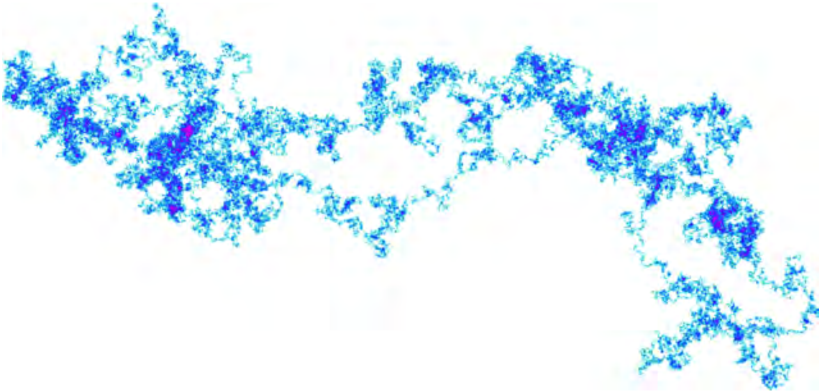


Figure : 100 million base-4 digits of $\pi$ coloured by number of returns to points.

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# The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

**1973** Richard Stoneham proved some of the following (nearly 'natural') constants are $b$-normal for relatively prime integers $b, c$:

$$\alpha_{b,c} := \frac{1}{cb^c} + \frac{1}{c^2 b^{c^2}} + \frac{1}{c^3 b^{c^3}} + \ldots.$$

Such super-geometric sums are Stoneham constants. To 10 places

$$\alpha_{2,3} = \frac{1}{24} + \frac{1}{3608} + \frac{1}{3623878656} + \ldots$$

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For every coprime pair of integers $b \geq 2$ and $c \geq 2$, the constant $\alpha_{b,c}$ is $b$-normal.

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- Since $3 < 2^{3-1} = 4$, $\alpha_{2,3}$ is 2-normal and 6-nonnormal !

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Figure : $\alpha_{2,3}$ is 2-normal (top) but 6-nonnormal (bottom). Is seeing believing?

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Figure : Is $\alpha_{2,3}$ 3-normal or not?

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# Main References




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- F.J. Aragón Artacho, D.H. Bailey, J.M. Borwein, P.B. Borwein: Walking on real numbers, The *Mathematical Intelligencer* **35** (2013), no. 1, 42–60.
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