Exploratory Experimentation: Digitally-Assisted Discovery and Proof

Jonathan Michael Borwein

Abstract The mathematical community (appropriately defined) faces a great challenge to re-evaluate the role of proof in light of the power of current computer systems, the sophistication of modern mathematical computing packages, and the growing capacity to data-mine on the internet. Added to those are the enormous complexity of many modern mathematical results such as the Poincaré conjecture, Fermat's last theorem, and the classification of finite simple groups. With great challenges come great opportunities. Here, I survey the current challenges and opportunities for the learning and doing of mathematics. As the prospects for inductive mathematics blossom, the need to ensure that the role of proof is properly founded remains undiminished. Much of this material was presented as a plenary talk in May 2009 at the National Taiwan Normal University Workshop for ICMI Study 19 "On Proof and Proving in Mathematics Education."

1 Digitally-assisted Discovery and Proof

> "[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.
>
> Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning."—George Polya (1887-1985) [28, (2) p. 128]

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1.1 Exploratory Experimentation

I share Polya's view that intuition precedes deductive reasoning. Nonetheless, Polya also goes on to say, proof should certainly be taught in school. I begin with some observations many of which have been fleshed out in *The Computer as Crucible* [12], *Mathematics by Experiment* [10], and *Experimental Mathematics in Action* [6]. My musings here focus on the changing nature of mathematical knowledge and in consequence asks the questions such as "How do we come to believe and trust pieces of mathematics?", "Why do we wish to prove things?" and "How do we teach what and why to students?"

While I have described myself in [6] and elsewhere as a "computationally assisted fallibilist", I am far from a social-constructivist. Like Richard Brown, I believe that Science "at least attempts to faithfully represent reality" [13, p. 7]. I am, though, persuaded by various notions of embodied cognition. As Smail writes:

> "[T]he large human brain evolved over the past 1.7 million years to allow individuals to negotiate the growing complexities posed by human social living." [33, p. 113]

In consequence, humans find various modes of argument more palatable than others, and are prone to make certain kinds of errors more than others. Likewise, Steve Pinker's observation about language as founded on

> ". . . the ethereal notions of space, time, causation, possession, and goals that appear to make up a language of thought" [27, p. 83]

remains equally potent within mathematics. The computer offers to provide scaffolding both to enhance mathematical reasoning and to restrain mathematical error.

To begin with let me briefly reprise what I mean by discovery and by proof in mathematics. The following attractive definition of *discovery* has the satisfactory consequence that a student can certainly discover results whether those results are known to the teacher or not.

> "In short, discovering a truth is coming to believe it in an independent, reliable, and rational way." [18, p. 50]

Nor is it necessary to demand that each dissertation be original (only independently discovered).

A standard definition[1] of *proof* follows.

> **PROOF**, n. a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established.

As a working definition of mathematics itself, I offer the following, in which the word "proof" does not enter. Nor should it; mathematics is much more than proof alone:

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[1] All definitions below are taken from the *Collin's Dictionary of Mathematics* which I co-authored. It is available as software — with a version of Student *Maple* embedded in it — at http://www.mathresources.com/products/mathresource/index.html.

> **MATHEMATICS**, n. a group of subjects, including algebra, geometry, trigonometry and calculus, concerned with number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.
> **DEDUCTION**, n. 1. the process of reasoning typical of mathematics and logic, in which a conclusion follows necessarily from given premises so that it cannot be false when the premises are true.
> **INDUCTION**, n. 3. (Logic) a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them. Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.

It awaited the discovery of Australia to confound the seemingly compelling inductive conclusion that all swans are white. Typically, mathematicians take for granted the distinction between *induction* and *deduction* and rarely discuss their roles with either colleagues or students. Despite the conventional identification of Mathematics with deductive reasoning, in his 1951 Gibbs Lecture Kurt Gödel (1906-1978) said:

> "If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics."

He held this view until the end of his life, despite the epochal deductive achievement of his incompleteness results. And this opinion has been echoed or amplified by logicians as different as Willard Quine and Greg Chaitin. More generally, one discovers a substantial number of great mathematicians from Archimedes and Galileo — who apparently said "All truths are easy to understand once they are discovered; the point is to discover them." — to Poincaré and Carleson who have emphasized how much it helps to "know" the answer. Over two millennia ago Archimedes wrote to Eratosthenes in the introduction to his long-lost and recently re-constituted *Method of Mechanical Theorems*:

> "I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding proofs of these same theorems.
> For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. *For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.*" [My emphasis] [25]

Think of the *Method* as an ur-precursor to today's interactive geometry software — with the caveat that, for example, *Cinderella* actually does provide certificates for much Euclidean geometry. As 2006 Abel Prize winner Leonard Carleson described in his 1966 ICM speech on his positive resolution of Luzin's 1913 conjecture, about the pointwise convergence of Fourier series for square-summable functions, after many years of seeking a counter-example he decided none could exist. The importance of this confidence he expressed as follows:

> "The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so."

1.2 Digitally Mediated Mathematics

I shall now assume that all proofs discussed are "non-trivial" in some fashion appropriate to the level of the material since the issue of using inductive methods is really only of interest with this caveat. Armed with these terms, it remains to say that by *digital assistance* I intend the use of such *artefacts* as

- *Modern Mathematical Computer Packages* — be they Symbolic, Numeric, Geometric, or Graphical. I would capture all as "modern hybrid workspaces". One should also envisage much more use of stereo visualization, *haptics*[2], and auditory devices.
- *More Specialist Packages* or *General Purpose Languages*, such as Fortran, C++, CPLEX, GAP, PARI, SnapPea, Graffiti, and MAGMA. The story of the *SIAM 100-Digits Challenge* [9] illustrates the degree to which mathematicians now start computational work within a hybrid platform such as *Maple*, *Mathematica* or MATLAB and make only sparing recourse to more specialist packages when the hybrid work spaces prove too limited.
- *Web Applications*, such as Sloane's Online Encyclopedia of Integer Sequences, the Inverse Symbolic Calculator, Fractal Explorer, Jeff Weeks' Topological Games, or Euclid in Java.[3]
- *Web Databases*, including Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, Digital Library of Mathematical Functions (DLMF), MacTutor, Amazon, and many more sources that are not always viewed as part of the palette. Nor is necessary that one approve unreservedly, say of the historical reliability of MacTutor, to acknowledge that with appropriate discrimination in its use it is a very fine resource.

All the above entail *data-mining* in various forms. Franklin [17] argues that what Steinle has termed "exploratory experimentation" facilitated by "widening technology" as in pharmacology, astrophysics, and biotechnology, is leading to a reassessment of what is viewed as a legitimate experiment, in that a "local model" is not a prerequisite for a legitimate experiment. Hendrik Sørenson [34] cogently makes the case that *experimental mathematics* — as "defined" below — is following similar tracks:

> "These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation

[2] With the growing realization of the importance of gesture in mathematics "as the very texture of thinking," [32, p. 92] it is time to seriously explore tactile devices.

[3] A cross-section of such resources is available through http://ddrive.cs.dal.ca/∼isc/portal/.

> for an exploratory approach to experiments that includes concept formation also pertain to mathematics."

Danny Hillis is quoted as saying recently that:

> "Knowing things is very 20th century. You just need to be able to find things."[4]

about how *Google* has already changed how we think. This is clearly not yet true and will never be, yet it catches something of the changing nature of cognitive style in the 21st century. Likewise, in a provocative article [1], Chris Anderson, the Editor-in-Chief of *Wired*, recently wrote

> "There's no reason to cling to our old ways. It's time to ask: What can science learn from Google?"

In consequence, the boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting blurrier. This is discussed at some length by Jeremy Avigad [2]. A very useful discussion of similar issues from a more explicitly pedagogical perspective is given by de Villiers [14] who also provides a quite extensive bibliography.

1.3 Experimental Mathodology

We started *The Computer as Crucible* [12] with then United States Supreme court Justice Potter Stewart's famous, if somewhat dangerous, 1964 Supreme Court judgement on pornography:

> "I know it when I see it." [12, p. 1]

I complete this subsection by reprising from [10] what somewhat less informally we mean by *experimental mathematics*. I say 'somewhat', since I do not take up the perhaps vexing philosophical question of whether a true *experiment* in mathematics is even possible — without adopting a fully realist philosophy of mathematics — or if we should rather refer to 'quasi-experiments'? Some of this is discussed in [6, Chapter 1] and [10, Chapters 1,2, and 8], wherein we further limn the various ways in which the term 'experiment' is used and underline the need for mathematical experiments with predictive power.

What is experimental mathematics?

1. Gaining insight and *intuition*.
   Despite my agreement with Polya, I firmly believe that — in most important senses — intuition, far from being "knowledge or belief obtained neither by reason nor by perception," as the Collin's English Dictionary and Kant would have

[4] In Achenblog http://blog.washingtonpost.com/achenblog/ of July 1 2008.

it, is acquired not innate. This is well captured by Lewis Wolpert's 2000 title *The Unnatural Nature of Science*, see also [19].

2. *Discovering* new relationships
I use "discover" in Giaquinto's terms as quoted above.
3. *Visualizing* math principles.
I intend the fourth Random House sense of "to make perceptible to the mind or imagination" not just Giaquinto's more direct meaning.
4. *Testing* and especially *falsifying* conjectures.
Karl Popper's "critical rationalism" asserts that induction can never lead to truth and hence that one can only falsify theories [13]. Whether one believes this is the slippery slope to *Post modernist interpretations of science* (Brown's term abbreviated *PIS*) or not is open to debate, but Mathematics, being based largely on deductive science, has little to fear and much to gain from more aggressive use of falsification.
5. *Exploring* a possible result to see if it *merits* formal proof.
"Merit" is context dependent. It may mean one thing in a classroom and quite another for a research mathematician.
6. *Suggesting* approaches for *formal proof*.
I refer to computer-assisted or computer-directed proof which is quite far from completely *Formal Proof* — the topic of a special issue of the *Notices of the AMS* in December 2008.
7. *Computing* replacing lengthy hand derivations.
Hales' recent solution of the *Kepler problem*, described in the 2008 *Notices* article, pushes the boundary on when "replacement" becomes qualitatively different from, say, factoring a very large prime. In the case of factorization, we may well feel we understand the entire sequence of steps undertaken by the computer.
8. *Confirming* analytically derived results.
The *a posteriori* value of confirmation is huge, whether this be in checking answers while preparing a calculus class, or in confirming one's apprehension of a newly acquired fact.

Of these, the first five play a central role in the current context, and the sixth plays a significant one.

1.4 Cognitive Challenges

Let me touch upon the *Stroop effect*[5] illustrating *directed attention* or *interference*. This classic cognitive psychology test, discovered by John Ridley Stroop in 1935, is as follows.

Consider the picture in Figure 1, in which various coloured words are colored in one of the colors mentioned, but not necessarily in the same one to which the word refers.

[5] http://www.snre.umich.edu/eplab/demos/st0/stroopdesc.html has a fine overview.

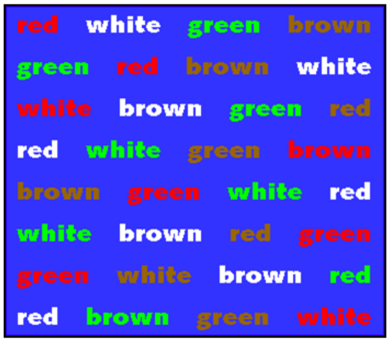

Fig. 1 An Illustration of the Stroop test.

First, say the **color** which the given **word mentions**.

Second, say the **color** in which the **word is written**.

Most people find the second task harder. You may find yourself taking more time for each word, and may frequently say the word, rather than the color in which the word appears. Proficient (young) multitaskers find it easy to suppress information and so perform the second task faster than traditionally. Indeed, Cliff Nass' work in the CHIME lab at Stanford suggests that neurological changes are taking place amongst the 'born-digital.'[6] If such cognitive changes are taking place there is even more reason to ensure that epistemology, pedagogy, and cognitive science are in concert.

1.5 Paradigm Shifts

> "Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. Moreover, the conviction persists-though history shows it to be a hallucination that all the questions that the human mind has asked are questions that can be answered in terms of the alternatives that the questions themselves present. But in fact intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them: we get over them.
> Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species"."—John Dewey (1859-1952)[7]

[6] See http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif.

[7] In Dewey's introduction to his book [15]. Dewey, a leading pragmatist (or instrumentalist) philosopher and educational thinker of his period, is also largely responsible for the Trotsky archives being at Harvard, through his activities on the *Dewey Commission*.

Thomas Kuhn (1922-1996) has noted that a true *paradigm shift* — as opposed to the cliché — is "a conversion experience."[8] You (and enough others) either have one or you don't. Oliver Heaviside (1850-1925) said in defending his operator calculus before it could be properly justified: "Why should I refuse a good dinner simply because I don't understand the digestive processes involved?"

But please always remember as Arturo Rosenblueth and Norbert Wiener wrote: "The price of metaphor is eternal vigilance."[9] I may not convince you to reevaluate your view of Mathematics as an entirely deductive science — if so indeed you view it — but in the next section I will give it my best shot.

2 Mathematical Examples

I continue with various explicit examples. I leave it to the reader to decide how much or how frequently he or she wishes to exploit the processes I advertise. Nonetheless they all controvert Picasso's "Computers are useless they can only give answers."[10] and confirm Hamming's "The purpose of computing is insight not numbers."[11] As a warm-up illustration, consider Figure 2. The lower function in both graphs is $x \mapsto -x^2 \log x$ The left-hand graph compares $x \mapsto x - x^2$ while the right-hand graph compares $x \mapsto x^2 - x^4$ each on $0 \le x \le 1$.

Before the advent of plotting calculators if asked a question like *"Is* $-x^2 \log x$ *less than* $x - x^2$ *on the open interval between zero and one?"* one immediately had recourse to the calculus. Now that would be silly, clearly they cross. In the other case, if there is a problem it is at the right-hand end point. 'Zooming' will probably persuade you that $-x^2 \log x \le x^2 - x^4$ on $0 \le x \le 1$ and may even guide a calculus proof if a proof is needed.

The examples below contain material on sequences, generating functions, special functions, continued fractions, partial fractions, definite and indefinite integrals, finite and infinite sums, combinatorics and algebra, matrix theory, dynamic geometry and recursions, differential equations, and mathematical physics, among other things. So they capture the three main divisions of pure mathematical thinking: algebraic-symbolic, analytic, and topologic-geometric, while making contact with more applied issues in computation, numerical analysis and the like.

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[8] This was said in an interview in [30], not only in Kuhn's 1962 *The Structure of Scientific Revolutions*, which Brown notes is "the single most influential work in the history of science in the twentieth century." In Brown's accounting [13] Kuhn bears more responsibility for the slide into PIS than either Dewey or Popper. An unpremeditated example of digitally assisted research is that — as I type — I am listening to *The Structure of Scientific Revolutions*, having last read it 35 years ago.

[9] Quoted by R. C. Leowontin, in *Science* p. 1264, Feb 16, 2001 (the *Human Genome Issue*).

[10] Michael Moncur's (Cynical) Quotations #255 http://www.quotationspage.com/collections.html

[11] Richard Hamming's philosophy of scientific computing appears as preface to his influential 1962 book [21].

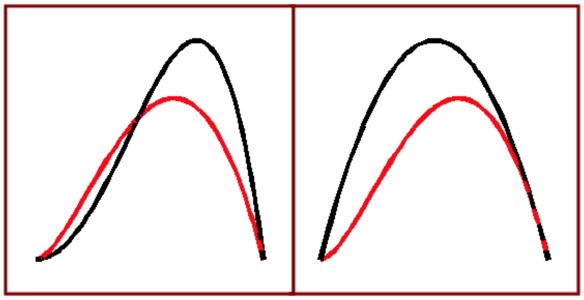

Fig. 2 Try Visualization or Calculus First?

Example I: What Did the Computer Do?

> "This computer, although assigned to me, was being used on board the International Space Station. I was informed that it was tossed overboard to be burned up in the atmosphere when it failed."—anonymous NASA employee[12]

In my own work, computer experimentation and digitally-mediated research now invariably play a crucial part. Even in many seemingly non-computational areas of functional analysis and the like, there is frequently a computable consequence whose verification provides confidence in the result under development. Moreover, the process of specifying my questions enough to program with them invariably enhances my understanding and sometimes renders the actual computer nearly superfluous. For example, in a recent study of expectation or "box integrals" [7] we were able to evaluate a quantity which had defeated us for years, namely

$$\mathscr{K}_1 := \int_3^4 \frac{\operatorname{arcsec}(x)}{\sqrt{x^2-4x+3}}\,dx$$

in *closed-form* as

$$\begin{aligned}\mathscr{K}_1 = {} & \mathrm{Cl}_2(\theta) - \mathrm{Cl}_2\left(\theta+\frac{\pi}{3}\right) - \mathrm{Cl}_2\left(\theta-\frac{\pi}{2}\right) + \mathrm{Cl}_2\left(\theta-\frac{\pi}{6}\right) \\ & - \mathrm{Cl}_2\left(3\theta+\frac{\pi}{3}\right) + \mathrm{Cl}_2\left(3\theta+\frac{2\pi}{3}\right) - \mathrm{Cl}_2\left(3\theta-\frac{5\pi}{6}\right) + \mathrm{Cl}_2\left(3\theta+\frac{5\pi}{6}\right) \\ & + \left(6\theta-\frac{5\pi}{2}\right)\log\left(2-\sqrt{3}\right). \end{aligned} \tag{1}$$

where $\mathrm{Cl}_2(\theta) := \sum_{n=1}^{\infty} \sin(n\theta)/n^2$ is the *Clausen function*, and

$$3\theta := \arctan\left(\frac{16-3\sqrt{15}}{11}\right) + \pi.$$

[12] *Science*, August 3, 2007, p. 579: "documenting equipment losses of more than $94 million over the past 10 years by the agency."

Along the way to the evaluation above, after exploiting some insightful work by George Lamb, there were several stages of symbolic computation, at times involving an expression for $\mathcal{K}_1$ with over $28,000$ characters (perhaps 25 standard book pages). It may well be that the closed form in (1) can be further simplified. In any event, the very satisfying process of distilling the computer's $28,000$ character discovery, required a mixture of art and technology and I would be hard pressed to assert categorically whether it constituted a conventional proof. Nonetheless, it is correct and has been checked numerically to over a thousand-digit decimal precision. $\trianglelefteq$

I turn next to a mathematical example which I hope will reinforce my assertion that there is already an enormous amount to be mined mathematically on the internet. And this is before any mathematical character recognition tools have been made generally available and when it is still very hard to search mathematics on the web.

Example II: What is That Number?

> "The dictum that everything that people do is 'cultural' … licenses the idea that every cultural critic can meaningfully analyze even the most intricate accomplishments of art and science. … It is distinctly weird to listen to pronouncements on the nature of mathematics from the lips of someone who cannot tell you what a complex number is!"—Norman Levitt[13]

In 1995 or so Andrew Granville emailed me the number

$$\alpha := 1.4331274267223\ldots \tag{2}$$

and challenged me to identify it; I think this was a test I could have failed. I asked *Maple* for its continued fraction. In the conventional concise notation I was rewarded with

$$\alpha = [1,2,3,4,5,6,7,8,9,10,11,...]. \tag{3}$$

Even if you are unfamiliar with continued fractions, you will agree that the changed representation in (3) has exposed structure not apparent from (2)! I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{4}$$

where I_0 and I_1 are *Bessel functions* of the first kind. Actually I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In 2009 there are at least three "zero-knowledge" strategies:

1. Given (3), type "arithmetic progression", "continued fraction" into *Google*.

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[13] In *The flight From Science and Reason.* See *Science*, Oct. 11, 1996, p. 183.

2. Type "1, 4, 3, 3, 1, 2, 7, 4, 2" into *Sloane's Encyclopedia of Integer Sequences*.[14]
3. Type the decimal digits of α into the *Inverse Symbolic Calculator*.[15]

I illustrate the results of each strategy.

1. On October 15, 2008, on typing "arithmetic progression", "continued fraction" into *Google*, the first three hits were those shown in Figure 3. Moreover, the Math-World entry tells us that any arithmetic continued fraction is of a ratio of Bessel functions, as shown in the inset to Figure 3, which also refers to the second hit in Figure 3. The reader may wish to see what other natural search terms uncover (4) — perhaps in the newly unveiled *Wolfram Alpha*.

***Continued Fraction* Constant -- from Wolfram MathWorld**

- 3 visits - 14/09/07Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. **...**
*mathworld.wolfram.com/**ContinuedFraction**Constant.html - 31k*

HAKMEM -- *CONTINUED FRACTIONS* -- DRAFT, NOT YET PROOFED

The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is I (2/D) A/D [A+D, A+2D, A+3D, . **...**
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

On simple *continued fractions* with partial quotients in *arithmetic* ...

0. This means that the sequence of partial quotients of the *continued fractions* under. investigation consists of finitely many *arithmetic progressions* (with **...**
www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the MathWorld entry includes

$$[A+D, A+2D, A+3D, \ldots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$.

Fig. 3 What Google and MathWorld offer.

2. Typing the first few digits into Sloane's interface results in the response shown in Figure 4. In this case we are even told what the series representations of the requisite Bessel functions are, we are given sample code (in this case in *Mathematica*), and we are lead to many links and references. Moreover, the site is carefully moderated

[14] See http://www.research.att.com/ njas/sequences/.

[15] The online *Inverse Symbolic Calculator* http://ddrive.cs.dal.ca/∼isc was newly web-accessible in the same year, 1995.

and continues to grow. Note also that this strategy only became viable after May 14th 2001 when the sequence was added to the database which now contains in excess of $158,000$ entries.

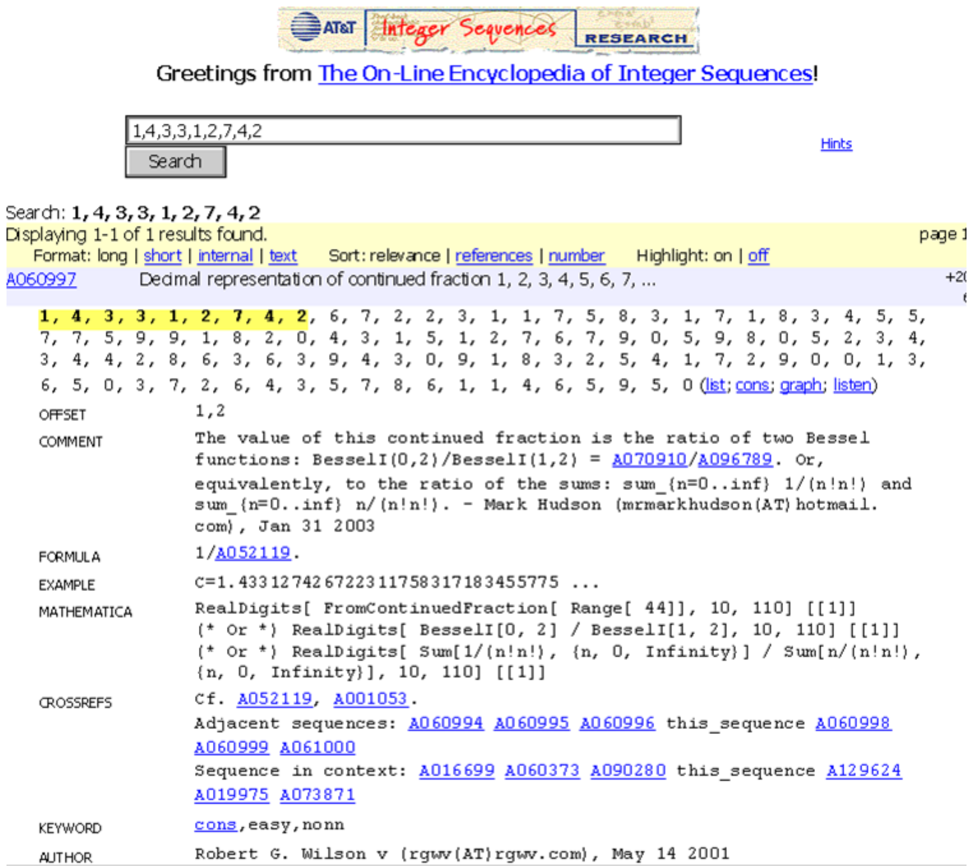

Fig. 4 What *Sloane's Encyclopedia* offers.

3. If one types the decimal representation of α into the Inverse Symbolic Calculator (ISC) it returns

```
Best guess: BesI(0,2)/BesI(1,2)
```

Most of the functionality of the ISC is built into the "identify" function in versions of *Maple* starting with version 9.5. For example, `identify(4.45033263602792)` returns $\sqrt{3}+e$. As always, the experienced user will be able to extract more from this tool than the novice for whom the ISC will often produce more. $\triangleleft$

Example III: From Discovery to Proof

> "Besides it is an error to believe that rigor in the proof is the enemy of simplicity."—David Hilbert[16]

The following integral was made popular in a 1971 *Eureka*[17] article

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x = \frac{22}{7} - \pi \tag{5}$$

as described in [10]. As the integrand is positive on $(0,1)$ the integral yields an area and hence $\pi < 22/7$. This problem was set on a 1960 Sydney honours mathematics final exam (5) and perhaps originated in 1941 with the author of the 1971 article — Dalzeil who chose not reference his earlier self! Why should we trust this discovery? Well *Maple* and *Mathematica* both 'do it'. But this is *proof by appeal to authority* less imposing than, say, von Neumann [24] and a better answer is to ask *Maple* for the indefinite integral

$$\int_0^{\mathbf{t}} \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x = \ ?$$

The computer algebra system (CAS) will return

$$\int_0^t \frac{x^4\,(1-x)^4}{1+x^2}\mathrm{d}x = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t)\,, \tag{6}$$

and now differentiation and the *Fundamental theorem of calculus* proves the result.

This is probably not the proof one would find by hand, but it is a totally rigorous one, and represents an "instrumental use" of the computer. The fact that a CAS will quite possibly be able to evaluate an indefinite integral or a finite sum whenever it can evaluate the corresponding definite integral or infinite sum frequently allows one to provide a certificate for such a discovery. In the case of a sum, the certificate often takes the form of a mathematical induction (deductive version). Another interesting feature of this example is that it appears to be quite irrelevant that $22/7$ is an early, and the most famous, continued-fraction approximation to π [26]. Not every discovery is part of a hoped-for pattern. $\trianglelefteq$

Example IV: From Concrete to Abstract

> "The plural of 'anecdote' is not 'evidence'."—Alan L. Leshner[18]

We take heed of Leshner's caution but still celebrate accidental discovery.

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[16] In his *23 Mathematische Probleme* lecture to the Paris International Congress, 1900 [35].

[17] *Eureka* was an undergraduate Cambridge University journal.

[18] Leshner, the publisher of *Science*, was speaking at the Canadian Federal Science & Technology Forum, October 2, 2002.

1. In April 1993, Enrico Au-Yeung, then an undergraduate at the University of Waterloo, brought to my attention the result

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 k^{-2} = 4.59987\ldots \approx \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360}$$

He had spotted from six place accuracy that $0.047222\ldots = 17/360$. I was very skeptical, but Parsevals identity computations affirmed this to high precision. This is effectively a special case of the following class

$$\zeta(s_1,s_2,\cdots,s_k) = \sum_{n_1>n_2>\cdots>n_k>0}\prod_{j=1}^{k} n_j^{-|s_j|}\sigma_j^{-n_j},$$

where s_j are integers and $\sigma_j = \text{signum} s_j$. These can be rapidly computed as implemented at www.cecm.sfu.ca/projects/ezface+. [11]. In the past 20 years they have become of more and more interest in number theory, combinatorics, knot theory and mathematical physics. A marvellous example is Zagier's conjecture, found experimentally and now proven in [11], viz;

$$\zeta\left(\overbrace{3,1,3,1,\cdots,3,1}^{n}\right) = \frac{2\pi^{4n}}{(4n+2)!}. \tag{7}$$

Along the way to finding the proof we convinced ourselves that (7) held for many values including $n = 163$ which required summing a slowly convergent 326-dimensional sum to $1,000$ places with our fast summation method. Equation (7) is a remarkable non-commutative counterpart of the classical formula for $\zeta(2n)$ [11, Ch. 3].

2. In the course of proving empirically-discovered conjectures about such multiple zeta values [10] we needed to obtain the coefficients in the *partial fraction* expansion for

$$\frac{1}{x^s(1-x)^t} = \sum_{j\geq 0}\frac{a_j^{s,t}}{x^j} + \sum_{j\geq 0}\frac{b_j^{s,t}}{(1-x)^j}. \tag{8}$$

It transpires that

$$a_j^{s,t} = \binom{s+t-j-1}{s-j}$$

with a symmetric expression for $b_j^{s,t}$. This was known to Euler and once known is fairly easily proved by induction. But it can certainly be discovered in a CAS by considering various rows or diagonals in the matrix of coefficients — and either spotting the pattern or failing that by asking Sloane's Encyclopedia. Partial fractions like continued fractions and Gaussian elimination are the sort of task that *once* *mastered* are much better performed by computer while one focusses on the more conceptual issues they expose.

3. We also needed to show that $M := A+B-C$ was invertible where the $n \times n$ matrices A, B, C respectively had entries

$$(-1)^{k+1}\binom{2n-j}{2n-k}, \quad (-1)^{k+1}\binom{2n-j}{k-1}, \quad (-1)^{k+1}\binom{j-1}{k-1}. \tag{9}$$

Thus, A and C are triangular while B is full. For example, in nine dimensions M is displayed below

$$\begin{bmatrix} 1 & -34 & 272 & -1360 & 4760 & -12376 & 24752 & -38896 & 48620 \\ 0 & -16 & 136 & -680 & 2380 & -6188 & 12376 & -19448 & 24310 \\ 0 & -13 & 105 & -470 & 1470 & -3458 & 6370 & -9438 & 11440 \\ 0 & -11 & 88 & -364 & 1015 & -2093 & 3367 & -4433 & 5005 \\ 0 & -9 & 72 & -282 & 715 & -1300 & 1794 & -2002 & 2002 \\ 0 & -7 & 56 & -210 & 490 & -792 & 936 & -858 & 715 \\ 0 & -5 & 40 & -145 & 315 & -456 & 462 & -341 & 220 \\ 0 & -3 & 24 & -85 & 175 & -231 & 203 & -120 & 55 \\ 0 & -1 & 8 & -28 & 56 & -70 & 56 & -28 & 9 \end{bmatrix}$$

After messing around futilely with lots of cases in an attempt to spot a pattern, it occurred to me to ask *Maple* for the *minimal polynomial* of M.

```
> linalg[minpoly](M(12),t);
```

returns $-2+t+t^2$. Emboldened I tried

```
> linalg[minpoly](B(20),t);
> linalg[minpoly](A(20),t);
> linalg[minpoly](C(20),t);
```

and was rewarded with $-1+t^3, -1+t^2, -1+t^2$. Since a typical matrix has a full degree minimal polynomial, we are quite assured that A, B, C really are roots of unity. Armed with this discovery we are lead to try to prove

$$A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2 \tag{10}$$

which is a nice combinatorial exercise (by hand or computer). Clearly then we obtain also

$$B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I \tag{11}$$

and the requisite formula

$$M^{-1} = \frac{M+I}{2}$$

is again a fun exercise in formal algebra; in fact, we have

$$\begin{aligned} M^2 &= AA+AB-AC+BA+BB-BC-CA-CB+CC \\ &= I+C-B-A+I \\ &= 2I-M. \end{aligned}$$

It is also worth confirming that we have discovered an amusing presentation of the symmetric group S_3. $\trianglelefteq$

Characteristic or minimal polynomials, entirely abstract for me as a student, now become members of a rapidly growing box of concrete symbolic tools, as do many matrix decomposition results, the use of Groebner bases, Robert Risch's 1968 decision algorithm for when an elementary function has an elementary indefinite integral, and so on.

Example V: A Dynamic Discovery and Partial Proof

> "Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ... I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane."—Augustus De Morgan (1806–1871) [31, p. 540]

In a wide variety of problems (protein folding, 3SAT, spin glasses, giant Sudoku, etc.) we wish to find a point in the intersection of two sets A and B where B is non-convex but "divide and concur" works better than theory can explain. Let $P_A(x)$ and $R_A(x) := 2P_A(x) - x$ denote respectively the *projector* and *reflector* on a set A, as shown in Figure 5, where A is the boundary of the shaded ellipse. Then "divide and concur" is the natural geometric iteration "reflect-reflect-average":

$$x_{n+1} \Longrightarrow \frac{x_n + R_A\left(R_B(x_n)\right)}{2}. \tag{12}$$

Consider the simplest case of a line A of height α (all lines may be assumed horizontal) and the unit circle B. With $z_n := (x_n, y_n)$ we obtain the explicit iteration

$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n, \quad (\theta_n := \arg z_n). \tag{13}$$

For the infeasible case with $\alpha > 1$ it is easy to see the iterates go to infinity vertically. For the tangent $\alpha = 1$ we provably converge to an infeasible point. For $0 < \alpha < 1$ the pictures are lovely but proofs escape me and my collaborators. Spiraling is ubiquitous in this case. The iteration is illustrated in Figure 6 starting at $(4.2, -0.51)$ with $\alpha = 0.94$. Two representative *Maple* pictures follow in Figure 7.

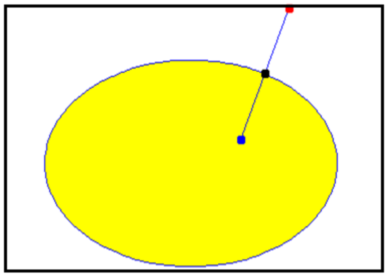

Fig. 5 Reflector (interior) and Projector (boundary) of a point external to an ellipse.

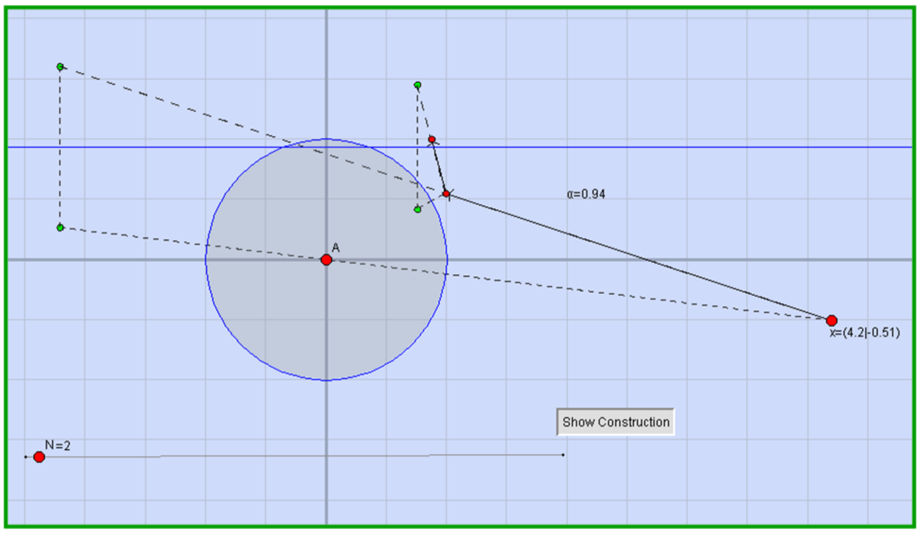

Fig. 6 The first three iterates of (13) in *Cinderella*.

For $\alpha = 0$ we can prove convergence to one of the two points in $A \cap B$ if and only if we do not start on the vertical axis, where we provably have *chaos*.

Let me sketch how the interactive geometry *Cinderella*[19] leads one both to discovery and a proof in this equatorial case. Interactive applets are easily made and the next two figures come from ones that are stored on line at

A1. http://users.cs.dal.ca/∼jborwein/reflection.html; and
A2. http://users.cs.dal.ca/∼jborwein/expansion.html respectively.

Figure 8 illustrates the applet **A1.** at work: by dragging the trajectory (with $N = 28$) one quickly discovers that

(i) as long as the iterate is outside the unit circle the next point is *always* closer to the origin;

[19] Available at http://www.cinderella.de.

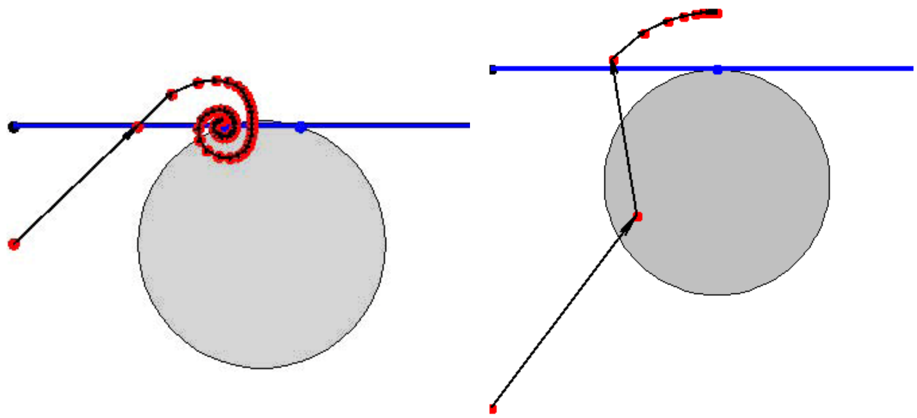

Fig. 7 The behaviour of (13) for $\alpha = 0.95$ (L) and $\alpha = 1$ (R).

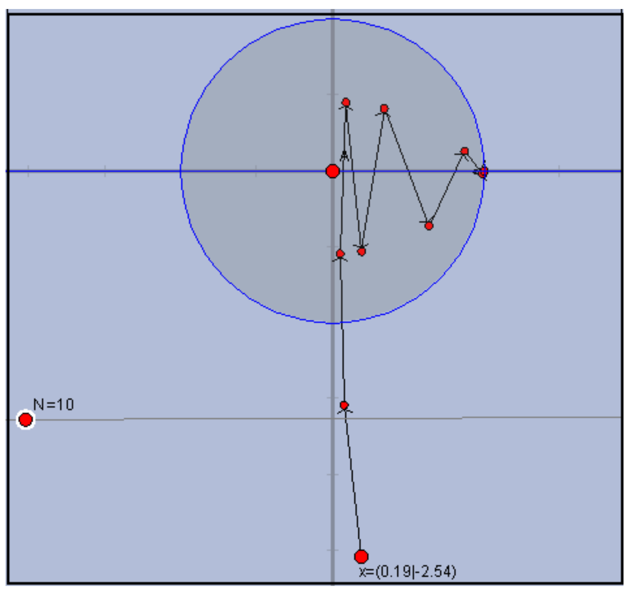

Fig. 8 Discovery of the proof with $\alpha = 0$.

(ii) once inside the circle the iterate *never* leaves;
(iii) the angle now *oscillates* to zero and the trajectory hence converges to $(1,0)$.

All of this is quite easily made algebraic in the language of (13).

Figure 9 illustrates the applet **A2.** which takes up to $10,000$ starting points in the rectangle $\{(x,y)\colon\, 0 \le x \le 1, |y-\alpha|| \le 1\}$ coloured by distance from the vertical axis with red on the axis and violet at $x = 1$, and produces the first hundred iterations in gestalt. Thus, we see clearly but I cannot yet prove, that all points not on the y-axis are swept into the feasible point $(\sqrt{1-\alpha^2}, \alpha)$. It also shows that to accurately record the behaviour *Cinderella*'s double precision is inadequate and hence provides a fine if unexpected starting point for a discussion of numerical analysis and instability.

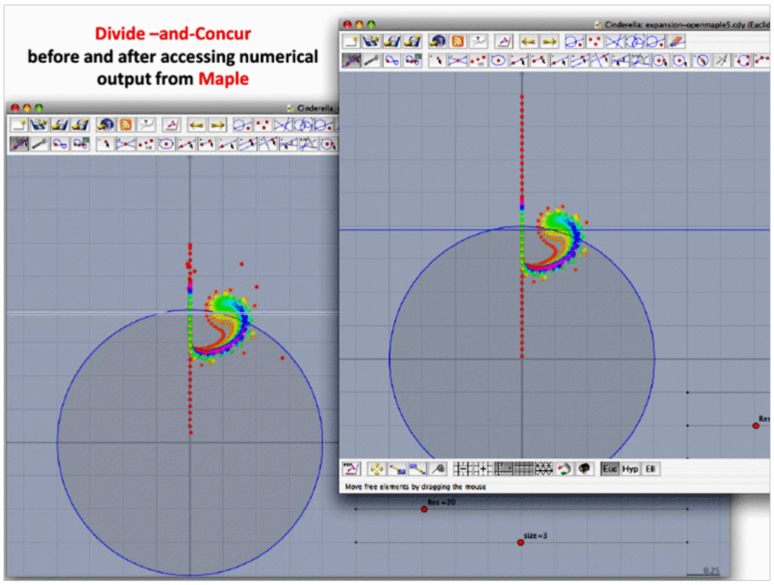

Fig. 9 Gestalt of 400 third steps in *Cinderella* without (L) and with *Maple* data (R).

Here we have a fine counter-example to an old mathematical bugaboo:

> "A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself)."—J. E. Littlewood, (1885-1977)[20]

Á la Littlewood, I find it hard to persuade myself that the applet **A2.** does not constitute a *generic proof* of what it displays in Figure 10. *Cinderella*'s numerical instability is washed away in this profusion of accurate data. For all intents and purposes, we have now run the algorithm from all relevant starting points.

We have also considered the analogous differential equation since asymptotic techniques for such differential equations are better developed. We decided

$$x'(t) = \frac{x(t)}{r(t)} - x(t)r(t) := \sqrt{x(t)^2 + y(t)^2}\,] \tag{14}$$
$$y'(t) = \alpha - \frac{y(t)}{r(t)}$$

[20] From p. 53 of the 1953 edition of Littlewood's *Miscellany* and so said long before the current fine graphic, geometric, and other visualization tools were available; also quoted in [24].

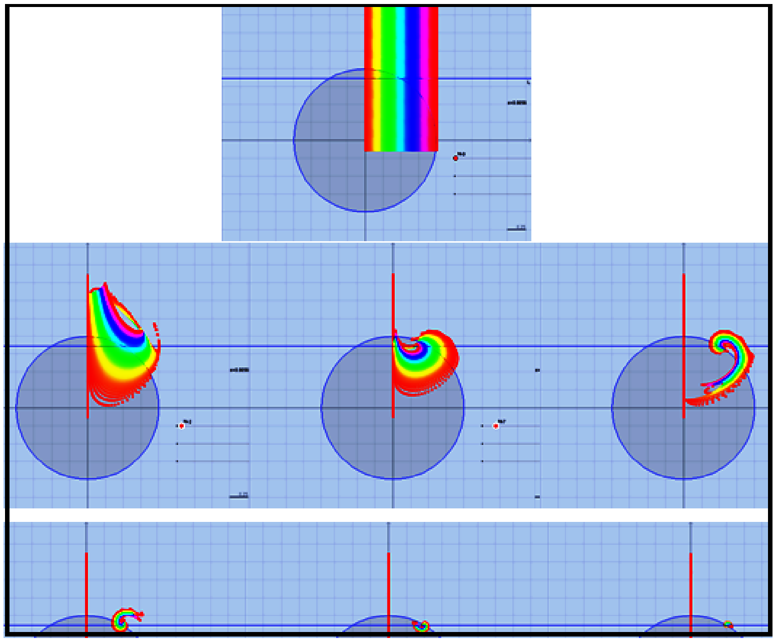

Fig. 10 Snapshots of 10,000 points after $0, 2, 7, 13, 16, 21$, and 27 steps in *Cinderella*.

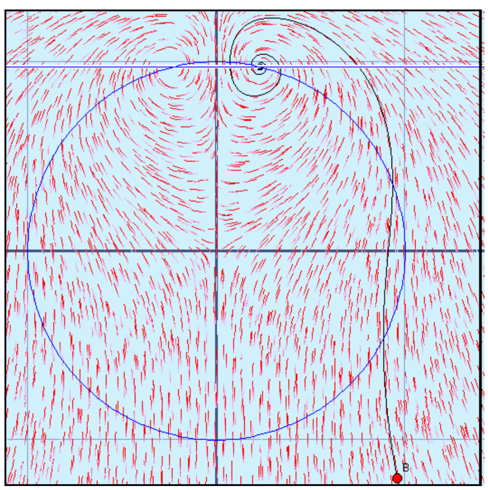

Fig. 11 ODE solution and vector field for (14) with $\alpha = 0.97$ in *Cinderella*.

was a reasonable counterpart to the Cartesian formulation of (13) — we have replaced the difference $x_{n+1} - x_n$ by $x'(t)$, etc. — as shown in Figure 11. Now we have a whole other class of discoveries without proofs. For example, the differential equation solution clearly performs like the discrete iteration solution.

This is also an ideal problem to introduce early under-graduates to research as it involves only school geometry notions and has many accessible extensions in two or three dimensions. Much can be discovered and most of it will be both original and unproven. Consider what happens when B is a line segment or a finite set rather than a line or when A is a more general conic section.

Corresponding algorithms, like "project-project-average", are representative of what was used to correct the Hubble telescope's early optical abberation problems. $\trianglelefteq$

Example VI: Knowledge without Proof

"All physicists and a good many quite respectable mathematicians are contemptuous about proof."—G. H. Hardy (1877-1947)[21]

A few years ago Guillera found various Ramanujan-like identities for π, including three most basic ones:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n} \tag{15}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n} \tag{16}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}. \tag{17}$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

As far as we can tell there are no analogous formulae for $1/\pi^N$ with $N \geq 4$. There are, however, many variants based on other Pochhammer symbols.

Guillera proved (15) and (16) in tandem, by using very ingeniously the *Wilf–Zeilberger algorithm* for formally proving hypergeometric-like identities [10, 6]. He ascribed the third to Gourevich, who found it using *integer relation methods* [10, 6]. Formula (17) has been checked to extreme precision. It is certainly true but has no proof, nor does anyone have an inkling of how to prove it, especially as experiment suggests that it has no mate, unlike (15) and (16).

My by-now-sophisticated intuition on the matter tells me that if a proof exists it is most probably more a verification than an explication and so I for one have stopped looking. I am happy just to know the beautiful identity is true. It may be

---

[21] In his famous *Mathematician's Apology* of 1940. I can not resist noting that modern digital assistance often makes more careful referencing unnecessary and sometimes even unhelpful!

so for no good reason. It might conceivably have no proof and be a very concrete Gödel statement. $\trianglelefteq$

Example VII. A Mathematical Physics Limit

"Anyone who is not shocked by quantum theory has not understood a single word."—Niels Bohr[22]

The following N-dimensional integrals arise independently in mathematical physics, indirectly in statistical mechanics of the *Ising Model* and as we discovered later more directly in *Quantum Field Theory*:

$$C_N = \frac{4}{N!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^N (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_N}{u_N}. \tag{18}$$

We first showed that C_N can be transformed to a 1-D integral:

$$C_N = \frac{2^N}{N!} \int_0^\infty t K_0^N(t)\, dt \tag{19}$$

where K_0 is a *modified Bessel function* — Bessel functions which we met in Example I are pervasive in analysis.

We then computed 400-digit numerical values. This is impossible for $n \geq 4$ from (18) but accessible from (19) and a good algorithm for K_0. Thence, we found the following, now proven, results [4]:

$$C_3 = \mathrm{L}_{-3}(2) := \sum_{n \geq 0} \left\{ \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right\}$$
$$C_4 = 14\,\zeta(3).$$

We also observed that

$$C_{1024} = 0.630473503374386796122040192710878904354587\ldots$$

and that the limit as $N \to \infty$ was the same to many digits. We then used the Inverse Symbolic Calculator, the aforementioned online numerical constant recognition facility, at http://ddrive.cs.dal.ca/ isc/portal which returned

```
Output: Mixed constants, 2 with elementary transforms.
.6304735033743867 = sr(2)^2/exp(gamma)^2
```

from which we discovered that

[22] As quoted in [8, p. 54] with a footnote citing *The Philosophical Writings of Niels Bohr* (1998). (1885–1962).

$$C_{1024} \approx \lim_{n\to\infty} C_n = 2e^{-2\gamma}.$$

Here $\gamma = 0.57721566490153\ldots$ is *Euler's constant* and is perhaps the most basic constant which is not yet proven irrational [20]. The limit discovery showed the Bessel function representation to be fundamental. Likewise $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ the value of the Riemann zeta-function at 3, also called Apéry's constant, was only proven irrational in 1978 and the irrationality of $\zeta(5)$ remains unproven.

The discovery of the limit value, and its appearance in the literature of Bessel functions, persuaded us the Bessel function representation (19) was fundamental — not just technically useful — and indeed this is the form in which C_N, for odd N appears in quantum field theory [4]. ⊴

Example VIII: Apéry's formula

> "Another thing I must point out is that you cannot prove a vague theory wrong. ... Also, if the process of computing the consequences is indefinite, then with a little skill any experimental result can be made to look like the expected consequences."—Richard Feynman (1918–1988)

Margo Kondratieva found the following identity in the 1890 papers of Markov [6]:

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n\,(n!)^6}{(2n+1)!} \frac{\left(5\,(n+1)^2 + 6\,(a-1)\,(n+1) + 2\,(a-1)^2\right)}{\prod_{k=0}^{n} (a+k)^4}. \tag{20}$$

Apéry's 1978 formula

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}, \tag{21}$$

which played a key role in his celebrated proof of the irrationality of $\zeta(3)$, is the case with $a = 0$.

Luckily, by adopting Giaquinto's accounting of discovery we are still entitled to say that Apéry discovered the formula (21) which now bears his name.

We observe that *Maple* 'establishes' identity (20) in the hypergeometric formula

$$-\frac{1}{2}\Psi(2,a) = -\frac{1}{2}\Psi(2,a) - \zeta(3) + \frac{5}{4}\,{}_4\mathrm{F}_3\left(\left. \begin{matrix} 1,1,1,1 \\ 2,2,\frac{3}{2} \end{matrix} \right| -\frac{1}{4} \right),$$

that is, it has reduced it to a form of (21). ⊴

Like much of mathematics, this last example leads to something whose computational consequences are very far from indefinite. Indeed, it is the rigidity of much algorithmic mathematics that makes it so frequently the way hardware or software errors, such as the 'Pentium Bug', are first uncovered.

Example IX: When is easy bad?

Many algorithmic components of CAS are today extraordinarily effective when two decades ago they were more like ‘toys’. This is equally true of extreme-precision calculation — a prerequisite for much of my own work [3, 7] and others [9] — or in combinatorics.

Consider the *generating function* of the number of *additive partitions*, $p(n)$ of a natural number where we ignore order and zeroes. Thus,

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

and so $p(5) = 7$. The *ordinary generating function* (22) discovered by Euler is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)}. \tag{22}$$

This is easily obtained by using the geometric formula for each $1/(1-q^k)$ and observing how many powers of q^n are obtained. The famous and laborious computation by MacMahon of $p(200) = 3972999029388$ early last century, if done *symbolically and entirely naively* from (22) on a reasonable laptop took 20 minutes in 1991, and about 0.17 seconds today, while

$$p(2000) = 4720819175619413888601432406799959512200344166$$

took about two minutes in 2009.

Moreover, Richard Crandall was able, in December 2008, to calculate $p(10^9)$ in 3 seconds on his laptop, using Hardy-Ramanujan and Rademacher’s ‘finite’ series along with FFT methods. The current ease of computation of $p(500)$ directly from (22) raises the question of what interesting mathematical discoveries does easy computation obviate? $\trianglelefteq$.

Likewise, the record for computation of π has gone from under 30 million decimal digits in 1986 to over 5 trillion places this year [10].

3 Concluding Remarks

> “We [Kaplansky and Halmos] share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.”—Paul Halmos (1916–2006) [16]

Theory and practice should be better comported!

The students of today live, as we do, in an information-rich, judgement-poor world in which the explosion of information, and of tools, is not going to diminish. So we have to teach judgement (not just concern with plagiarism) when it comes to using what is already possible digitally. This means mastering the sorts of tools I have illustrated. Additionally, it seems to me critical that we mesh our software design — and our teaching style more generally — with our growing understanding of our cognitive strengths and limitations as a species (as touched upon in Section 1). Judith Grabner, in her contribution to this volume, has noted that a large impetus for the development of modern rigor in mathematics came with the Napoleonic introduction of regular courses: Lectures and text books force a precision and a codification that apprenticeship obviates.

As Dave Bailey noted to me recently in email:

> "Moreover, there is a growing consensus that human minds are fundamentally not very good at mathematics, and must be trained as Ifrah points out [23]. Given this fact, the computer can be seen as a perfect complement to humans — we can intuit but not reliably calculate or manipulate; computers are not yet very good at intuition, but are great at calculations and manipulations."

We also have to acknowledge that most of our classes will contain students with a very broad variety of skills and interests (and relatively few future mathematicians). Properly balanced, discovery and proof, assisted by good software, can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion. Impediments to the assimilation of the tools I have illustrated are myriad as I am only too aware from my own recent teaching experiences. These impediments include our own inertia and organizational and technical bottlenecks (this is often from poor IT design - not so much from too few dollars). The impediments certainly include under-prepared or mis-prepared colleagues and the dearth of good material from which to teach a modern syllabus.

Finally, it will never be the case that quasi-inductive mathematics supplants proof. We need to find a new equilibrium. Consider the following empirically-discovered identity

$$\sum_{n=-\infty}^{\infty} \operatorname{sinc}(n)\operatorname{sinc}(n/3)\operatorname{sinc}(n/5)\cdots\operatorname{sinc}(n/23)\operatorname{sinc}(n/29) \tag{23}$$
$$= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)\cdots\operatorname{sinc}(x/23)\operatorname{sinc}(x/29)\,dx$$

where the denumerators range over the primes.

Provably, the following is true: The analogous "sum equals integral" identity remains valid for more than the first $10,176$ primes but stops holding after some larger prime, and thereafter the 'sum minus integral" is positive but *much less than one part in a googolplex* [3]. It is hard to imagine that inductive mathematics alone will ever be able to handle such behaviour . Nor, for that matter, is it clear to me what it means psychologically to digest equations which are false by a near infinitesimal amount.

That said, we are only beginning to scratch the surface of a very exciting set of tools for the enrichment of mathematics, not to mention the growing power of formal proof engines. I conclude with one of my favourite quotes from George Polya and Jacques Hadamard:

> This "quasi-experimental" approach to proof can help to de-emphasise a focus on rigor and formality for its own sake, and to instead support the view expressed by Hadamard when he stated "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." [28, (2) p. 127]

Unlike Frank Quinn [29] perhaps, I believe that in the most complex modern cases certainty, in any reasonable sense, is unattainable through proof. I do believe that even then quasi-inductive methods and experimentation can help us improve our level of certainty. Like Reuben Hersh [22], I am happy to at least entertain some "non-traditional forms of proof." Never before have we had such a cornucopia of fine tools to help us develop and improve our intuition. The challenge is to learn how to harness them, how to develop and how to transmit the necessary theory and practice.

Acknowledgements

I owe many people thanks for helping refine my thoughts on this subject over many years. Four I must mention by name: my long-standing collaborators Brailey Sims, Richard Crandall and David Bailey, and my business partner Ron Fitzgerald from *MathResources*, who has taught me a lot about balancing pragmatism and idealism in educational technology — among other things. I also thank Henrik Sørenson whose thought-provoking analysis gave birth to the title and the thrust of the paper, and my student Chris Maitland who built most of the *Cinderella* applets.

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