Banach Spaces of Type (NI) and Monotone Operators on Non-reflexive Spaces

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Abstract

A convex set $C \subseteq X^{**} \times X^*$ admits the variant Banach-Dieudonné property (VBDP) if the weak*-strong closure $\overline{C}^{w^* \times \|\cdot\|}$ is the smallest set containing C that is closed to all limits of its *bounded* and weak* $\times \|\cdot\|$ convergent nets. We show in particular, that all convex sets in $X^{**} \times X^*$ admit the VBDP when $E^* := X^* \times X^{**}$ is weakly-compactly generated (WCG) and hence if E is either a dual separable or a reflexive Banach space.

This allows us to answer some outstanding problems regarding the embedding of various representative functions, including what we call the Penot function (but which seems first to have been clearly identified by Svaiter) , for nonreflexive spaces [15, 7]. In particular, we show that the conjugate of the Fitzpatrick representative function $\widehat{\mathcal{F}_T}^* : X^* \times X^{**} \to \overline{\mathbf{R}}$ and the Penot representative function $\widehat{\mathcal{P}_T}^* : X^* \times X^{**} \to \overline{\mathbf{R}}$ are themselves representative functions when $T \subseteq X \times X^*$ is maximal monotone and E is as above. It follows that in such spaces all maximal monotone operators are well-behaved and the classical sum rule holds.

Notice: We no longer trust Theorem 5 and so the results herein should be taken as conjectured not proven.

1 Introduction

Our basic notation is consistent with [16] and [18]. Given a Banach space Z with dual Z^*, denote the closed ball of radius r in Z by $B_Z(r) = \{z \mid \|z\| \leq r\}$ and the closed dual ball by $B_{Z^*}(r) = \{z^* \mid \|z^*\|^* \leq r\}$. When it is clear in context we will omit reference to the space and write simply $B(r)$.

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We may view $Z\times Z^*$ paired with $Z^*\times Z$ using the coupling $\langle (z,z^*),(x^*,x)\rangle = \langle z,x^*\rangle + \langle x,z^*\rangle$ and the norm $\|(z,z^*)\|^2 = \|z\|^2 + \|z^*\|^2$. At times we will assume $Z\times Z^*$ is endowed with the product topology $s\times \mathrm{bw}^*(Z^*,Z)$ formed from the norm or strong topology on Z and the bounded-weak* topology on Z^* (see [13] page 150–154)—we also write $\|\cdot\|\times \mathrm{bw}^*$ as appropriate. We will pair this space with $Z^*\times Z$ endowed with the product topology $\mathrm{bw}^*(Z^*,Z)\times s$. This is a valid pairing since when Z is Banach the bw^*- continuous linear functionals on Z^* are canonically isomorphic to Z. When $Z = X^*$ we obtain the pairing of $s\times \mathrm{bw}^*(X^*,X^{**})$ with $\mathrm{bw}^*\times s(X^{**},X^*)$, which is of central importance in the study of maximal monotone operators using representative functions.

When we pair the space $s\times \mathrm{bw}^*(Z,Z^*)$ with $\mathrm{bw}^*\times s(Z^*,Z)$ the associated convex conjugation operation of a proper convex function $f\in \Gamma_{s\times \mathrm{bw}^*}(Z,Z^*)$ is denoted by $f^*\in \Gamma_{\mathrm{bw}^*\times s}(Z^*,Z)$. When we associate $X\times X^*$ with $X^*\times X^{**}$ using the weak topology on $X\times X^*$ and the weak* topology on $X^*\times X^{**}$ we denote the conjugate of $f\in\Gamma_{\mathrm{w}}(X,X^*)$ by $\widehat{f}^* := \left(\widehat{f}\right)^*\in \Gamma_{\mathrm{w}^*}(X^*,X^{**})$. For a subset $V\subseteq X^*\times X$, denote by $J_{X^*\times X}(V)$ the imbedding of V into $X^*\times X^{**}$. We also use the shorthand $J_X(X) = \widehat{X}$ along with $J_X(x) = \hat{x}$ for the embedding into X^{**}.

Definition 1 *We say a convex subset $C\subseteq X^{**}\times X^*$ admits the* variant Banach-Dieudonné property *(VBDP) when its* $\mathrm{bw}^*\times\|\cdot\|$ *closure can be characterised as the smallest convex set $Q\supseteq C$ that contains all limits (x^{**},x^*) of bounded nets $\left\{\left(x^{**}_\beta,x^*_\beta\right)\right\}_{\beta\in\Lambda}\subseteq Q$ with x^{**}_β converging weak* to x^{**} and x^*_β converging strongly to x^*.*

For epigraphs of functions $f: X\times X^*\to \overline{\mathbf{R}}$ we must extend these notions to a subset $C\subseteq X^*\times X\times \mathbf{R}$ and consider the strong topology imposed on $X\times\mathbf{R}$ but the bw^*-topology only on X^*. With this slight abuse of notation all properties shown for $C\subseteq X^*\times X$ extend to this case.

We recall that a multifunction T mapping X to X^* is *monotone* if $\langle y^*-x^*, y-x\rangle \ge 0$ for all $y^*\in T(y)$ and $x^*\in T(x)$, and that T is *maximal monotone* if its graph is maximal amongst monotone graphs ordered by set inclusion. We call a $s\times\mathrm{bw}^*$-closed convex function $\mathcal{H}_T$ on $X\times X^*$ a *representative function* of a monotone mapping T on X when $\mathcal{H}_T(y,y^*)\ge\langle y,y^*\rangle$ for all $(y,y^*)\in X\times X^*$ while $\mathcal{H}_T(y,y^*) = \langle y,y^*\rangle$ when $y^*\in T(y)$. If T is not specified we say a $s\times\mathrm{bw}^*$ closed, proper convex function f is *representative* when $f(y,y^*)\ge\langle y,y^*\rangle$ for all $(y,y^*)\in X\times X^*$.

For a function $f: X\times X^*\to\overline{\mathbf{R}}$ define the natural embedding of f via $\operatorname{epi}\widehat{f} := \widehat{\operatorname{epi} f}$. Thence $\operatorname{epi}\widehat{f}\subseteq X^{**}\times X^*\times\mathbf{R}$ and the *lower level set* $\operatorname{lev}_\alpha\widehat{f}\subseteq X^{**}\times X^*$.

We shall show that, for example, when $E^* := X^*\times X^{**}$ is *weakly-compactly generated*—there is a weak-compact convex set whose span is norm-dense in the space—all convex sets admit the VBDP. In this setting we are then able to deduce that both $\operatorname{epi}\widehat{f}$ and $\operatorname{epi}\widehat{(f^*)}$ also possess the VBDP. In these cases for $\widehat{T} = J_{X\times X^*}(T)$ (the embedding of T into $X^{**}\times X^*$) the function

$$\mathcal{F}_{\widehat{T}}(x^{**},x^*) = \sup_{(\hat{y},y^*)\in\widehat{T}}\left\{\langle\hat{y},x^*\rangle + \langle x^{**},y^*\rangle - \langle\hat{y},y^*\rangle\right\}$$

is representative and hence we are able to deduce that T is of *type (NI)* [17]. Indeed T is type (NI) exactly when

$$\mathcal{F}_{\widehat{T}}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle$$

for all $x^{**} \in X^{**}, x^* \in X^*$, and we take this as our definition. In [21] $\mathcal{F}_{\widehat{T}}$ is then called *strongly representative*—see [3, 6] for further discussion this concept.

We shall call X a *Banach space of type (NI)* if every maximal monotone operator on X is of type (NI). To summarize, we shall show that there are many non-reflexive Banach spaces of type (NI).

2 The Variant Banach-Dieudonné Property

Recall that the *Banach-Dieudonné* (or *Krein-Šmulian*) theorem shows that for a Banach space Z a *convex subset* $D \subseteq Z^*$ is bw*-closed iff D is w*-closed iff for all $M > 0$ we have $D \cap B_{Z^*}(M)$ is w* -closed for every closed dual ball [9]. We shall investigate conditions on X or C guaranteeing equivalence of the following for a convex set $C \subseteq X^{**} \times X^*$:

A1. C is bw$^* \times s$-closed; A2. C is w$^* \times s$-closed;

A3. $C \cap B_{(X^* \times X)^*}(M)$ is w$^* \times s$-closed, for all closed dual balls $B_{(X^* \times X)^*}(M)$ in $X^{**} \times X^*$ and all $M > 0$.

It is clear that A1. is equivalent to A2. in any Banach space, since when X is complete we have two locally convex topologies for the same dual pair [13, 9]). Clearly A2. implies A3., leaving the final and harder implication to be addressed. It seems plausible that A1., A2. and A3. are equivalent in many Banach spaces. This is clarified by the next result.

Proposition 2 *Given a Banach space X, A1., A2., and A3, coincide if and only if X admits the VBDP. That is, if and only if the* bw$^* \times s$ *closure of a convex set C is the smallest convex set $Q \supseteq C$ containing all limits $(x^{**}, x*)$ of norm bounded nets $\{(x^{**}_\beta, x^*_\beta)\}_{\beta \in \Lambda} \in Q$ with x^*_β converging strongly to x^* and x^{**}_β converging weak* to x^{**}.*

Proof. Assume X possess the VBDP and that $C \cap B(M)$ is w$^* \times s$ -closed for all closed dual balls $B(M)$. We need to show that C is bw$^* \times s$ closed. By the VBDP property this closure is characterized as the smallest convex set containing C that is closed with respect to bounded w$^* \times s$ convergent nets. But indeed by hypothesis C is the smallest such set.

Conversely assume A3. is equivalent to A1. Now the set $\overline{C}^{\mathrm{bw}^* \times s}$ is clearly convex, contains C and is closed with respect to bounded weak$^* \times s$ convergent nets. Denote by Q the smallest such set. Then we clearly have $\overline{C}^{\mathrm{bw}^* \times s} \supseteq Q \supseteq C$. Now Q satisfies A3. This follows from the definition of Q and the fact that the dual ball is bw* closed. As A3 is equivalent to A1. we have Q bw$^* \times s$ - closed. As Q also contains C it follows that $Q \supseteq \overline{C}^{\mathrm{bw}^* \times s}$ as the later set is the smallest such convex set. Consequently $Q = \overline{C}^{\mathrm{bw}^* \times s}$ and X possess the VBDP.

■

Remark 3 Norm boundedness of x_β is actually superfluous, since $x_\beta \to x$ strongly.

We shall also have call to use the following deep result, well described in [20].

Theorem 4 (Spaces not containing ℓ^1) *For a separable Banach space X, the following coincide:*

1. *X does not contain any isomorphic copy if ℓ^1.*
2. *Every bounded subset of X^{**} is w^* -sequentially dense in its w^*-closure.*

In the non-separable case, 2. implies 1. but not conversely [20, Ch. 4]. Combining Theorem 4 with the classical Banach-Dieudonné theorem [9, p.125] allows us to handle unbounded convex sets.

Theorem 5 *Let X be a separable Banach space without any copy of ℓ^1 and let $C \subset X^{**}$ be convex. Then*

$$\overline{C}^{\mathrm{w}^*} = \{x^{**} \mid \exists \textit{ a sequence } y_n^{**} \rightharpoonup_* x^{**}, y_n^{**} \in C\}\,. \tag{1}$$

Proof. Recall that in Banach space w^*-convergent sequences are norm bounded and denote the right hand side of (1) by $\overline{C}^{\mathrm{w}^*_\sigma}$. Throughout $B(K) = B_{X^{**}}(K)$.
Being bounded $C \cap B\,(K)$ is w^*-sequentially dense in $\overline{C \cap B\,(K)}^{\mathrm{w}^*}$. Thus for all $K > 0$

$$\begin{aligned} C \cap B\,(K) \subseteq \overline{C \cap B\,(K)}^{\mathrm{w}^*} &= \{x^{**} \mid \exists \text{ a sequence } y_n^{**}(\,\in C \cap B\,(K)\,) \rightharpoonup_* x^{**}\} \\ &\subseteq \overline{C}^{\mathrm{w}^*_\sigma} \cap B\,(K) \subseteq \overline{C}^{\mathrm{w}^*} \cap B\,(K)\,. \end{aligned}$$

We now show that

$$\overline{C}^{\mathrm{w}^*_\sigma} = \bigcup_{K>0} \overline{C \cap B\,(K)}^{\mathrm{w}^*}. \tag{2}$$

Clearly

$$\overline{C}^{\mathrm{w}^*_\sigma} = \bigcup_{K>0} \left[\overline{C}^{\mathrm{w}^*_\sigma} \cap B\,(K)\right] \supseteq \bigcup_{K>0} \overline{C \cap B\,(K)}^{\mathrm{w}^*}$$

and if $x^{**} \in \overline{C}^{\mathrm{w}^*_\sigma}$ there is $y_m^{**}(\in C) \rightharpoonup_* x^{**}$. By norm boundedness there exists $K > 0$ such that $\{y_m^{**}\} \subseteq C \cap B\,(K)$ implying $x^{**} \in \overline{C \cap B\,(K)}^{\mathrm{w}^*}$ giving (2).
Next note that (2) implies

$$\overline{C}^{\mathrm{w}^*_\sigma} \cap B(M) = \bigcup_{K>0} \left[\left[\overline{C \cap B\,(K)}^{\mathrm{w}^*}\right] \cap B(M)\right] \subseteq B(M).$$

To show closure, fix $M > 0$ and a net $\{x_\beta^{**}\}_{\beta \in \Lambda} \subseteq \overline{C}^{\mathrm{w}^*_\sigma} \cap B(M)$ with $x_\beta^{**} \rightharpoonup_* x^{**} \in B\,(K)$. Then

$$x_\beta^{**} \in \overline{C \cap B\,(K_\beta)}^{\mathrm{w}^*} \cap B(M) \quad \text{ for each } \beta \in \Lambda. \tag{3}$$

Property 2 of Theorem 4 shows $\left\{x_\beta^{**} \mid \beta \in \Lambda\right\}$ is weak*-sequentially dense in its weak* closure. Take a sequence $x_{\beta_n}^{**} \rightharpoonup_* x^{**}$ with $x_{\beta_n}^{**} \in \overline{C \cap B\left(K_{\beta_n}\right)}^{\mathrm{w}^*}$. We now use property 2 of Theorem 4 again to deduce that $C \cap B\left(K_{\beta_n}\right)$ is weak*-sequentially dense in its weak* closure. Thus there exists a sequence $x_{n_k}^{**} \rightharpoonup_* x_{\beta_n}^{**}$ as $k \to \infty$. Use a diagonalisation to obtain a sequence $x_{n_{k_n}}^{**} \rightharpoonup_* x^{**}$ with $x_n^{**} := x_{n_{k_n}}^{**} \in C \cap B\left(K_{\beta_{n_{k_n}}}\right)$ for all n. This diagonalisation can be achieved because the dual space is weak-star separable. As weak*-sequentially convergent sequences are bounded there exists a $K > 0$ independent of n such that

$$x_n^{**} \in C \cap B\left(K\right)$$

for all n. Consequently $x^{**} \in \overline{C \cap B\left(K\right)}^{\mathrm{w}^*}$ and also by (3) $\|x^{**}\| \leq M$. Thus, using (2) again

$$x^{**} \in \overline{C}^{\mathrm{w}_\sigma^*} \cap B\left(M\right).$$

Hence $\overline{C}^{\mathrm{w}_\sigma^*} \cap B\left(M\right)$ is weak* closed for all $M > 0$. Clearly $\overline{C}^{\mathrm{w}_\sigma^*}$ is convex and so by the Krein-Šmulian theorem that $\overline{C}^{\mathrm{w}_\sigma^*}$ is w*-closed. As $C \cap \overline{B}\left(K\right) \subseteq \overline{C}^{\mathrm{w}_\sigma^*} \cap B\left(K\right)$ for all $K > 0$ we have $\overline{C}^{\mathrm{w}^*} \subseteq \overline{C}^{\mathrm{w}_\sigma^*}$ with equality ensuing. ∎

In particular we obtain a refined description of the w*-closure of a convex set in $X^{**} \times X^*$.

Corollary 6 *Let $E = X \times X^*$ be a separable Banach space containing no copy of ℓ^1 (as holds if X^{**} is separable). Let C be a convex subset of $X^{**} \times X^*$. Then*

$$\overline{C}^{\mathrm{w}^* \times \|\cdot\|} = \left\{(x^{**}, x^*) \mid \exists (y_\beta^{**}, y_\beta^*) \in C \text{ bounded }, y_\beta^{**} \rightharpoonup_* x^{**}, y_\beta^* \to_{\|\cdot\|} x^*\right\}.$$

Consequently X admits the VBDP.

Proof. Fix $(x^{**}, x^*) \in \overline{C}^{\mathrm{w}^* \times \|\cdot\|}$. We apply Theorem 5 to $E := X \times X^*$ and $C \subset X^{**} \times X^*$ and deduce that there is a sequence in C converging w* to (x^{**}, x^*) in $E^{**} = X^{**} \times X^{***}$. Since the second coordinate lies in X^* we deduce that (x^{**}, x^*) is the limit of a $\mathrm{w}^* \times \mathrm{w}$ -convergent sequence $(z_n, z_n^*) \in C$. Since the convex hull of this bounded sequence has the same $\mathrm{w}^* \times \|\cdot\|$-closure, we are done.

The VBDP follows from an application of Proposition 2. ∎

More generally we have only used that the second dual ball for $E := X \times X^*$ is w*-*angelic* (every bounded set in E^{**} is sequentially dense in its w*-closure). This implies E contains no copy of ℓ^1 but not conversely.

Corollary 7 *Let $X \times X^*$ be a Banach space with a w* -angelic second dual ball. Then every convex subset $C \subset X^{**} \times X^*$ has the VBDP and so X admits the VBDP.*

This now captures the product of any reflexive space (weakly compact sets are angelic, see [9]) and a space with separable second dual (such as the James space [9]). Indeed, subspaces of WCG spaces are w*-angelic. Thus, it suffices that $E^* = X^* \times X^{**}$ be a subspace of a WCG space

Corollary 8 *Let $E^* = X^* \times X^{**}$ be a WCG Banach space. Then X admits the VBDP.*

Furthermore we may replace WCG by *weakly countably determined* (WCD). Every subspace of a WCG space is WCD, see [8, p. 120]. We have succeeded in identifying a sizeable class of Banach spaces with the VBDP that strictly includes reflexive spaces.

3 Representative Convex Functions

Define the *transpose operator* $\dagger$: $(x^*,x) \to (x,x^*)$ and $c_T(\cdot,\cdot) := \delta_T(\cdot,\cdot) + \langle \cdot,\cdot \rangle$, and denote the indicator function of the graph of T by δ_T (we use T for both the graph and the multifunction when no confusion is likely), $\mathcal{F}_T := (c_T^*)^\dagger$ and $\mathcal{P}_T := c_T^{**}$, where the conjugates are taken between the paired spaces $s \times \mathrm{bw}^*(X,X^*)$ with $\mathrm{bw}^* \times s\,(X^*,X)$. Recall that when we associate $X \times X^*$ with $X^* \times X^{**}$ using the weak topology on $X \times X^*$ and the weak* topology on $X^* \times X^{**}$ we denote the conjugate of $f \in \Gamma_{\mathrm{w}}(X,X^*)$ to be $\widehat{f}^* := \left(\widehat{f}\right)^* \in \Gamma_{\mathrm{w}^*}(X^*,X^{**})$ given by

$$\left(\widehat{f}\right)^*(x^*,x^{**}) := \sup_{(y,y^*)} \left\{\langle x^*,y\rangle + \langle x^{**},y^*\rangle - f(y,y^*)\right\},$$

where as always Γ_τ denotes the closed proper convex functions in the appropriate topology. The rationale for this notation is discussed later in this section.
Fitzpatrick [10] showed that the convex functions defined by

$$\operatorname{epi}\mathcal{P}_T = \left[\left(X^* \times \widehat{X} \times \mathbf{R}\right) \cap \left(\operatorname{epi}\widehat{\mathcal{F}_T}^*\right)\right]^\dagger \text{ and } \operatorname{epi}\mathcal{F}_T = \left[\left(X^* \times \widehat{X} \times \mathbf{R}\right) \cap \left(\operatorname{epi}\widehat{c_T}^*\right)\right]^\dagger$$

induce representative functions when T is maximal monotone, and in [4] it is shown that $\mathcal{P}_T$ is representative when T is merely monotone, as pointed out by Penot.
The *Fitzpatrick function* on $X^{**} \times X^*$ that captures the (NI) property is just

$$\mathcal{F}_{\widehat{T}}(x^{**},x^*) := \left(\widehat{c_T}^*\right)^\dagger(x^{**},x^*) \equiv \left(c_{\widehat{T}}\right)^*(x^{**},x^*) = \sup_{(\hat{y},y^*)\in\widehat{T}} \left\{\langle \hat{y},x^*\rangle + \langle x^{**},y^*\rangle - \langle \hat{y},y^*\rangle\right\}.$$

Denote by $\overline{f}$ the $\mathrm{bw}^* \times s$-closed convex function whose epigraph is $\overline{\operatorname{epi} f}^{\mathrm{bw}^* \times s}$. When $\operatorname{epi} f$ possess the VBDP then this closure corresponds to a set that is closed to limits of bounded and $\mathrm{w}^* \times s$-convergent nets. If we take $\left(x_\beta^{**},x_\beta^*\right)_{\beta\in\Lambda} \to^{\mathrm{w}^*\times s} (x^{**},x^*)$ with $\left\|x_\beta^{**}\right\| \le M$, for some $M > 0$ then we note the following important continuity property:

$$\left|\langle x_\beta^{**},x_\beta^*\rangle - \langle x^{**},x^*\rangle\right| \le \left|\langle x_\beta^{**} - x^{**},x^*\rangle\right| + M\left\|x_\beta^* - x^*\right\| \to 0. \tag{4}$$

We denote by $\overline{T}$ the *monotone closure* of the graph in $X^{**} \times X^*$ as studied by Gossez [4, 17]. That is,

$$\overline{T} := \{(x^{**},x^*) \mid \langle x^{**} - y, x^* - y^*\rangle \ge 0, \forall (y,y^*) \in T\}, \tag{5}$$

and the closure of the graph of with respect to τ-convergence by $\overline{T}^\tau$. Likewise $\operatorname{co} T$ denotes the convex hull of the graph of T. It is known that all maximal monotone

operators with convex graph are affine [6]. A monotone operator is said to be of *dense type* (D) if $\overline{T} = \overline{T}^{\mathrm{b}(\mathrm{w}^*\times s)}$ where $\mathrm{b}(\mathrm{w}^* \times s)$ requires convergence of all $\mathrm{w}^* \times s$ convergent and bounded nets. Note that while this is not a topology, for any monotone operator $\overline{T} \supseteq \overline{T}^{\mathrm{bw}^*\times s}$ while any monotone operator of dense type is (NI).
Various relationships that follow easily from the definitions of these objects are collected in the next proposition.

Proposition 9 *Let X be an arbitrary Banach space and let T be a monotone set in $X \times X^*$.*

1. *We have*

$$\mathcal{P}_{\widehat{T}}\left(x^{**}, x^*\right) := \mathcal{F}^*_{\widehat{T}}\left(x^*, x^{**}\right) \geq \widehat{\mathcal{F}_T}^*\left(x^*, x^{**}\right) \qquad \text{for all } \left(x^{**}, x^*\right). \tag{6}$$

2. *We have $\mathcal{P}_{\widehat{T}} = \overline{p_T}^{\mathrm{bw}^*\times s}$ where $p_T := \mathrm{co}\left(\langle\cdot,\cdot\rangle + \delta_T\right)$ and we use the bounded-weak* closure in X^{**} and strong closure in X^*. In particular*

$$\overline{T}^{\mathrm{b}(\mathrm{w}^*\times s)} \subseteq \left\{\left(y^{**}, y^*\right) \mid \mathcal{P}_{\widehat{T}}\left(y^{**}, y^*\right) = \langle y^{**}, y^*\rangle\right\} := M_{\mathcal{P}_{\widehat{T}}} \subseteq \mathrm{dom}\,\mathcal{P}_{\widehat{T}} \subseteq \overline{\mathrm{co}\,T}^{\mathrm{bw}^*\times s}. \tag{7}$$

3. *We have*

$$\widehat{\mathcal{F}_T}^*\left(x^*, x^{**}\right) \geq \mathcal{F}_{\widehat{T}}\left(x^{**}, x^*\right) \quad \text{for all } \left(x^{**}, x^*\right) \tag{8}$$

$$\text{and} \quad \widehat{\mathcal{F}_T}^*\left(x^*, x^{**}\right) \;\geq\; \widehat{\mathcal{P}_T}^*\left(x^*, x^{**}\right) \geq \mathcal{F}^{**}_{\widehat{T}}\left(x^{**}, x^*\right) = \overline{\left(\mathcal{F}_{\widehat{T}}\right)}\left(x^{**}, x^*\right) \qquad \text{for all } \left(x^{**}, x^*\right). \tag{9}$$

Thus $\widehat{\mathcal{F}_T}^*\left(x^*, \hat{x}\right) \geq \langle \hat{x}, x^*\rangle$ for all $(x, x^*) \in X \times X^*$, whenever T is maximal and

$$\widehat{\mathcal{F}_T}^*\left(x^*, x^{**}\right) \geq \langle x^{**}, x^*\rangle$$

for all $(x^{**}, x^*) \in X^{**} \times X^*$ exactly when T is (NI).

4. *Let T be a monotone operator on X. Then*

$$\mathcal{P}_{\widehat{T}}\left(x^{**}, x^*\right) \geq \widehat{\mathcal{F}_T}^*\left(x^*, x^{**}\right) \geq \widehat{\mathcal{P}_T}^*\left(x^*, x^{**}\right) \geq \mathcal{P}^*_{\widehat{T}}\left(x^*, x^{**}\right) = \mathcal{F}_{\widehat{T}}\left(x^{**}, x^*\right) \;. \tag{10}$$

Proof. Identity (6) The equality is well known [15, 7]. The inequality is definitional.
(8) We have

$$\mathcal{F}_{\widehat{T}}\left(x^{**}, x^*\right) := c^*_{J_{X\times X^*}(T)}\left(x^*, x^{**}\right) = \sup_{(\hat{y}, y^*) \in J_{X\times X^*}(T)} \left\{\langle \hat{y}, x^*\rangle + \langle x^{**}, y^*\rangle - \langle \hat{y}, y^*\rangle\right\}$$

and so

$$\mathcal{P}_{\widehat{T}}\left(x^{**}, x^*\right) := \mathcal{F}^*_{\widehat{T}}\left(x^*, x^{**}\right) = c^{**}_{J_{X\times X^*}(T)}\left(x^{**}, x^*\right).$$

By definition, for all $(y, y^*) \in T$ we have

$$\widehat{\mathcal{F}_T}^*(x^*, x^{**}) \geq \langle x^{**}, y^* \rangle + \langle x^*, y \rangle - \langle y, y^* \rangle$$

and so

$$\widehat{\mathcal{F}_T}^*(x^*, x^{**}) + \langle y, y^* \rangle \geq \langle x^{**}, y^* \rangle + \langle x^*, y \rangle.$$

On taking $\sum_i \lambda_i (y_i, y_i^*, 1) = (y, y^*, 1)$, with $(y_i, y_i^*) \in T$ then we find

$$\widehat{\mathcal{F}_T}^*(x^*, x^{**}) + p_{\widehat{T}}(\widehat{y}, y^*) \geq \langle x^{**}, y^* \rangle + \langle x^*, \widehat{y} \rangle.$$

Taking the bw$^* \times s$ closure within $X^{**} \times X^*$ we obtain for all $(y^{**}, y^*) \in X^{**} \times X^*$ that

$$\widehat{\mathcal{F}_T}^*(x^*, x^{**}) + \mathcal{P}_{\widehat{T}}(y^{**}, y^*) \geq \langle x^{**}, y^* \rangle + \langle x^*, y^{**} \rangle. \tag{11}$$

Consequently

$$\widehat{\mathcal{F}_T}^*(x^*, x^{**}) \geq \mathcal{P}^*_{\widehat{T}}(x^*, x^{**}) = \mathcal{F}_{\widehat{T}}(x^{**}, x^*)$$

establishing the inequality in (8). So when T is maximal we have $\widehat{\mathcal{F}_T}^*(x^*, \hat{x}) \geq \mathcal{F}_{\widehat{T}}(\hat{x}, x^*) = \mathcal{F}_T(\hat{x}, x^*) \geq \langle \hat{x}, x^* \rangle$ for all (x, x^*). Directly from the definition we also have T of type (NI) exactly when $\mathcal{F}_{\widehat{T}}$ is a representative function on $X^{**} \times X^*$.

(9) We note that the first inequality follows from $\mathcal{P}_T \geq \mathcal{F}_T$ while

$$\mathcal{P}_T(x, x^*) = \widehat{\mathcal{F}_T}^*(x^*, \widehat{x}) = \left(\widehat{\mathcal{F}_T}\right)^*(x^*, \widehat{x})^\dagger \quad \text{or equivalently} \quad \mathcal{P}_T = \widehat{\left(\widehat{\mathcal{F}_T}\right)^*}^\dagger.$$

As $\widehat{\mathcal{F}_T} \geq \mathcal{F}_{\widehat{T}}$ it follows that

$$\widehat{\mathcal{P}_T}^* \geq \mathcal{F}^{**}_{\widehat{T}}.$$

The function $\mathcal{P}_{\widehat{T}}$ is the smallest bw$^* \times s$ closed convex function that interpolates $\langle \cdot, \cdot \rangle$ on $J_{X \times X^*}(T)$, that is, $\overline{p_T}^{\mathrm{bw}^* \times s}$. Consequently $\operatorname{dom} \overline{p_T}^{\mathrm{bw}^* \times s} \subseteq \overline{\operatorname{co} T}^{\mathrm{bw}^* \times s}$. Clearly $\widehat{T} \subseteq M_{\mathcal{P}_{\widehat{T}}}$ (as $\mathcal{P}_{\widehat{T}}(\widehat{y}, y^*)$ is a representative function for T on $X \times X^*$).

(7) The second inclusion is obvious and the third is left to the reader. For the first inclusion take $(x^{**}, x^*) \in \overline{T}^{\mathrm{b}(\mathrm{w}^* \times s)}$ and a norm bounded net $\left(x^{**}_\beta, x^*_\beta\right)_{\beta \in \Lambda} \in T$ with $\left(\widehat{x}_\beta, x^*_\beta\right) \to_\beta (x^{**}, x^*)$. Thus for all $\beta \in \Lambda$ we have $\mathcal{P}_{\widehat{T}}\left(\widehat{x}_\beta, x^*_\beta\right) = \mathcal{P}_{\widehat{T}}\left(x_\beta, x^*_\beta\right) = \langle x_\beta, x^*_\beta \rangle$ and by the w$^* \times s$ lower semi–continuity of the function $\mathcal{P}_{\widehat{T}}$ and inequalities (6) and (8) we obtain

$$\langle x^{**}, x^* \rangle \geq \mathcal{P}_{\widehat{T}}(x^{**}, x^*) = \mathcal{F}^*_{\widehat{T}}(x^*, x^{**}) \geq \widehat{\mathcal{F}_T}^*(x^*, x^{**}) \ \geq \mathcal{F}_{\widehat{T}}(x^{**}, x^*).$$

When $(x^{**}, x^*) \in \overline{T}^{\mathrm{b}(\mathrm{w}^* \times s)}$ as above we have existence of $\left(\widehat{x}_\beta, x^*_\beta\right) \in T$ with $\langle \widehat{x}_\beta - x^{**}, x^*_\beta - x^* \rangle \to_\beta 0$ and so

$$\begin{aligned} \mathcal{F}_{\widehat{T}}(x^{**}, x^*) &= \langle x^{**}, x^* \rangle - \inf_{(\hat{y}, y^*) \in \hat{T}} \langle \widehat{y} - x^{**}, y^* - x^* \rangle \\ &\geq \langle x^{**}, x^* \rangle - \lim_\beta \langle \widehat{x}_\beta - x^{**}, x^*_\beta - x^* \rangle \geq \langle x^{**}, x^* \rangle, \end{aligned}$$

as in (4). Thus (7) holds.

(10) This final set of inequalities is a consequence of the others. ∎

Remark 10 We observe that all inequalities in (10) may be strict and that when T is (NI) all the functions therein are representative on $X^{**} \times X^{*}$. Moreover, for an arbitrary maximal monotone operator $\mathcal{P}_{\widehat{T}}$ is representative on $X^{**} \times X^{*}$; the proof relies on the existence of a "self-conjugate" representative function minorizing $\mathcal{P}_{\widehat{T}}$ and is a more delicate version of the reflexive case given in [15]. ■

The next corollary follows from (7) and (10).

Corollary 11 *Let T be a monotone operator on a Banach space X. Then*

$$\overline{T} = \left\{ (y^{**}, y^{*}) \mid \mathcal{F}_{\widehat{T}} (y^{**}, y^{*}) \leq \langle y^{**}, y^{*} \rangle \right\}, \tag{12}$$

and if T is (NI) then

$$\overline{T} = \left\{ (y^{**}, y^{*}) \mid \mathcal{F}_{\widehat{T}} (y^{**}, y^{*}) = \langle y^{**}, y^{*} \rangle \right\}. \tag{13}$$

By contrast, if T is affine

$$\overline{T}^{\mathrm{b}(\mathrm{w}^{*} \times s)} \subseteq \left\{ (y^{**}, y^{*}) \mid \mathcal{P}_{\widehat{T}} (y^{**}, y^{*}) = \langle y^{**}, y^{*} \rangle \right\} \subseteq \overline{T}^{\mathrm{bw}^{*} \times s} \tag{14}$$

Below we clarify the slight abuse of notation flagged in the introduction. We denote

$$\operatorname{lev}_{\alpha} \widehat{f} := \left\{ (\hat{x}, x^{*}) \mid \widehat{f} (\hat{x}, x^{*}) \leq \alpha \right\}.$$

We give the next result in some detail because it is neither purely topological nor purely sequential.

Lemma 12 *If $\operatorname{lev}_{\alpha} \widehat{f}$ has the VBDP for all $\alpha \in \mathbf{R}$, then $\operatorname{epi} \widehat{f}$ has the VBDP.*

Proof. Let $C \supseteq \operatorname{epi} \widehat{f}$ be the smallest convex set that is closed with respect to norm bounded $\mathrm{w}^{*} \times s$-convergent nets. It is clear that

$$\overline{\operatorname{epi} \widehat{f}}^{\mathrm{bw}^{*} \times s} \supseteq C \supseteq \operatorname{epi} \widehat{f}$$

and $\overline{\operatorname{epi} \widehat{f}}^{\mathrm{bw}^{*} \times s}$ is the epigraph of the proper $\mathrm{bw}^{*} \times s$-closed convex function $\overline{\widehat{f}}$. Clearly C can be viewed as an epigraph, $\operatorname{epi} g$, of a convex function $g \leq \widehat{f}$ with

$$\operatorname{lev}_{\alpha} g = \{ (x^{**}, x^{*}) \mid (x^{**}, x^{*}, \gamma) \in C \text{ and } \gamma \leq \alpha \}.$$

Take a norm bounded net $\left(x_{\beta}^{**}, x_{\beta}^{*}\right)_{\beta \in \Lambda} \in \operatorname{lev}_{\alpha} g$ with $\left(x_{\beta}^{**}, x_{\beta}^{*}\right) \to^{\mathrm{w}^{*} \times s} (x^{**}, x^{*})$. Then for each $\beta \in \Lambda$ there exists $\left(x_{\beta}^{**}, x_{\beta}^{*}, \gamma_{\beta}\right) \in C$. Now $\gamma_{\beta} \leq \alpha$ and if we suppose $\gamma_{\beta} \downarrow -\infty$ we find that $\overline{\widehat{f}} (x^{**}, x^{*}) \leq g (x^{**}, x^{*}) = -\infty$ and $\overline{\widehat{f}}$ is not proper, a contradiction. Thus $\{\gamma_{\beta}\}_{\beta \in \Lambda}$ is bounded and hence there exists a subnet with $\left(x_{\beta_{\delta}}^{**}, x_{\beta_{\delta}}^{*}, \gamma_{\beta_{\delta}}\right) \in C$ and $\gamma_{\beta_{\delta}} \to \gamma \leq \alpha$ or $(x^{**}, x^{*}) \in \operatorname{lev}_{\alpha} g$. Consequently $\operatorname{lev}_{\alpha} g$ is closed with respect to norm-bounded $\mathrm{w}^{*} \times s$-convergent nets. As $\operatorname{lev}_{\alpha} \widehat{f} \subseteq \operatorname{lev}_{\alpha} g$ it follows from the VBDP of $\operatorname{lev}_{\alpha} \widehat{f}$ that for all real α we have

$$\operatorname{lev}_{\alpha} \widehat{f} \subseteq \overline{\operatorname{lev}_{\alpha} \widehat{f}}^{\mathrm{bw}^{*} \times s} \subseteq \operatorname{lev}_{\alpha} g.$$

Now for all α and $\varepsilon > 0$ it easily follows that $\operatorname{lev}_{\alpha}\overline{\widehat{f}} \subseteq \overline{\operatorname{lev}_{\alpha+\varepsilon}\widehat{f}}^{\mathrm{bw}^* \times s}$ and so $\operatorname{lev}_{\alpha}\overline{\widehat{f}} \subseteq \operatorname{lev}_{\alpha+\varepsilon} g$. Hence we have $\operatorname{epi} \overline{\widehat{f}} \subseteq \operatorname{epi} g = C$. ∎

Since a representative function satisfies $f \leq \langle \cdot, \cdot \rangle + \delta_T = c_T$ we have $\widehat{c_T^*} \leq \widehat{f}^*$. Thus, when T is maximal

$$\langle x, x^* \rangle \leq \mathcal{F}_T(x, x^*) \ \leq \widehat{f}^*(x^*, \widehat{x}) = f^*(x^*, x).$$

The $\mathrm{bw}^* \times s$ -closure of $\widehat{f}$ is the convex function $\overline{\widehat{f}} : X^{**} \times X^* \to \overline{\mathbf{R}}$ with epi-graph $\operatorname{epi} \overline{\widehat{f}} = \overline{\operatorname{epi} \widehat{f}}^{\mathrm{bw}^* \times s} \equiv \overline{\operatorname{epi} \widehat{f}}^{\mathrm{w}^* \times s}$ where the w^* topologies are placed on X^{**} only. The following results show that when f is a representative function such that $\operatorname{epi} \widehat{f}$ also has the VBDP then $\overline{\widehat{f}}$ is representative as well. Such functions are always proper since $\overline{\widehat{f}} \geq \langle \cdot, \cdot \rangle > -\infty$.

Lemma 13 *Let X be a Banach space. Suppose $f : X \times X^* \to \mathbf{R}$ is a representative function and that the convex set $\operatorname{epi} \widehat{f} \subseteq X^{**} \times (X^* \times \mathbf{R})$ has the VBDP. Then the $\mathrm{bw}^* \times s$ -closure of $\widehat{f}$ is a representative function. That is*

$$\overline{\widehat{f}}(y^{**}, y^*) \geq \langle y^{**}, y^* \rangle, \tag{15}$$

for all $(y^{**}, y^*) \in X^{**} \times X^*$.

Proof. Note that

$$\operatorname{epi} \widehat{f} \subseteq \{(x^{**}, x^*, \alpha) \in X^{**} \times X^* \mid \langle x^{**}, x^* \rangle \leq \alpha\}.$$

Using the VBDP we take a bounded net $\left(x_\beta^{**}, x_\beta^*\right)_{\beta \in \Lambda} \to^{\mathrm{w}^* \times s} (x^{**}, x^*)$ and so $\langle x_\beta^{**}, x_\beta^* \rangle \to \langle x^{**}, x^* \rangle$. Thus, $\{(x^{**}, x^*, \alpha) \in X^{**} \times X^* \mid \langle x^{**}, x^* \rangle \leq \alpha\}$ is closed with respect to limits of $\mathrm{w}^* \times s$-convergent bounded nets. Hence, the smallest such set $\mathrm{bw}^* \times s$ closed set C containing $\operatorname{epi} \widehat{f}$ satisfies

$$\operatorname{epi} \overline{\widehat{f}} = C \subseteq \{(x^{**}, x^*, \alpha) \in X^{**} \times X^* \mid \langle x^{**}, x^* \rangle \leq \alpha\}.$$

∎

To show an operator is (NI) we shall use the fact that when T is maximal we have $\mathcal{F}_T \geq \langle \cdot, \cdot \rangle$ and

$$\widehat{\mathcal{F}_T}(\widehat{x}, x^*) = \mathcal{F}_{\widehat{T}}(\widehat{x}, x^*) \quad \text{(the restriction of } \mathcal{F}_{\widehat{T}} \text{ to } X \times X^*).$$

We aim to show that when the VBDP is present the $\mathrm{bw}^* \times s$ closure of this function must also be representative; and so to deduce that $\mathcal{F}_{\widehat{T}}$ is representative. Hence, we must show that $\overline{\widehat{\mathcal{F}_T}} = \mathcal{F}_{\widehat{T}}$. By (9) we have $\mathcal{F}_{\widehat{T}}^{**} = \overline{(\mathcal{F}_{\widehat{T}})} = \mathcal{F}_{\widehat{T}}$—so this seem plausible. Yet, (9) is insufficient as we require the identity $\left(\widehat{\mathcal{F}_T}\right)^{**} = \mathcal{F}_{\widehat{T}}$. Because $\mathcal{F}_T = c_T^*$ one may as well study the relationship between $\overline{\left(\widehat{(c_T^*)}\right)}$ and $\mathcal{F}_{\widehat{T}} = \widehat{c_T}^*$. We will in fact show in Proposition 15 $\overline{\left(\widehat{(f^*)}\right)} = \widehat{f}^*$.

By $\widehat{(f^*)}$ denote as before the embedding of the conjugate function $f^* : X^* \times X \to \mathbf{R}$ into $X^* \times X^{**}$. Again, for a representative function f on $X \times X^*$ we denote its conjugate in the dual space $(X \times X^*)^*$ by $\widehat{f}^*$, and its conjugate between the paired spaces $s \times \mathrm{bw}^* \,(X \times X^*)$ and $\mathrm{bw}^* \times s\,(X^* \times X)$ by f^*. This notation is consistent in that $\left(\widehat{f}\right)^* (x^*, x^{**}) = \widehat{f}^* (x^*, x^{**})$ and so $\widehat{f}^* (x^*, \widehat{x}) = f^* (x^*, x)$, or $\widehat{(f^*)} (x^*, \hat{x}) = \left(\widehat{f}\right)^* (x^*, \widehat{x})$. With this notation, we have:

Proposition 14 *Suppose $f : X \times X^* \to \overline{\mathbf{R}}$ is proper, $s \times \mathrm{bw}^*$-closed convex function. Then for all $(\widehat{y}, y^*) \in \operatorname{dom} \widehat{f} \subseteq \widehat{X} \times X^*$ we have*

$$\overline{\widehat{f}} (\widehat{y}, y^*) = \liminf_{\left(\widehat{y_\beta}, y^*_\beta\right) \to^{\mathrm{w}^* \times s}_\beta (\widehat{y}, y^*)} \widehat{f} \left(\widehat{y_\beta}, y^*_\beta\right) = \widehat{f} (\widehat{y}, y^*) . \tag{16}$$

Proof. Since f is $s \times \mathrm{bw}^*$-closed, convex and proper, it also norm-closed. Being convex f is also weakly closed and so weak$\times$strong-closed. Now if we take $\left(\widehat{y_\beta}, y^*_\beta\right) \to^{\mathrm{w}^* \times s}_\beta (\widehat{y}, y^*)$ (within $X^{**} \times X^*$) then $\left(y_\beta, y^*_\beta\right) \to^{\mathrm{w} \times s}_\beta (y, y^*)$. Thus

$$\liminf_{\left(\widehat{y_\beta}, y^*_\beta\right) \to^{\mathrm{w}^* \times s}_\beta (\widehat{y}, y^*)} \widehat{f} \left(\widehat{y_\beta}, y^*_\beta\right) \geq \widehat{f} (\widehat{y}, y^*) .$$

The reverse inequality is immediate, using a constant net, and we obtain the second equality in (16). The first equality in (16) is a standard representation of the lower semi–continuous hull. ■

Finally, we establish the promised consistency of closure and conjugation operations.

Proposition 15 *Suppose $f : X \times X^* \to \overline{\mathbf{R}}$ is a proper, $s \times \mathrm{bw}^*$-closed convex function. Then*

$$\overline{\widehat{(f^*)}} = \left(\widehat{f}\right)^* . \tag{17}$$

Proof. By definition $\overline{\operatorname{epi} \widehat{(f^*)}} \subseteq \operatorname{epi} \widehat{f}^*$. Hence, $\widehat{(f^*)} \geq \widehat{f}^*$ and $\overline{\widehat{f}^*} \geq \widehat{f}^*$ since $\operatorname{epi} \widehat{f}^*$ is $s \times \mathrm{bw}^*$ -closed. Thus, $\overline{\widehat{(f^*)}} (x^*, x^{**}) \geq \widehat{f}^* (x^*, x^{**})$ for all (x^*, x^{**}) in $X^* \times X^{**}$.

Conversely take $(x^*, x^{**}) \in \operatorname{dom} \left(\widehat{f}\right)^*$ and r such that $\overline{\widehat{f}^*} (x^*, x^{**}) > r$. The $s \times \mathrm{w}^*$-lower semicontinuity of $\widehat{f}^*$ implies the set

$$U := \left\{ (z^*, z^{**}) \mid \left(\widehat{f}\right)^* (z^*, z^{**}) > r \right\}$$

is a $s \times \mathrm{w}^*$- open neighbourhood in $X^* \times X^{**}$. Likewise, $s \times \mathrm{w}^*$-lower semi-continuity of $\overline{\widehat{f}^*}$ yields a convex $s \times \mathrm{w}^*$- open neighbourhood V of (x^*, x^{**}) in $X^* \times X^{**}$ such that

$$\begin{aligned} \overline{\widehat{f}^*} (z^*, z^{**}) &> r \quad \text{for all } (z^*, z^{**}) \in V \\ \text{and hence } \left(\widehat{f}\right)^* (z^*, \widehat{z}) &> r \quad \text{for all } (z^*, \widehat{z}) \in V \cap \left(X^* \times \widehat{X}\right) \end{aligned}$$

where the last inequality holds since $\left(\widehat{f}\right)^* (z^*, \widehat{z}) = \widehat{f}^* (z^*, \widehat{z}) = \overline{\widehat{f}^*} (z^*, \widehat{z})$ as shown in Proposition 14.

Thence we have a net $\left(x_\beta^*, \widehat{x}_\beta\right) \to^{s\times w^*} (x^*, x^{**})$ such that $\widehat{f^*}\left(x_\beta^*, \widehat{x}_\beta\right) \to \overline{\widehat{f^*}}(x^*, x^{**})$. Eventually this net lies in $V \cap \left(X^* \times \widehat{X}\right)$ and so

$$\emptyset \neq V \cap \left(X^* \times \widehat{X}\right) \subseteq \left\{(z^*, z^{**}) \mid \left(\widehat{f}\right)^* (z^*, z^{**}) > r\right\} = U.$$

It follows—by Lemma 18 given below for completeness—that

$$\overline{V}^{s\times w^*} = \overline{V \cap \left(X^* \times \widehat{X}\right)}^{s\times w^*} \subseteq \overline{U}^{s\times w^*}. \tag{18}$$

Thus, $(x^*, x^{**}) \in V \subset \operatorname{int} \overline{U} = U$ and so for all r such that $\overline{\widehat{(f^*)}}(x^*, x^{**}) > r$ we have

$$\left(\widehat{f}\right)^* (x^*, x^{**}) > r.$$

It follows that $\overline{\widehat{(f^*)}}(x^*, x^{**}) \leq \left(\widehat{f}\right)^* (x^*, x^{**})$, giving the asserted equality. ∎

Lemma 16 *Suppose W is an open convex subset of a locally convex topological vector space X and A is a dense convex subset of X. Then $\overline{W \cap A} = \overline{W}$.*

Proof. Without loss we assume $0 \in W$. Let γ_W denote the Minkowski function of W given by $\gamma_W(x) := \inf_{t>0}\{t \mid x \in t\overline{W}\}$ and for $\lambda \in X^*$ consider the program $\mu := \inf\{\lambda(x) \mid \gamma_W(x) < 1, x \in A\}$. Slater's condition holds since W meets A. Thus, there is a positive number p such that $\lambda(x) + p(\gamma_W(x) - 1) \geq \mu$ for all $x \in A$. Since the function $\lambda + p \cdot \gamma_W$ is semicontinuous and A is dense,

$$\inf\{\lambda(x) \mid \gamma_W(x) < 1, x \in A\} = \mu = \inf\{\lambda(x) \mid \gamma_W(x) \leq 1\},$$

which is equivalent to the desired conclusion. ∎

We now show the conjugates of the Fitzpatrick and Penot functions are representatives when $Z = X^*$ is a dual space.

Theorem 17 *Suppose T is maximal monotone on X. Let $E := X \times X^*$ and suppose E^* has an angelic dual ball as happens if (a) E^* is a subspace of a WCG Banach space or (b) if E is a separable Banach space containing no copy of ℓ^1.*

Then $\widehat{\mathcal{P}_T}^*$ and $\widehat{\mathcal{F}_T}^*$ are representative functions on $X^* \times X^{**}$. Indeed the hypotheses imply that $\widehat{f}^*$ is representative on $X^* \times X^{**}$ for any representative function for T.

Proof. Under the given assumptions $\operatorname{epi}\widehat{(\mathcal{P}_T^*)}$ and $\operatorname{epi}\widehat{(\mathcal{F}_T^*)}$ are embedded as convex subsets of $X^* \times X^{**} \times \mathbf{R}$, with $\mathcal{P}_T^*$ and $\mathcal{F}_T^*$ representative on $X^* \times \widehat{X}$. Since X possess the VBDP by Lemma 13 we deduce that $\overline{\widehat{(\mathcal{P}_T^*)}}$ and $\overline{\widehat{(\mathcal{F}_T^*)}}$ are representative on $X^* \times X^{**}$. Now apply Proposition 15 to deduce that $\overline{\widehat{(\mathcal{P}_T^*)}} = \widehat{\mathcal{P}_T}^*$ and $\overline{\widehat{(\mathcal{F}_T^*)}} = \widehat{\mathcal{F}_T}^*$ are representative functions on $X^* \times X^{**}$.

Finally, suppose f is a representative function for T. Then $\mathcal{P}_T \geq f \geq \mathcal{F}_T$ implying

$$\left(\widehat{f}\right)^* \geq \widehat{\mathcal{P}_T}^* \geq \langle \cdot, \cdot \rangle.$$

As $\left(\widehat{f}\right)^*$ is clearly proper and closed, we are done. ∎

4 Applications to Maximal Monotone Operators

We begin this section by reprising what it is that we have now established:

Corollary 18 *Let $E := X \times X^*$. Suppose E^* has an angelic dual ball as happens if E^* is a subspace of a WCG Banach space or if E is a separable Banach space containing no copy of ℓ^1. Then X is of type (NI), that is, every maximal monotone operator on X is type (NI).*

Proof. We need to show that

$$\mathcal{F}_{\widehat{T}}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle$$

for all elements of $X^{**} \times X^*$. This follows from Lemma 13 and Theorem 15 which justify the following identities

$$\left(\widehat{\mathcal{F}_T}\right)^{**} = \overline{\widehat{\mathcal{F}_T}} = \overline{\left(\widehat{(c_T^*)}\right)} = \widehat{c_T}^* = \mathcal{F}_{\widehat{T}}.$$

■

Thanks to the many striking recent results in [1, 21, 18] as reprised in [6, Ch. 6] we can now assert the following theorem. All the classes mentioned in the result which follows are carefully described by Simons who introduced many of them [17].
A schematic illustration of known and open relationships is given in Figure 1.

Theorem 19 *Let X be a Banach space of type (NI). Then every maximal monotone operator T is of type (NI) and in consequence:*

1. *The range and domain of T have convex norm-closure [1, 21, 18];*
2. *T has the Brønsted-Rockafellar property (BR) [1];*
3. *T is almost negatively aligned (ANA) [21];*
4. *T is both locally maximal monotone (LMM) or (FP) and maximally locally monotone (FPV) [21].*

Moreover, if S is another maximal monotone operator and

$$0 \in \operatorname{core}\{\operatorname{dom} T - \operatorname{dom} S\}$$

then $T + S$ is again maximal monotone [21].

Remark 20 While Corollary 18 shows that there are many non-reflexive Banach spaces of type (NI), the angelic class we have identified is almost certainly a small subset of the full class of (NI) Banach spaces. Indeed, all known examples of maximal operators which fail to be type (NI) lie in spaces containing a *complemented* copy of ℓ^1, see [3]. This even excludes $C[0,1]$ which—while universal for separable spaces—contains only non-complemented copies of ℓ^1.

For example, the continuous linear map $S:\ell^1\to\ell^\infty$ given by

$$(Sx)_n := -\sum_{k<n} x_k + \sum_{k>n} x_k, \quad \forall x=(x_k)\in\ell^1, n\in N,$$

is called the *Gossez operator*. We record that $\mp S$ is a skew bounded linear operator, for which S^* is not monotone but $-S^*$ is. Hence, $-S$ is of *dense type* (D) [17, 3] and has all the consequent properties of Theorem 19, while S is maximal monotone locally but is not of type (NI) or (FP), but it does have a unique monotone extension to the second dual [17, 3]. Corollary 11 applied to the Gossez operator highlights the distinct roles of $\mathcal{F}_{\widehat{T}}$ and $\mathcal{P}_{\widehat{T}}$. ■

Thus, it would be reasonable to conjecture that a separable Banach space is type (NI) if (and perhaps only if) it contains no copy of ℓ^1 and that all Asplund spaces are type (NI). An excellent start would be to determine the status of c_0 which contains no copy of ℓ^1—being separable and Asplund—but the space does not satisfy the hypotheses of Corollary 18.

Remark 21 It is now known that every maximal monotone operator of type (NI) that is of dense type (D) [2] and that both classes coincide with Simon's class (ED) [19]. ■.

Remark 22 There are various conditions on an individual operator in an arbitrary Banach space that easily assure it is type (NI). For example, a surjective maximal monotone operator is type (NI) and hence locally maximal monotone. This thus recaptures a result proved directly by Fitzpatrick and Phelps [17]. By contrast dom $T = X$ does not imply local maximality as is shown by the various skew examples. It is similarly easy to confirm that T is type (NI) if it has bounded domain and norm-dense range. More difficult but true is that an operator is (NI) if the convex hull of its range has non-empty interior. ■

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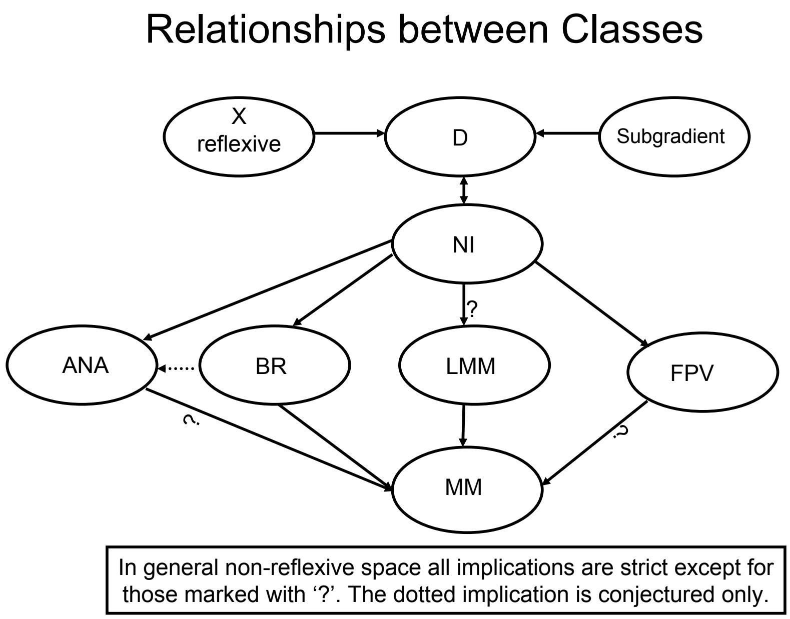

Figure 1: **Relations between classes of operators**.

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