Deriving new sinc results from old

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Abstract

From previously established results in [2] we develop a simple proof of Keith Ball's expression in [1] for the volume of the intersection of an $(n-1)$-dimensional hyperplane with an n-dimensional cube, as well as a simple proof of the formula given by Frank and Riede in [5] for that volume.

In our study in [2] of sinc integrals over a decade ago we established results—some of which were recapitulated in [4]—that lead by differentiation, as we now show, to a simple proof of K. Ball's formula in [1] for the volume of the intersection of an $(n-1)$-dimensional hyperplane with an n-dimensional cube (Ball's original proof is itself quite direct), as well as a simple proof of the formula given very recently for that volume by Frank and Riede in [5].

We proved the following theorem in [2, Theorem 2, Remarks 1]. To assist in reading this note, we sketch that proof below.

Theorem 1. *Suppose that $b > 0$ and $a_k > 0$ for $k = 1, 2, \ldots, n$. For each of the 2^n ordered n-tuples $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \{-1, 1\}^n$ define*

$$\beta_\gamma = \beta_\gamma(b) := b + \sum_{k=1}^{n} \gamma_k a_k, \qquad \epsilon_\gamma := \prod_{k=1}^{n} \gamma_k.$$

Then

$$\begin{aligned}\int_0^\infty \left(\prod_{k=1}^{n} \operatorname{sinc}(a_k x)\right) \operatorname{sinc}(bx)\, dx &= \frac{\pi}{2^{n+1} n!\, b a_1 a_2 \cdots a_n} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \beta_\gamma^n \operatorname{sgn}(\beta_\gamma) \\ &= \frac{\pi}{2^{n+1} b} \operatorname{Vol}_n(P_n), \end{aligned} \tag{1}$$

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where $\operatorname{sinc}(t) := \sin(t)/t$ *and* $P_n = P_n(b)$ *is the* central polyhedral slab *given by*

$$P_n := \{(x_1, x_2, \cdots, x_n) \big| -b \leq \sum_{k=1}^{n} a_k x_k \leq b, -1 \leq x_k \leq 1 \text{ for } k = 1, 2, \ldots, n\}$$

Here and subsequently Vol_m *denotes the* m*-dimensional volume for any positive integer* m *and* sgn *the sign of a real number.*

Proof. From first principles we have

$$\begin{aligned}
\sin(bx) \prod_{k=1}^{n} \sin(a_k x) &= \frac{1}{(2i)^{n+1}} (e^{ibx} - e^{-ibx}) \prod_{k=1}^{n} (e^{ia_k x} - e^{-ia_k x}) \\
&= \frac{1}{(2i)^{n+1}} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \left(e^{i\beta_\gamma x} - (-1)^n e^{-i\beta_\gamma x} \right) \\
&= \frac{1}{2^n} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \cos\left(\beta_\gamma x - \frac{\pi}{2}(n+1) \right),
\end{aligned}$$

Hence

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x} \right) \frac{\sin(bx)}{x} \, dx = \frac{1}{2^n} \int_0^\infty x^{-n-1} c_n(x) \, dx,$$

where $c_n(x) := \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \cos\left(\beta_\gamma x - \frac{\pi}{2}(n+1) \right)$. Because $c_n(x)$ is an entire function, bounded for all real x, with a zero of order $n+1$ at $x = 0$, we can legitimately integrate the right-hand side by parts n times to get

$$\begin{aligned}
\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x} \right) \frac{\sin(bx)}{x} \, dx &= \frac{1}{2^n n!} \int_0^\infty \frac{dx}{x} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \beta_\gamma^n \sin(\beta_\gamma x) \\
&= \frac{1}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \beta_\gamma^n \int_0^\infty \frac{\sin(\beta_\gamma x)}{x} \, dx \\
&= \frac{\pi}{2} \frac{1}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \beta_\gamma^n \operatorname{sgn}(\beta_\gamma),
\end{aligned}$$

from which the first identity in (1) follows.

For the proof of the second identity in (1) we use standard Fourier analysis (for details see [2]) to write

$$\begin{aligned}\int_0^\infty \left(\prod_{k=1}^n \operatorname{sinc}(a_k x)\right) \operatorname{sinc}(bx)\,dx &= \frac{\pi}{b}\frac{1}{2^n a_1 a_2\cdots a_n}\int_0^{\min(s_n,b)} \chi_{a_1} * \chi_{a_2} * \cdots * \chi_{a_n}\,dx \\ &= \frac{\pi}{2^{n+1}b}\operatorname{Vol}_n(P_n), \qquad (2)\end{aligned}$$

where $s_n := a_1 + a_2 + \cdots + a_n$, χ_a is the characteristic function of the interval $(-a, a)$, and $*$ indicates convolution. That is

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-t) f_2(t)\,dt.$$

A key element in this analysis is the foundational fact that the Fourier cosine transform of χ_a is $\widehat{\chi_a} = a\sqrt{\frac{2}{\pi}}\operatorname{sinc}(ax)$. That is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi_a(t)\cos(xt)\,dt = a\sqrt{\frac{2}{\pi}}\operatorname{sinc}(ax).$$

The first equality in (2) is then a consequence of careful inductive application of *Parseval's identity* in the form

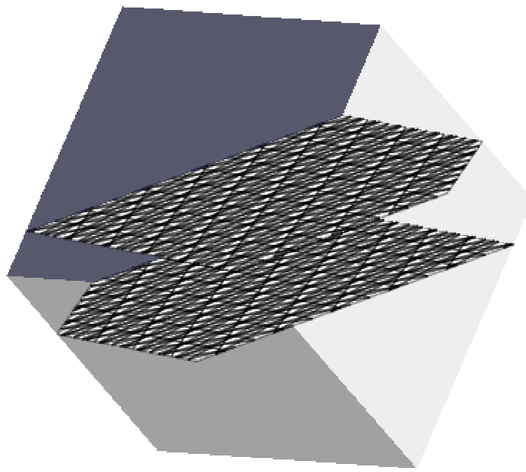
$$\int_{\mathbb{R}} f\,g = \int_{\mathbb{R}} \widehat{f}\,\widehat{g}$$

with $f(x) := \operatorname{sinc}(bx)$ and $g(x) := \prod_{k=1}^n \operatorname{sinc}(a_k x)$. See also [3, Lemma 3] where a more general result is established in extenso.

The final equality in (2) is now accessible. With $\mu_n := \min(s_n, b)$, we appeal to Fubini's theorem to write the n-fold convolution as a multiple integral and obtain:

$$\begin{aligned}&\int_0^{\min(s_n,b)} \chi_{a_1} * \chi_{a_2} * \cdots * \chi_{a_n}\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \left(\chi_{a_1}(x_1)\chi_{a_2}(x_2)\cdots\chi_{a_n}(x_n)\chi_{\mu_n}(x_1 + x_2 + \cdots + x_n)\right) dx_1\,dx_2\cdots dx_n \\ &= \frac{a_1 a_2\cdots a_n}{2}\int_{\mathbb{R}^n} \left(\chi_1(x_1)\chi_1(x_2)\cdots\chi_1(x_n)\chi_{\mu_n}(a_1x_1 + a_2x_2 + \cdots + a_nx_n)\right) dx_1\,dx_2\cdots dx_n \\ &= \frac{a_1 a_2\cdots a_n}{2}\int_{[-1,1]^n} \chi_{\mu_n}(a_1x_1 + a_2x_2 + \cdots + a_nx_n)\,dx_1\,dx_2\cdots dx_n \\ &= a_1 a_2\cdots a_n \frac{\operatorname{Vol}_n(P_n)}{2}.\end{aligned}$$

A comparison to (2) shows that this is the desired evaluation. $\square$



The central slab with $|2x+y+z| \le 2/3, |x| \le 1, |y| \le 1, |z| \le 1$.

It is necessary to treat proofs of results like Theorem 1 with some care. We recently showed in [4] how purely formal application of Fourier analysis techniques led to a result that was almost always wrong – the requisite side conditions rarely all applied at the same time.

From Theorem 1 we will deduce the following theorem, the first conclusion of which is Ball's identity proved in [1] and restated as [5, Theorem 1] and the second conclusion is [5, Theorem 2].

Theorem 2. *Suppose that the same assumptions hold as in Theorem 1. Let $C_n := [-1,1]^n$ be the n-dimensional hypercube and, for all real c, let*

$$H(c) := \{(x_1, x_2, \cdots, x_n) \big| \sum_{k=1}^{n} a_k x_k = c\}$$

be the $(n-1)$-dimensional hyperplane. Then

$$\begin{aligned}\mathrm{Vol}_{n-1}(C_n \cap H(b)) =& \frac{2^{n-1}|a|}{\pi} \int_{-\infty}^{\infty} \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{a_k x} \right) \cos(bx)\, dx \\ =& \frac{|a|}{2(n-1)!\, a_1\, a_2 \cdots a_n} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma \beta_\gamma^{n-1} \mathrm{sgn}(\beta_\gamma),\end{aligned} \tag{3}$$

where $|a| := \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

Proof. It follows from (1) that, for fixed n,

$$\begin{aligned}
T(b) :=& b\int_{-\infty}^{\infty}\left(\prod_{k=1}^{n}\operatorname{sinc}(a_k x)\right)\operatorname{sinc}(bx)\,dx \\
=& \frac{\pi}{2^n n!\,a_1a_2\cdots a_n}\sum_{\gamma\in\{-1,1\}^n}\epsilon_\gamma\beta_\gamma^n\operatorname{sgn}(\beta_\gamma) \\
=& \frac{\pi}{2^n}V_n(b), \qquad (4)
\end{aligned}$$

where

$$V_n(b) := \mathrm{Vol}_n(P_n(b))$$

is the n-dimensional volume of the part of the hypercube C_n lying between the parallel hyperplanes $H(\pm b)$. Now the distance $d(\varepsilon)$ between the parallel hyperplanes $H(c+\varepsilon)$ and $H(c)$ with $\varepsilon > 0$ is given by $d(\varepsilon) = \varepsilon/|a|$. It follows that

$$V_n{}'(b) = \lim_{\varepsilon\to 0}\frac{V_n(b+\varepsilon)-V_n(b)}{\varepsilon} = \frac{2}{|a|}\mathrm{Vol}_{n-1}(C_n\cap H(b)). \qquad (5)$$

Here we have used the fact that $V_n(b+\varepsilon)$ exceeds $V_n(b)$ by the volume of the two parallel strips—each volume being approximately equal to $d(\varepsilon)\mathrm{Vol}_{n-1}(C_n\cap H(b))$. Differentiating (4) partially with respect to b and applying (5) yields

$$\begin{aligned}
T'(b) =& \int_{-\infty}^{\infty}\left(\prod_{k=1}^{n}\operatorname{sinc}(a_k x)\right)\cos(bx)\,dx \\
=& \frac{\pi}{2^n(n-1)!a_1a_2\cdots a_n}\sum_{\gamma\in\{-1,1\}^n}\epsilon_\gamma\beta_\gamma^{n-1}\operatorname{sgn}(\beta_\gamma) \\
=& \frac{\pi}{2^{n-1}|a|}\mathrm{Vol}_{n-1}(C_n\cap H(b)), \qquad (6)
\end{aligned}$$

from which the desired conclusion (3) follows. □

This recaptures both Theorem 1 and Theorem 2 in [5].

Remark 1. An extension of Theorem 1 to more general polyhedra is given in [3]. In principle it can be similarly used to compute the $(n-k)$-dimensional volume of the intersection of k hyperplanes with the unit n-cube. ♢

Remark 2. There is large literature on the volumes of slices, slabs and the like which goes back to the 19th century. Some seminal results, similar to Theorem 1, relating volumes appeared in George Pólya's 1912 PhD thesis and were published in [7]. They were established by complex analytic methods. ♢

Remark 3. Various related sinc integral evaluations appeared on the Cambridge tripos [2, 8]. Much of this literature is very nicely recapitulated in [6] which also provides an explicit combinatorial formula for the volume of slices and thence of slabs. Finally we mention that [2] and [4] record various of the striking and originally unexpected identities lying in Theorem 1. These results have gotten a life of their own[1] ♢

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[1]See, for instance, http://en.wikipedia.org/wiki/Borwein_integral and http://mathworld.wolfram.com/BorweinIntegrals.html.

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