On delta-convex functions

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CARMA
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http://carma.newcastle.edu.au/jon/dctalk.pdf

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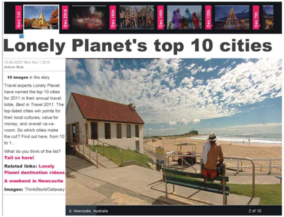

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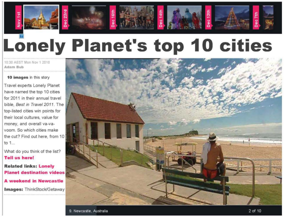

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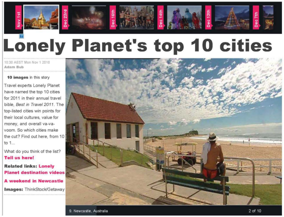

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Outline

1. Basic structure of DC functions
2. Examples of DC functions
   - Polynomials in several variables
   - Variational analysis
   - Nash equilibria
   - Eigenvalues
   - Further operator theory
3. Finer structure of DC functions
   - Differentiability
   - Composition of DC mappings
   - Toland duality
4. Negative results
   - Composition of DC mappings
   - Finite vs infinite dimensions
   - Differentiability
5. Distance functions

Definition of DC functions

Definition (DC functions)

Let X be a normed linear space. A function $f : X \to \mathbb{R}$ is delta-convex (or DC) (on an open Ω) if there exist convex continuous functions f_1, f_2 on X such that $f = f_1 - f_2$ (on Ω).

- Can typically assume $f_1, f_2 \geq 0$ by adding affine minorants.

Conjecture

Delta-convex functions first appeared in the paper:

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DC mappings between Euclidean spaces

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A mapping $F = (F_1, \ldots, F_m) : \mathbb{R}^n \to \mathbb{R}^m$ is *DC* if all the components $F_1, \ldots, F_m$ are DC functions.

- $f : [a, b] \to \mathbb{R}$ is DC if and only if f is absolutely continuous (AC) and f' has bounded variation (BV) – precisely a difference of two nondecreasing functions.

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$$y^* \circ F + \widetilde{f}$$

is convex (on Ω) for *all* $y^* \in Y^*$, with $\|y^*\| = 1$.

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Order DC mappings

Recall that $F : \Omega \subset X \mapsto Y$ is S-convex (order-convex) when

$$\mathrm{Epi}_S(F) := \{(x, y) : F(x) \in y + S, x \in \Omega\}$$

is convex and $S \subset Y$ us a convex cone.

- If $G = F_1 - F_2$ with F_1, F_2 both S-convex, we say G is S-DC or *order-DC*.

Theorem (Order Convexity)

Suppose S is a convex cone whose dual S^+ has nonempty interior.

- *Then every S-DC operator is DC. (Can vary the S.)*
- *In particular, $\mathbb{R}^N_+$-DC and DC coincide in $\mathbb{R}^N$.*

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Structural properties

Theorem (Structure)

The real-valued DC functions on an open set form a subspace of locally Lipschitz functions and:

1. *a vector space;*
2. *an algebra (closed under multiplication);*
3. *a lattice (closed under finite maxima/minima).*

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Mixing properties — 'convex under switching'

Theorem (Mixing, Veselý-Zajíček, 2001)

Let $g_1, g_2, \ldots, g_n$ be DC on Ω. Any continuous selection σ with

$$\sigma(x) \in \{g_1(x), g_2(x), \ldots, g_n(x)\}$$

for all $x \in \Omega$ is also a DC function.
In particular, each piecewise linear and continuous function is DC.

A nice (partial) converse is:

Theorem (Absoluteness)

Let f be continuous, real-valued. Then $|f|$ is DC if and only if f is.

- This converse fails for $\|f\|$.

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Examples of DC functions

We now present various examples of DC functions arising naturally:

- Polynomials in several variables
- Variational analysis
- Non-cooperative game theory
- Spectral theory
- Operator theory

Polynomials in several variables

Theorem (Polynomials)

Polynomials on $\mathbb{R}^N$ are DC: each polynomial p can be decomposed as $p = q - r$ where r, q are nonnegative convex functions.

- Hence, DC functions are dense uniformly in $C(\Omega)$ for compact Ω — there are too many of them.

- Easy induction: $x^{2n-1} = (x^+)^{2n-1} - (x^-)^{2n-1}$ and x^{2n} are DC in an algebra (Structure Thm), as positive convex squares are convex and: $\pm 2fg = (|f| + |g|)^2 - |f|^2 - |g|^2$.

Conjecture

There is a concise explicit determinantal decomposition in $\mathbb{R}^N$.

- I found one 35 years ago but have lost it!

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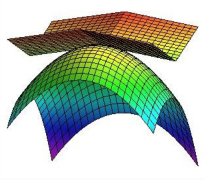
Definition

A function $f : X \to \mathbb{R}$ is *paraconvex* if there is $\lambda \geq 0$ such that $f + \frac{\lambda}{2}\|\cdot\|^2$ is continuous and convex; $-f$ is *paraconcave*.

Example

Clearly, paraconvex and paraconcave functions are 'very' DC.

- On Hilbert space, locally paraconvex = lower-$\mathcal{C}^2$.



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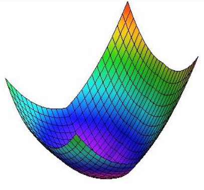

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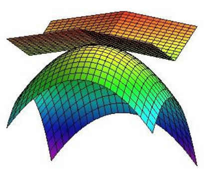

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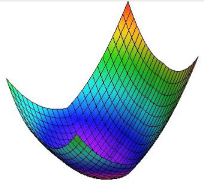

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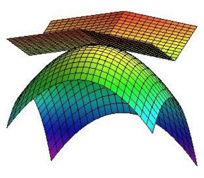

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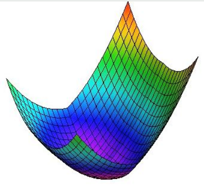
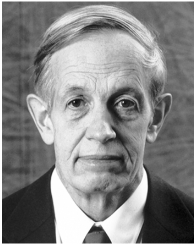


Non-cooperative game theory

n-player games



Player i has:

- pure strategies $(\pi_{i\alpha})_\alpha$
- mixed strategies S_i = convex combination
- pay-off function $p_i(\pi_{1\alpha_1}, \ldots, \pi_{i\alpha_i}, \ldots, \pi_{n\alpha_n})$

Definition (Equilibrium)

An n-tuple $s = (s_1, \ldots, s_n)$, where $s_i \in S_i$, is an *equilibrium point* of the game if for each $1 \leq i \leq n$ we have

$$p_i(s) = \max_{t_i \in S_i} p_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n).$$

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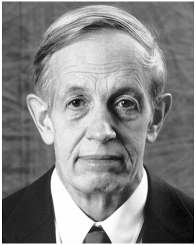

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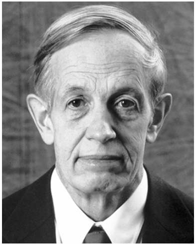

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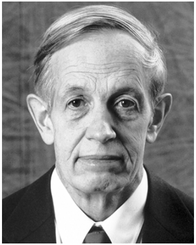

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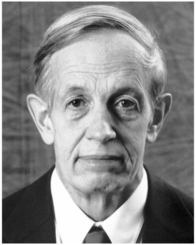

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Theorem (Nash, 1951)

Assuming convexity of all $t_i \mapsto p_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$, *every Nash game admits an equilibrium point.*

Sketch of Nash's proof.

Denote $p_{i\alpha}(s) := p_i(s, \pi_{i\alpha})$, and define DC functions

$$\varphi_{i\alpha}(s) := \max\{0, p_{i\alpha}(s) - p_\alpha(s)\} \quad i = 1, \ldots, n.$$

Define $T : s \mapsto s'$ componentwise by

$$s_i' := \frac{s_i + \sum_\alpha \varphi_{i\alpha}\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}\pi_{i\alpha}}.$$

Equilibria are fixed points of T, which exist (Brouwer). □

- T is DC as a DC ratio (not so useful; only in Euclidean space).
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Spectral theory in finite dimensions

Denote by $\mathcal{S}_N$ the set of real symmetric N by N matrices.

Theorem (Lewis, 1995)

The k^{th}-largest eigenvalue function

$$\lambda_k : A \to \lambda_k(A)$$

is DC on the space of symmetric matrices $\mathcal{S}_N$. Indeed,

$$\lambda_k = \sigma_k - \sigma_{k-1}$$

where σ_k, the sum of the k largest eigenvalues, is convex for all k.

- Try proving *directly* that λ_k is locally Lipschitz.

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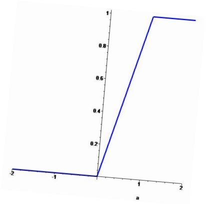
The 3×3 case

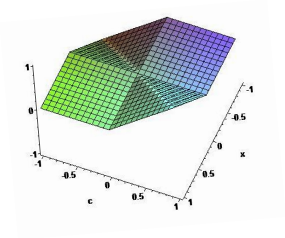
There are three eigenvalues: $\lambda_1, \lambda_2, \lambda_3$, and $\text{Trace} = \lambda_1 + \lambda_2 + \lambda_3$.

Now $\lambda_1(A) = \lambda_{MAX}(A) = \max_{\|x\|=1}\langle Ax, x\rangle$ is convex (Rayleigh-Ritz) and $\lambda_3 = \lambda_{MIN} = -\lambda_{MAX}(-\cdot)$ is concave (R-R). Then

$$\boxed{\lambda_2 = \text{Trace} - \lambda_1 - \lambda_3}$$

is a DC decomposition.





One-D and two-D cross-sections of λ_2

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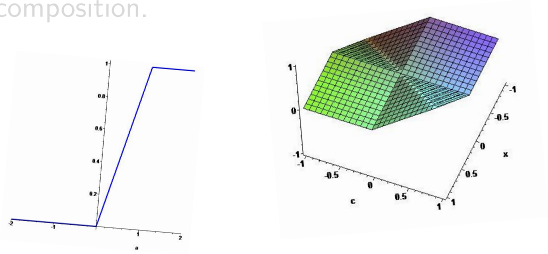

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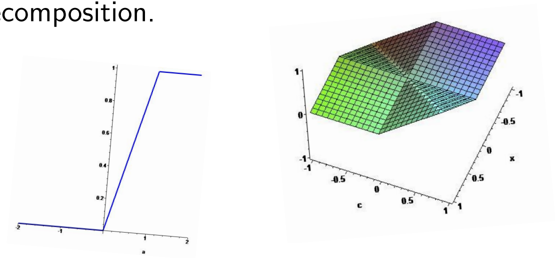

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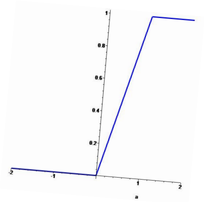

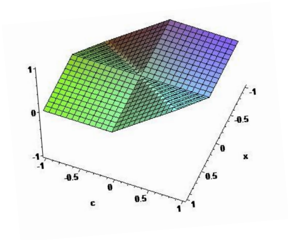

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Spectral theory in infinite dimensions

Denote by $\mathcal{B}_{\mathsf{sa}}$ the self-adjoint bounded linear operators on $\ell^2_{\mathbb{C}}$.

Definition (Schatten classes)

$A \in \mathcal{B}_{\mathsf{sa}}$ belongs to the 0-*Schatten class* if it is compact, and belongs to the p-*Schatten class*, $\mathcal{B}_p$, for $p \in [1, +\infty)$, if

$$\|A\|_p := \left(\mathrm{Trace}\left(|A|^p\right)\right)^{1/p} < \infty,$$

where $|A| := (A^*A)^{1/2}$.

- Then $\mathcal{B}_2$ is the *Hilbert-Schmidt operators* — a Hilbert space — and $\mathcal{B}_1$ is the *trace class* or *nuclear operators*.

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Theorem (B-Z, 2005)

For $p \in \{0\} \cup [1, +\infty)$ the k^{th}-largest eigenvalue function $\lambda_k : A \to \lambda_k(A)$ is DC on the set of positive operators of p-Schatten class.

Example

Despite not living on the nuclear operators — as induced by $\sum_i t_i - \log(1 + t_i)$ — we have :

$$A \mapsto \mathrm{Trace}(A) - \log\det(I + A)$$

is a convex barrier on $\mathcal{B}_2$, for $\{A : I + A \geq 0\}$.

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Further operator theory

Let X be a Banach space. Each symmetric bounded linear operator $T : X \to X^*$ generates a quadratic form on X by $x \mapsto \langle Tx, x \rangle$.

- When is a quadratic form DC?
- X is a UMD space if this holds for all symmetric T?

Theorem (Kalton-Konyagin-Veselý, 2008)

The quadratic form

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- All UMD spaces are super-reflexive;
- W_p and $\mathcal{B}_p$ — and so L_p is UMD — for $1 < p < \infty$. Hence:

Proposition

Let T be a symmetric bounded linear operator on a Hilbert space. Then the function $x \mapsto \langle Tx, x\rangle$ is DC on X.

- Alternative proof: Clearly $\langle T\cdot, \cdot\rangle$ is $C^{1,1}$, which in Hilbert spaces implies DC.
- A stronger result: $\langle T\cdot, \cdot\rangle$ is a difference of two nonnegative quadratic forms (necessarily convex): $T = \frac{|T|+T}{2} - \frac{|T|-T}{2}$.
  - "X is type (DCQ)" $\Leftarrow$ type $p \geq 2$; $\ell^p (p < 2)$ is not (DCQ).

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Finer structure of DC functions

JMB and MB

Differentiability properties

- The Clarke subdifferential on $\mathbb{R}^N$.

Theorem (Euclidean properties)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be DC with a decomposition $f = f_1 - f_2$. Then,

1. $\partial_C f(x) \subset \partial_C f_1(x) - \partial_C f_2(x)$ *for all* $x \in \mathbb{R}^n$;
2. $\partial_C f$ *reduces to* ∇f *a.e. on* $\mathbb{R}^n$; *so a.e. strictly differentiable;*
3. f *has a second-order Taylor expansion a.e. on* $\mathbb{R}^n$.

Proof of 1.

$$(f-g)^o(x;h) \leq (f)^o(x;h) + (-g)^o(x;h) = (f)'(x;h) + (-g)'(x;h).$$

(Find a minimal decomposition with equality?)

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(Find a minimal decomposition with equality?)

- $\partial_C f(x)$ need not be singleton when f is differentiable at $x \in \mathbb{R}^n$ i.e., DC need not be *regular*

Differentiability properties

- The Clarke subdifferential on $\mathbb{R}^N$.

Theorem (Euclidean properties)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be DC with a decomposition $f = f_1 - f_2$. Then,

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Let X be a Banach space and $A \subset X$ an open convex subset. Suppose $f : A \to \mathbb{R}$ is locally DC.

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Differentiability of the control function.

Proposition (Veselý-Zajíček, 2001)

Let X be a normed linear space and $A \subset X$ open and convex. Suppose $f : A \to \mathbb{R}$ is DC on A with a control function $\widetilde{f}$.

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Recall: f is DC if and only if there exists a continuous convex function $\widetilde{f}$ such that both $\pm f + \widetilde{f}$ are convex:

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Composition of DC mappings

Theorem (Hartman, 1959)

Let $A \subset \mathbb{R}^m$ be convex and either open or closed. Let $B \subset \mathbb{R}^n$ be convex and open. If $F : A \to B$ and $g : B \to \mathbb{R}$ are DC, then $g \circ F$ is a locally DC function on A.

Theorem (Veselý, Zajíček, 1987, 2009)

Let X, Y be normed linear spaces, $A \subset X$ a convex set, and $B \subset Y$ open convex. If $F : A \to B$ and $g : B \to \mathbb{R}$ are locally DC, then $g \circ F$ is locally DC on A.

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Toland duality, 1978

For a function $f : X \to (-\infty, \infty]$ on a Banach space X define its *conjugate function* by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \qquad x^* \in X^*.$$

Theorem (Ellaia and Hiriart-Urruty, 1986)

Let X be a Banach space, $h : X \to \mathbb{R}$ be convex continuous, and $g : X \to (-\infty, \infty]$ any function. Then for each $x^* \in \operatorname{dom} g^*$,

$$(g-h)^*(x^*) = \sup_{y^* \in \operatorname{dom} h^*} \{g^*(x^* + y^*) - h^*(y^*)\}$$

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Corollary

If we assume both g, h are continuous convex, and so $g - h$ is DC on X, we arrive at (1) along with

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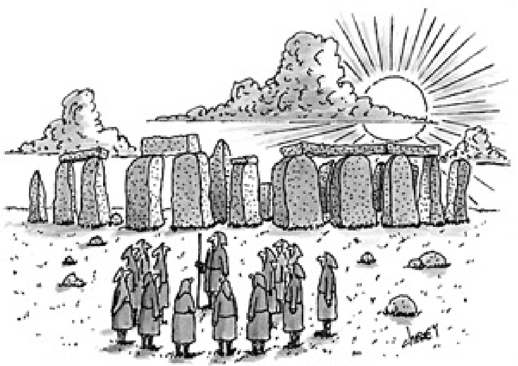
Negative results



"Now that we can tell time, I'd like to suggest that we begin imposing deadlines."

Counterexamples to composition

- A composition of DC functions that is not DC:

Example (Hartman, 1959)

The composition of DC functions need not be DC even in $\mathbb{R}$. Consider

$$f : (0,1) \to [0,1) : x \mapsto |x - 1/2|,$$

and

$$g : [\mathbf{0},1) \to \mathbb{R} : y \mapsto 1 - \sqrt{y}.$$

Then $g \circ f$ is not DC at $1/2$.

- Note: $0 \notin \operatorname{int}[0,1)$.

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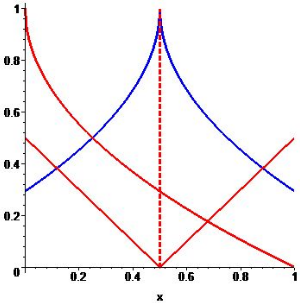

Figure: $g \circ f = 1 - \sqrt{|\cdot - 1/2|}$ is not DC around $1/2$.

- One-sided derivatives of $g \circ f$ infinite at $1/2$ (DC have finite limits).
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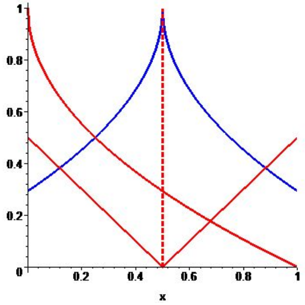

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What follows is a very general method of constructing composition counter-examples:

Theorem (Veselý-Zajíček, 2009)

Let X, Y be infinite-dimensional normed linear spaces. Let $A \subset X$ and $B \subset Y$ be convex with A open.
Suppose $g : B \to \mathbb{R}$ is unbounded on some bounded subset of B.
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Theorem (Veselý, Zajíček, 2009)

Let X be a normed linear space and $A \subset X$ open convex set. Then the following are equivalent.

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- Reciprocals of convex functions yield a striking variant.

Theorem (Holický et al, 2007)

X is reflexive (resp. finite dim.) if and only if every positive continuous convex (resp. DC) function on X has $1/f$ DC.

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There exists a function on ℓ_2 which is DC on each bounded convex subset of ℓ_2 but is not DC on ℓ_2.

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Counterexamples to differentiability theorems

Theorem (Kopecká, Malý, 1990)

There exists a DC function on $\mathbb{R}^2$ which is strictly Fréchet differentiable at the origin but which does not admit a control function that is Fréchet differentiable at the origin.

Theorem (Pavlica, 2005)

There exists a DC function on $\mathbb{R}^2$ which belongs to the class $\mathcal{C}^1$ but does not admit a control function that is Fréchet differentiable at the origin.

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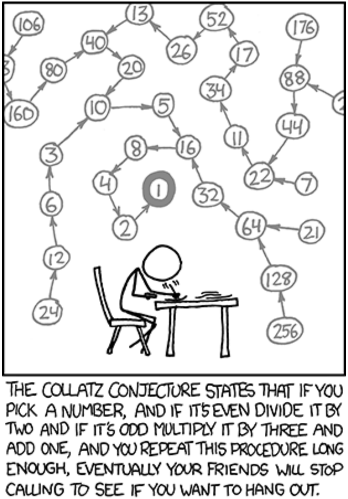
Distance functions



Distance functions: positive results

Observation (Asplund, 1969)

d_C^2 is paraconcave and so DC for $C \subset X$ closed in Hilbert space:

$$d_C^2(x) = -\sup_{c \in C} -\|x-c\|^2 = \|x\|^2 - [\sup_{c \in C} 2\langle x, c\rangle - \|c\|^2].$$

- The *smooth variational principle* produces:

Theorem (Borwein 1991, Borwein-Zhu, 2005)

For $C \subset X$ closed in Hilbert space, d_C is locally DC on $X \setminus C$ while $\partial_C d_C$ is a minimal CUSCO on X.

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Example (Borwein-Moors, 1997)

There is a closed set $C \subset \mathbb{R}^2$ with d_C not (locally) DC on $\mathbb{R}^2$.

Proof: Let $C := C_1 \times C_1 \subset \mathbb{R}^2$ for $C_1 \subset [0,1]$ be a Cantor set of positive measure.

d_C is not strictly differentiable anywhere on $\mathrm{bd}(C) = C$.

So d_C is not locally DC; as DC functions are a.e. strictly Fréchet.

- In particular, the operation $\sqrt{\cdot}$ does not preserve DC.
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- d_C is a very rich tool for building counter-examples.

Question

If the norm on a Banach space X is sufficiently nice, is d_C^2 DC locally for all closed sets C on X (d_C on $X \setminus C$)?

References — and many thanks to Regina and Yalcin

This talk was based on the paper:

- M. Bačák[1] and J.M. Borwein, "On difference convexity of locally Lipschitz functions." *Optimization*, 2011. (For Alfredo Iusem at 60.)
- Preprint available at: http://carma.newcastle.edu.au/jon/dc-functions.pdf

Additional information is to be found in:

- J.M. Borwein and J. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, CUP, 2010.
- Website: http://carma.newcastle.edu.au/ConvexFunctions/

[1] Now at Max Planck Institute, Leipzig