MONOTONE OPERATORS AND NON-LINEAR FUNCTIONAL ANALYSIS

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C O N T E N T S

SECTION I

Introduction

When one begins to investigate non-linear functionals and mappings there are several natural restrictions that one can place on the mappings under examination. One of the sharpest and most fruitful of these concepts is that of a monotone operator from a Banach space X to its dual space X*. The theory of monotone operators has particular application to the existence theory for non-linear partial differential equations and boundary value problems, and to the theory of integral equations.

It is the purpose of this survey to describe some useful results on the existence of solutions for non-linear functional equations and in particular to describe the fundamental theory of monotone operators. This will be done with emphasis on the most useful proof methods that have been employed and on the type of problems they solve - both abstract and practical - culminating in an example from integral equations theory.

Although we have chosen to give an example from integral equations theory because less terminology and machinery is needed, almost all the results that are discussed in this paper have been applied to the theory of partial differential equations. The close relationship between non-linear partial differential equations and non-linear functional analysis has, to quote Browder, "been evident historically starting with the work of Schauder in the late 1920's on the development and theory of compact non-linear operators in Banach spaces, culminating in the celebrated paper of Leray-Schauder of 1934 on the theory of the topological degree in Banach spaces for compact displacements and its application to non-linear elliptical equations of second order."

We will discuss Leray-Schauder degree theory and some of its modern generalizations and extensions that occur in the work of Browder, Nagume, Petryshyn and other both because of its own interest, and because of its repeated use as a proof method in monotone operator theory. It is worth noting, as Browder does, that part of the attraction of monotone operator theory depends on its applicability in absence of a prior estimates on the size of possible solutions. This stands in very strong contradistinction to the application of Leray-Schauder degree theory. We will also look briefly at a fixed point theorem for non expansive operators and remark that S is monotone if 1 - S is non expansive.

Finally, the bibliography has been arranged by section with works that have been refered to in more than one section listed in that section in which they have been used most fully.

We begin by giving the definition of monotone[1] and non expansive operators in a real reflexive Banach Space X with dual X*

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[1] The explicit definition of a monotone operator was first given by R.J.Kacuruski (1960).

We denote $y(x)$ by (y,x) where $y \in X^*$ and $x \in X$.

Definition 1: We say an operator T, with domain D contained in X and range in X*, is monotone over D if

$$(T(u) - T(v),\ u-v) \geqslant 0,$$

whenever u and v belong to D.

Definition 2: We say an operator T, with domain D contained in X and range in Y, another Banach space, is non expansive over D if

$$\|T(u)-T(v)\|_Y \leqslant \|u-v\|_X$$

whenever u and v belong to D.

If we replace (y,x) by Re (y,x) the spaces in question may also be complex. In general, results stated below for (y,x) will often remain valid for Re (y,x).

With these definitions in mind, we turn to Minty's development of the fundamental theory of monotone operators in a Hilbert space.

SECTION 2

Monotone Operators in Hilbert Space

Both Browder (1963) and Minty (1962) develop theorems on the existence of $(I+F)^{-1}$. Browder's method is perhaps more compact than Minty's but the following string of results leading to Minty's Theorem 4 and corollary are most revealing of the relationship between expansiveness and monotonicity.

The M and L Relations: We let S be a set and R be a relation on S. We call (S,R) a relation space and say that two relation spaces (S,R) and (S',R') are isomorphic if there is a 1-1 map ϕ of S onto S', such that for $x,y \in S$ $xRY \to \phi(x)R'\phi(y)$. If $A \subset S$ is such that $x,y \in A \to xRy$ we say A is totally R-related and we say A is a maximal totally-R-related set if it is not properly contained in another totally-R-related set.

We define two specific relations on H+H where H is a Hilbert Space and H+H is its product with itself.

We say $(x_1,y_1)M(x_2,y_2)$ if $\text{Re}\,(x_1-x_2,y_1-y_2) \geqslant 0$.

We say $(x_1,y_1)L(x_2,y_2)$ if $\| x_1-x_2 \| \geqslant \| y_1-y_2 \|$.

Armed with these definitions we have immediately:

Lemma 1: $\phi(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right)$ is a unitary, self inverse, Hermitian operator mapping (H+H,M) onto (H+H,L) isomorphically.

and then:

<u>Lemma 2</u>: Let $P: H+H \to H$ by $P(x,y)=x$. If F is totally-L- related and P' is P restricted to F then P' is a homeomorphism between F and P' (F). If F is maximal totally-L- related P' is onto.

<u>Proof</u>: If (x_1,y_1) and $(x_2,y_2) \in F$ then $x_1=x_2$, $y_1=y_2$ (by L-relatedness). Hence F can be regarded as the graph of a nonexpansive function. P' is clearly 1-1 and is continuous since the metric on F is given by

$$d(\,(x_1,y_1),\ (x_2,y_2)\,)^2 = \|x_1-x_2\|^2 + \|y_1-y_2\|^2 \leqslant 2\|x_1-x_2\|^2 .$$

This assures P' is metric decreasing and, since $(P')^{-1}$ increases the metric by 2 at most, it is also continuous. The final contention follows from the next theorem.

<u>Theorem 1</u>: If F is a non expansive map from $D \subset X$ to X then F has a non expansive extension from X to X.

<u>Proof</u>: Both Minty and Browder prove this using a Zorn's Lemma argument applied to a theorem of Kirsbraun and Valentine. (generalized by Mickle (1949) and Shoenberg).

<u>Theorem 2</u>: (Mickle and Shoenberg). If $\{S_\alpha\}_{\alpha \in A}$ and $\{S'_\alpha\}_{\alpha \in A}$ are families of spheres with (1) $\bigcap S_\alpha \neq \emptyset$. (2) rad (S_α) = rad (S'_α) and (3) $\delta(S_\alpha, S_\beta) \geq \delta(S'_\alpha, S'_\beta)$ then $\bigcap S' \neq \emptyset$. (by $\delta(S_\alpha, S_\beta)$ we mean the distance between the respective centres).

<u>Proof</u>: The proof is straightforward relying only on the <u>weak compactness</u> of the S and the truth of the theorem in <u>finite dimensions</u>. Browder develops the argument in great detail.

<u>Theorem 3</u>: Let F be a totally-M-related set in H+H. Then the map $\bar{\phi}: F \to X$ by $\bar{\phi}(x,y)=x+y$ is a homeomorphism, moreover, if F is a maximal totally-M-related set, $\bar{\phi}$ is onto.

<u>Proof</u>: $\bar{\phi}(x,y) = \sqrt{2}P' \circ \phi(x,y)$ and if F is (maximal) totally-M-related then $\phi(F)$ is maximal) totally-L-related.

<u>Theorem 4</u>: Let $F:D \to X$ be continuous and monotone with open domain. If F is maximal (has no proper monotone extension) then the graph of F, G(F), is a maximal totally-M-related set.

<u>Proof</u>: Suppose $(x_0,y_0)M(x,F(x))$, $x \in D$, $F(x_0) = y_0$. Then $\exists \delta > 0$ so that $\|x-x_0\| < \delta$ implies $x \in D$ and (1) $\|F(x)-F(x_0)\| \leqslant \frac{1}{2}\|y_0-F(x_0)\|$.

We now set x equal to $t\,(y_0-F(x_0)) + x_0$ with t sufficiently small for (1) to hold. Now $(x-x_0, F(x)-y_0) \geqslant 0$ by assumption which, by our choice of x and t, reduces to $(y_0-F(x_0), F(x)-F(x_0)) \geqslant \|y_0-F(x_0)\|^2$ in contradiction with (1).

<u>Corollary</u>: If F is a continuous monotone operator, then $(I+F)^{-1}$ exists, is continuous on its domain and monotone. If in addition F is maximal with open domain, then $(I+F)^{-1}$ is everywhere defined.

<u>Proof</u>: Let F have domain D and let $F':D \to H+H$ be the map $x \to (x,F(x))$. We let 0 and P' be as above then $1+F = 2P' \circ \phi \circ F$ of which as we know from above has inverse $\frac{1}{\sqrt{2}} P \circ \phi \circ (P')^{-1}$. All functions concerned are known to be continuous and monotonicity can be verified. The preceding 2 theorems give the remainder.

<u>Remark</u>: It is instructive to examine the corollary in the case that H=R and monotonicity becomes $x \geq y \to f(x) \geq f(y)$.

We can see now the way in which the introduction of M- and L-relatedness enables us to reduce the existence of solutions to $x+F(x)=u$ to the extendability of nonexpansive maps.

Browder proves the same result by essentially the same methods and then strengthens it as follows.

<u>Theorem 5</u>: If G is continuous and $\mathrm{Re}(G(x)-G(x_1), x-x_1) \geq C(\max(\|x\|, \|x_1\|))\ \|x-x_1\|^2$ where $C(r)$ is positive non-decreasing and $\int_1^{oo} C(r) = + oo$ then G^{-1} is everywhere defined and continuous. (Setting C = I and F=G-I, we obtain Minty's result for the case D(F)=H).

<u>Proof</u>: On each open ball in H G may be written as $G=\alpha(I+F)$ with F monotone and $\alpha > 0$ and depending on the ball. Using Minty's result G is open and hence has open range. G is clearly 1-1 and will have a continuous inverse defined on its range R. Suppose $y \in H-R$ with $\|y\|$ the infimum over $y \in H-R$. Since R is open there is a line segment $y(t)$ in R approaching y as t approaches 1. Since $y \notin R$ and G^{-1} is continuous $x(t)=G^{-1}(y(t))$ does not tend to a limit as t tends to 1 and indeed is unbounded. We can find a sequence $\{t_k\}$, $t_k \leq t_{k+1}$ with $x(t_k) = k$, $k > k_0$. Hence $\|y(t_{k+1}) - y(t_k)\| \geq C(k+1)\ \|x(t_{k+1})-x(t_k)\| \geq C(k+1)$, and $\sum_k C(k) \leq \sum \|y(t_{k+1})-y(t_k)\| < \infty$ which contradicts $\int_1^{oo} C(r)=oo$.

Thus R=H and the theorem is proved.

<u>Remark</u>: If we suppose that G has a continuous Fréchet derivative L_x though H we obtain

$$\mathrm{Re}\ (L_x(y),y)\ +o(1) \geqslant C\|y\|^2 \quad \text{as} \quad \epsilon \to 0$$

in place of

$$\mathrm{Re}\ (G(x)-G(x_1),x-x_1) \geqslant C\|x-x_1\|^2$$

Thus

$$C\|y\|^2 = \mathrm{Re}\ (L_x(y),y) \leqslant \|L_x y\|\,\|y\|$$

implying that $\|y\| \leqslant C^{-1}\|L_x(y)\|$. Thus $R(L_x)$ is closed, L is H with continuous inverse, and $R(L_x)$ is the orthogonal complement of $N(L^*)$. Since $\mathrm{Re}\ (L_x^*(y),y) \geqslant C\|y\|^2$, $N(L_x^*) = 0$ and $R(L_x) = H$.

Hence L_x has continuous inverse defined as all of H, and by the implicit function theorem G is open and the theorem valid.

This remark raises the question of when one can approximate by maps with continuous Fréchet derivatives. We have also, in passing, established the linear case of Theorem 4 which is essentially the Lax-Milgram Lemma.

Minty has given two simple sufficiency conditions for a mapping F over a domain D to be monotone. These are:

1. D is convex and F is a monotone over $D \cap \{x \mid \|x-x_0\| < \delta\}$ for all $x_0 \in D$ and some $\delta > 0$.
2. D is convex and $x, x_1 \in D$

   $$\frac{d}{dt}\,\mathrm{Re}\ (h,\ F(x+th))\Big|_{t=0} \geqslant 0$$

   for real t and $h = x_1 - x$. (1) is straightforward and (2) is proven by applying the Mean Value Theorem to

$$g(s) = \text{Re}\ (x_1-x_2,\ F(sx_1+(1-s)x_2))$$

He also notes that if the Gateaux derivative exists and is linear, then F is monotone over D when its derivative $F'(x,h)$ is dissipative. That is if:

$$(F'(x,h)y,y) \geqslant 0 \ .$$

Since a linear operator is its own differential this sheds light on the fact that dissipativeness and monotonicity coincide for linear operators.

In another paper Minty (1961) has shown that in a finite dimensional Hilbert Space the domain of a maximal monotone function must contain the interior of its convex hull. This forces the domain to be very nearly convex (indeed Minty calls this property almost convexity). The proof, using various results from convexity theory, is not particularly central to our discussion. The result, however, serves to illustrate the essential nature of convexity requirements of some kind.

SECTION 3

Monotone Mappings in Banach Space

The importance of dissipativeness is born out in papers by Minty (1963) and Browder (1965) which investigate the existence of solutions to $f(x)=0$ for monotone maps in reflexive Banach spaces, under a variety of continuity conditions.

We say a pair in XxY is M-related if Re (x_1-x_2, y_1-y_2) where $(\; , \;)$ is now the conjugate bilinear form on a reflexive Banach space X and its dual Y. With this change all the relevant definitions from the previous section are still applicable.

$f: B \subset X \to Y$ is said to hemicontinuous at $x_0 \in B$ if, for every line segment Γ with endpoint x_0, f is continuous considered as a mapping from $\Gamma \cap B$ to Y with the weak topology.

We say B surrounds x_0 (densely) if every line segment through x_0 contains points (arbitrarily closely) on either side of x_0. The symbol K(B) denotes the closed convex hull of B.

The following theorem of Minty serves to introduce a projection method and a technique Minty calls a 'Monotonicity Method' both of which are used again and again in the literature.

Theorem 1: Let $D \subset X$ be bounded and surround 0. Let $B \subset X$ surround K(D) densely. Let $f: B \to Y$ be monotonic and hemicontinuous throughout K(D), and suppose

$$x \in D \text{ implies } (x, f(x)) \geq 0 .$$

Then there exists $x \in K(D)$ such that $f(x)=0^*$.

Leading up to the proof we have:

Lemma 1: Let $x_o \in B \subset X$ with B surrounding x_o densely. Let $f: B \to Y$ be hemicontinuous at x_o. Let (x_o, y_o) be M-related to every point of G(f). Then $y_o = f(x_o)$.

Proof: Choose Z such that $(Z, f(x_o)-y_o) \geq \frac{1}{2} \| Z \| \| f(x_o)-y_o \|$. There exists t such that $x_o - tZ \in B$ and $(Z, f(x_o-tZ)-f(x_o)) \leq \frac{1}{3} \| Z \| \| f(x_o)-y_o \|$, since $(x_o, y_o) M(w, f(w))$, we have

$$(-Z, f(x_o-tZ)-f(x_o)) \geq (Z, f(x_o)-y_o)$$

which leads to a contradiction unless $f(x_o)-y_o$.

Lemma 2: The theorem holds if X is finite dimensional.

Proof: We can, as is often the case, assume X is a Hilbert space. For all positive integers n, we consider G_n the graph of the function nf(x). Now G_n is totally-M-related and, by the usual Zorn's Lemma argument, can be extended to a maximal totally-M-related set G_n' By Theorem 3 of the previous section $(x,y) \to x+y$ maps G_n' onto X. Thus, there exist $(x_n, y_n) \in G_n'$ such that $x_n + y_n = 0$. We shall show that $x_n \in K(D)$. Assume the contrary.

Then there is a $z_n \in D$ and $\lambda_n > 1$ with $x_n = \lambda_n z_n$ (this is possible since D surrounds 0 and $x_n \notin K(D)$). Either $x_n = 0$ or $\|x_n\|^2 + (x_n, y_n) = 0$ and $(x_n, y_n) < 0$, and hence $(z_n, y_n) < 0$.
Since $(z_n, nf(z_n)) \geq 0$

$$(\lambda_n - 1)(z_n, y_n) < (\lambda_n - 1)(z_n, nf(z_n))$$

* This theorem and many like it are applicable to non-reflexive Banach space Y with dual X providing the weak topology is replaced by the weak-star topology throughout.

which rearranges to:

$$(x_n - z_n, y_n - nf(z_n)) < 0$$

which contradicts $(x_n, y_n) \in G_n'$. Hence $x_n \in K(D)$.

By <u>Lemma 2</u>: $y_n = nf(x_n)$ and

$$\frac{x_n}{n} + f(x_n) = 0 .$$

We now appeal to the compactness of $K(D)$, which relies on the finite-dimensionality of X, to a convergent subsequence x_{n_i} with limit x. By the monotonicity of f

$$(x_o - x_{ni}, \ f(x_o) - f(x_{ni})) \geq 0$$

hence

$$(x_o - x_{ni}, f(x_o)) + (x_o, x_{ni}) / ni - \|x_{ni}\|^2 / ni \geq 0 .$$

Taking limits and applying Lemma 1, we have $f(x)=0$.

The finite dimensionality of X is, in fact, only invoked to ensure the existence of a convergent subsequence.

<u>Proof of Theorem One</u>: Since B contains and surrounds K(D) densely, it suffices by Lemma 1 to show that

$$\bigcap_{x_o \in B} \{ x: (x_o - x, f(x_o)) \geq 0, \ x \in K(D) \} \neq \phi .$$

Since K(D) is weakly compact we need only show that any finite subcollection has non empty intersection. We let x_1, x_2, x_m belong to B and let E be the subspace spanned by $x_1, \ \ , x_m$. We let j_E be the projection map of E into X with adjoint j^*_E and set $f_E U = j_E^* (f j_E U)$, $U \in E \cap B$. Then Lemma 2 can be applied to E, f_E,

$E \cap B$ and $K(E \cap D)$ and establishes the existence of x in $K(E \cap D)$ with $f_E(x)=0$ in E^*. Hence

$$(x_i - x,\ f_E(x_i) - f_E(x)\ ) = (j_E(x_i - x),\ f(j_E x_i) - f(j_E x)\ )$$

and since x_i and $x \in E$, for $i = 1, \quad m$.

$$(x_i - x,\ f(x_i) - f(x)) \geq 0$$

as desired.

We will see the projection method in a more fully analysed form in some later papers by Browder. In a somewhat different guise it is also basic to the generalised theory of topological degree.

Browder has extended the definition of monotonicity to multivalued maps as follows. We let $T: X \to 2^{X^*}$ then we say T is a <u>multivalued monotone map</u> if

$$(u_1 - v_2, v_1 - v_2) \geq 0 \quad \text{whenever } v_1 \in T(u_1) \text{ and } v_2 \in T(u_2)$$

He has also extended the concept of hemicontinuity and calls T <u>vaguely continuous</u> if $D(T) = X$ and for $u_o, u_1 \in D(T)$ there exists a sequence t_n tending to 0 as n tends to ∞ and there exists a $v_1 \in K(T(u_1)\ )$ such that if $u_n = t_n u + (1-t_n) u_o$ there exists a $v_n \in K(T(u_n)\ )$ such that v_n tends weakly to v_1 in X^*. By convention $D(T)$ is the set of u belonging to X for which $T(u) = 0$.

T has the following properties, if T is maximal:

(i) $T(u)$ is a closed convex set in X^*.

(ii) Let $u_k \to u_o$ strongly in X and $v_k \to v$ weakly in X_o. If $v_k \in T(u_k)$ then $v_o \in T(u_o)$.

(iii) Let D(T) be linear and dense. If for each line segments S_o in D(T) there exists a bounded set S_1 in X* with $T(u) \cap S_1 = 0$ for u in S_o, then T is vaguely continuous. The proof relies on the weak compactness of the unit ball in reflexive Banach Spaces.

As a partial converse to (iii) we have:

(iv) If T is monotone and vaguely continuous with D(T)=X, and T(u) a closed convex set for each u in X, then T is maximal monotone.

The proof uses the Strict Separation Theorem applied to the closed convex set T(u) and any point v exterior to it.

These properties allow Browder to derive.

Theorem 2: Let T satisfy (iii) and let T(u) be bounded for all $u \in X$. If $(T(u),u) \geq 0$ for all $u \in S$, where S is some bounded set surrounding 0, then there exists $u_0 \in K(S) \ni 0 \in T(u_0)$.

Proof: The proof is analogous to Minty's proof of Theorem 1. It appears that Lemma 1 must be replaced by (iv) and that the relationship between vague continuity and maximal monotonicity is a very subtle one.

We can replace the hypothesis of (iii) by its conclusion in the statement of Theorem 2. We thus obtain a cleaner, if less revealing, theorem which reduces, when T is single valued, to Theorem 1 with B=X.

Remark: A single valued monotone operator can be maximal as a single valued operator without being maximal among multivalued operators. If $D(T)$ is linear and dense and T is hemicontinuous this is not possible.

Proof: Suppose $(v_0 - T(u), u_0 - u) \geq 0$ for $u \in D(T)$. If $u_0 \notin D(T)$ we can find a monotone extension in contradiction to hypothesis. Hence $u_0 \in D(T)$ and, using Lemma 1 of Section 5, $v_0 = T(u)$.

Theorem 2 leads to results on anihilators including the following generalization of a theorem of Beurling and Livingston (1961).

Theorem 3: Let T be a multivalued monotone map satisfying Theorem 2 without $(T(u),u)$ necessarily non negative, but satisfying instead the condition that

$$(T(u),u) \geq C(\|u\|) \quad (\|u\| \quad + \|T(u)\|) \text{ for } u \in Y \text{ a}$$

closed subspace of $D(T)$ and $C(r)$ a continuous extended real valued function such that $c(r) \to \infty$ when $r \to \infty$.

Then for all v_0 in X and w_0 in X^*

$$T(Y + v_0) \cap w_0 + Y^{1} \neq \phi .$$

Proof: Let $K(u) = j^* \quad (T(u+Vo)-wo)$ where $j: Y \to X$ is the projection. $K(u) = 0$ has a solution by Theorem 2 and this is equivalent to the desired result.

We complete this section with an example. Let O be a monotonic non decreasing map from R to R with $\phi(o)=0$ and $\phi(\infty)=\infty$.

We define, T_ϕ the <u>duality map</u> from X into X* with respect to ϕ by

$$T_\phi(u) = \{ v \mid (v,u) = \|v\|\|u\|, \ \|v\| = \phi(\|u\|) \} .$$

We have

<u>Lemma 3</u>: The duality map T_ϕ is a multivalued monotone map and

1. T_ϕ is vaguely continuous ,
2. $D(T_\phi) = X$,
3. $T_\phi(u)$ is a bounded closed convex subset for all $u \in X$.
4. $(T_\phi(u),u) \quad C(\|u\|)(\|u\| + \|T_\phi(u)\|)$, $C(r) = \frac{1}{2}\min(r, \phi(r))$.

These conditions imply that T is maximal and satisfies Theorem 3 for any closed subspace.

SECTION 4

Topological Degree

The theory of Topological degree found its basis "within combinational topology of the fixed points and degree of continuous mappings as originated by Brouwer".[1] Leray and Schauder (1934) developed the theory for completely continuous displacements in a Banach space. It has been generalized in two different directions. Nagumo (1951) Browder (1957) and Rothe have extended the results to locally convex topological vector spaces, as indeed Leray himself has. Petryshyn and Browder (1968) have, in contrast, dealt with a class of non compact operators in Banach space. In each case the theory rests upon approximation by finitedimensional mappings and appeal to Brouwer's theory. Nagumo's exposition of the fundamental properties of the degree is very detailed and we proceed to outline his development of the subject.

The Brouwer Degree: We let f be a mapping of a bounded open set $G \subset E^m$ into E^m such that each component is continuously differentiable on $\bar{G}$. We say a point x is critical if the Jacobian evaluated at x is zero and call the image under f of the critical points the crease of f on G. If a belongs neither to $f(\bar{G}-G)$ or to the crease we define an integer $A[a,G,f]$ to be the number of points x for which f(x)=a and the Jacobian is positive

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[1] Browder (1957)

less those for which it is negative. We call $A[a,G,f]$ the degree of f, at a, over G. A succession of analytic approximations enables us to extend the degree to any continuous mapping and only providing $a \notin f(\bar{G}-G)$. $A[a,G,f]$ has the following properties:

(i.) $A[a,G,I] = 1$ if $a \in G$ and 0 if $a \notin G$.

(ii.) If $A[a,G,f] \neq 0$ then there is a solution to $f(x)=a$ with $x \in G$.

(iii) If G is divided into open sets $G_1, \ldots, G_k$ ($\cup G_i \subset G, G = \cup G_i$, $G_i \cap G_j = \emptyset$ $(i \neq j)$), and if $a \notin f(\bar{G}_i - G_i)$ for any i, then
$A[a,G,f] = \sum_i A[a,G_i,f]$.

(iv.) If $a \notin f(\bar{G}-G)$ and X is the set of roots of $f(x)=a$ in G then if G_0 is any open set with $X \subset G_0 \subset G$
$A[a,G_0,F] = A[a,G,f]$.

(v.) If $f_t(x)-x$ is a bounded continuous function of (t,x) in $[0,1] \times G, x \in \bar{G}$, if $a(t)$ $(\in E^m)$ is continuous and if $a(t) \notin f_t(\bar{G}-G)$ for $0 \leq t \leq 1$, then $A[a(t),G,f_t]$ is constant for $0 \leq t \leq 1$.

Nagumo's Extension to Convex Spaces: A transformation f of M into a convex space is completely continuous or compact if f is continuous on M and f(M) is compact in E. By a completely continuous displacement we mean the transformation

$Tf(x)=x+f(x)$ with f completely continuous.

Theorem 1: Let M be closed in E and f be completely continuous on M. Then Tf(M) is closed in E.

<u>Theorem 2</u>: Let K be compact in E. For any neighbourhood U of the origin there exists a finite dimensional linear manifold $E^m \subset E$ and a continuous transformation S of K into E^m such that $S(x)-x \in U$ for $x \in K$.

These results enable us to define the degree in E. Let $a \notin Tf(\bar{G}-G)$ and, by Theorem 1, choose U+a not intersecting $Tf(\bar{G}-G)$. We use this neighbourhood U in Theorem 2 and $K \supset f(\bar{G})$ and find S and E^m as in the Theorem. Then $TSf(x)-Tf(x) \in U$, if $x \in G$ and $a \notin TSf(\bar{G}-G)$. Now $TSf\ (\bar{G} \cap E^m) \subset E^m$, $TSf(x)-x = Sf(x)$ is bounded on $G^m = G \cap E^m$, and $a \notin TSf(\bar{G}^m - G^m)$.

It follows that $A^m[a, G^m, TSf]$ is well defined in E^m. We then prove that this is indeed independent of S and E^m and call this the degree of Tf, with respect to a, over G. It is easily verified that (i), (ii), (iii) and (iv) remain valid for Tf. Property (v) can also be established for Tf_t if we require that f_t is continuous in (t,x) and $f_t\ (G)\ K$, K a given compact set. We prove the result for constant a(t) by applying Theorem 1 to the mapping $f^*(x,t) = (f\langle t\rangle\ (x), o)$, $[\langle t\rangle = o,\ t<o;\ \langle t\rangle = t,\ o \leq t \leq 1;$ and $\langle t\rangle = 1,\ t>1]$ in the convex space E x R, with G replaced by $G \times (-\infty, \infty)$ and K by $K \times \{0\}$. This gives us $U\times(-\delta,\delta)+(a,\tau) \cap Tf^*(\bar{G}^*-G^*)=\emptyset$ or equivalently (1) $U+a \cap Tf_t(\bar{G}-G)=\emptyset$ for $|t-\tau|<\delta$.

We use Theorem 2 again to find S and E^m. The definition, (I) and (v) in the initial form allow us to assert the constancy of the degree for $|t-\tau|<\delta$ and hence throughout the unit interval.

We can now establish the case in which a(t) is non constant by examining $f_t(x)-a(t)$, since $K+\{-a(t)|t\in[o,1]\}$ is compact. It is in fact to insure this that we require continuity of a (t). We need only show that $A[a,G,Tf] = A[o,G,Tf-a]$. If $a \notin Tf(\bar{G}-G)$. This follows by applying (v) again, in the finite case, to $F_t = Tf-ta$, $a(t)=(1-t)a$ and using the definition of degree in E.

The follow theorem on invariance of domain has also been proven by Browder (1957) for a more restricted class of mappings in a general convex space. Leray claims to have proven it without restriction.

Theorem 3: Let E be a complete metric space G an open set in E and f a completely continuous transformation of G into E. If Tf is 1-1 between $\bar{G}$ and Tf $(\bar{G})$, then Tf(G) is open and $A[b,G,Tf]$ $=\pm 1$ for any b in Tf(G). (Tf(x)=b will, of course, have solution by (i).

It is specifically property (v) which is used again and again in monotone operator theory. Browder (1957) does, however, develop some estimates for the number of solutions, and the literature is extensive.

The Topological Degree for Non Compact Operators. It is clearly desirable, within the monotone operator theory developed thus far, to extend the applicability of degree methods. This has been done, for the class of A-proper mappings, by Browder and Petryshyn.

<u>Definition</u>: By an <u>oriented approximation scheme</u> for mappings from X to Y (real Banach spaces) we mean: two sequences $\{X_n\}, \{Y_n\}$ of oriented finite dimensional spaces with $\dim X_n = \dim Y_n$ for all n and two sequences $\{P_n\}$ and $\{Q_n\}$ of continuous mappings with P_n mapping X_n into X and Q_n mapping Y into Y_n.

<u>Definition</u>: Let G be open in X and $T(G) \subset Y$. T is said to be <u>A-proper</u> on cl(G) with respect to $\Gamma = (\{X_n\}, \{Y_n\}, \{P_n\}, \{Q_n\})$ if and only if for any sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ and a corresponding sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ with $P_{n_j} x_{n_j}$ in G such that $\| Q_{n_j} T P_{n_j} x_{n_j} - Q_{n_j} y \| \to 0$ for some y in Y, there exists an infinite subsequence $x_{n_{j(k)}}$ and x in X such that $P_{n_{j(k)}} x_{n_{j(k)}} \to x$ and $T(x)=y$. We denote $Q_n T P_n$ by T_n.

<u>Definition</u>: Let T be an A-proper continuous mapping from $\bar{G}$ to Y with respect to Γ, let $G_n = P_n^{-1}(G)$ be bounded for all n and suppose $a \notin T(\bar{G} - G)$. We define deg(T,G,a) with respect to Γ as follows:

$$\deg(T,G,a) = \{k \mid k \in \mathbb{Z} \cup \{\pm\infty\}, \text{ there exists an infinite sequence } \{n_j\} \text{ of positive intergers tending to } \infty \text{ and } A[Q_{n_j} a, G_{n_j}, T_{n_j}] \text{ tends to } k\}.$$

This degree has analogous properties to the Leray-Schauder degree. Moreover, we have

(1) If $\deg(T,G,a) \neq \{0\}$ then $T(x)=a$ has solution in G, and the following

<u>Theorem 4</u>: Let C be compact and H be an A-proper homeomorphism satisfying several subsidiary conditions. If $P_n^{-1}(G)$ is bounded and $a \notin T(\bar{G}-G)$, then

(1) H+C is A-proper ,

$$\deg(T_n, G_n, Q_n a) = \deg(1+CH^{-1}, H(G), a), \quad n \geqslant n_0 .$$

In particular the degree is single valued and possesses the previously listed five properties. If X is a real Banach space with Y=X, P_n a linear injection of X_n into X and Q_n a linear projection of X into X_n in the approximation scheme Γ^*, then the identity is suitable and compact displacements have single valued degrees.

<u>Example</u>: Let H be a separable Hilbert space and T be continuous From H to H with

$$(T(x)-T(y), x-y) \geqslant \|x-y\|^2 \quad (x,y \in H)$$

Then T is A-proper with respect to Γ^*.

It is worth remarking that in many applications we only wish to know that $0 \notin \deg(T,G,a)$ and that the multivalued nature of the degree is then irrelevant.

SECTION 5

The Projection Method

Two papers by Browder (1963), (1965) on non linear elliptic boundary value problems contain theorems on the surjectivity of non linear functional equations in reflexive Banach spaces which, in Browder's words, "present a new non-linear version of the orthogonal projection method". The results make essential use of both monotonicity and Leray-Schauder degree theory. The later paper also contains some results for monotone operators in convex spaces which are along the lines of Theorem 1 of Section 3.

Theorem 1: Let X be a separable reflexive Banach Space with dual X^* and let G map X into X^* satisfying

(i) G+C is monotone for some fully continuous C.
i.e. from the weak topology on X to the strong topology on X^*,

(ii) $\mathrm{Re}(G(u),u) \geqslant c(\|u\|)\,\|u\|$ and $c(r) \to \infty$ as $r \to \infty$,

(iii) G is demicontinuous. i.e. from the strong topology on X to the weak topology on X^*.

Then G maps X onto X^*.

Lemma 1: Let G be a demicontinuous mapping of the open subset D of X into X^*. Suppose that for u_o in D and w in X^*

$$\mathrm{Re}\,(w-G(u),u_0-u) \quad 0$$

for all u in a dense subset Y of D.

Then $w=G(u)$.

Proof: The proof again relies on the Strict Separation Theorem and is essentially the same as that of Lemma 1 of Section 3.

Lemma 2: Let $\{F_j\}_{k=1}^{\infty}$ be a sequence of nested subspaces of X of finite dimension. Suppose P_1 is the projection of X on F_1.

Then

(a) There exists a commutative increasing family $\{P_j\}$ of projections with $P_jX=F_j$.

(b) If F_j' is the range of P_j^* then $\{P_j^*\}$ is a commutative increasing family of projections on $\{F_j'\}$, $F_j' \subset F_j'+1$, and the pairing (w,u) $w \in F_j'$ $u \in F_j$ yields an isomorphism of F_j^* and F_j'.

Proof: (a) Suppose recursively $P_1, P_2, \ldots P_r$ are given and that F_j has dimension 1 in F_j+1. Then the dimension of the nullspace of P_r restricted to F_{r+1} is 1 and is generated by some element u_o. Then there exists w in X* such that $(w,u_o)=1$ and $(w,u)=o$ for u in F_r. We let $P_{r+1}(u)=P_r(u)+(w,u)u_o$. (b) follows by calculation.

Lemma 3: Let G be demicontinuous from X to X*, $G=G_o+C_o$, with G_o monotone and C_o fully continuous. Let $\{F_j\}$ be a sequence of increasing subspace (of finite dimension) with UF_j dense in X. Let P_j be the orthogonal projection on F_j. Let u_k be such that u_k belongs to P_k, P_k^* $G(u_k)$ converges strongly in X* to w and u_k conveys weakly in X to u_o.

Then $w=G(u_o)$.

Proof: Let be fixed and u belong to F_j.

Then $\mathrm{Re}(u_k - P_j u,\ G_o(u_k) - G_o\ (P_j u)) \geq 0$. By rearrangement and taking limits we obtain

$$\mathrm{Re}(u_o - u,\quad w - C_o\ (u_o)\quad - G_o(u)\ ) \geq 0$$

for all u in UF_j. Applying Lemma 1, $w - C_o\ (u_o) = G_o(u_o)$ and $w = G(u_o)$.

Lemma 4: Let G be a continuous map of a finite dimensional Banach space Y to its dual Y* such that $\mathrm{Re}(Gu,u) \geq C(\|u\|)\|u\|$ with $C(r) \to \infty$ as $r \to \infty$. Then G is onto.

Proof: (This is the finite dimensional version of Theorem 1 without (i)). There is no loss of generality in suppose $Y = Y^* = H$ a Hilbert space. Let $w \in H$, $G_t = I + t(G-I) = tG + (1-t)I$. For u in H and $0 \leq t \leq 1$

$$\mathrm{Re}(G_t(u) - w, u) = t\mathrm{Re}(G(u),u) + (1-t)\|u\|^2 - \|w\|\|u\| \geq \tfrac{1}{2}\ \|u\|$$

for $\|u\| \geq M$. Thus $G_t(u) = w$ for $\|u\| \geq M$.

We can easily verify that G_t satisfies the remaining conditions of property (v) of the Leray-Schauder degree for $a(t) \equiv w$ and $G = Q = \{x \mid \|x\| < M\}$. Hence

$$A[w,Q,G_t] = A[w,Q,I] = 1 \text{ for } 0 \leq t \leq 1.$$

Appealing to property (1) we have a solution to $G_1(x) = w$. (This is essentially an argument of Visik's (1961)).

Proof of Theorem 1: Let $w \in X^*$ and let F_1' be the vector subspace generated by w with P_1' the associated projection. Since X is separable we can find a sequence $\{F_j\}$ of increasing subspaces such that F_j is dense in X. By Lemma 2 we can construct an increasing commutative family of projections $\{P_j\}$ with $P_jX=F_j$.

Now, if we define $G_k(u)=P_k{*}G(u)$ for u in F_k, we have

$$Re(G_k(u),u) = Re(P_k{*}G(u),u) = Re(G(u),P_k(u)) = Re(G(u), u)$$

so $Re(G_k(u),u) \geqslant C(\|u\|)\|u\|$ and G_k is certainly continuous as a map from F_k to F_k', which is isomorphic to F_k^*.

We are in a position to apply Lemma 4 to produce $u_k \in F_k$ such that

$$G_k(u_k)= P_k{*}G(u_k)=w\in F_k' \quad k \geqslant 1.$$

so

$$Re(w,u_k)=Re(P_k^*G(u_k),u_k)=Re(G(u_k),u_k) \geqslant C(\|u_k\|)\|u_k\| .$$

From this we deduce that $\|w\| \geqslant C(\|u_k\|)$. Since $C(r)\to\infty$ as $r\to\infty$ we have that $\{u_k\}$ is bounded. Since X is reflexive the unit ball is weakly compact and $\{u_k\}$ has a weakly convergent subsequence which we may take to be $\{u_k\}$. Since $P_k^* G(u\ )=w$ and u_k tends weakly to u_o we can assert, using Lemma 3, that $G(u_o)=w$.

Remarks: The general strategy is made very clear. We prove the result in finite dimension using both the Hilbert space structure and Leray-Schauder degree theory. We then use the projection families to produce a sequence of potential solutions. Because of the coercivity condition (ii) we can get an a priori estimate on the size of any solution and by appealing to compactness we find a sequence converging to a solution. It is only at this point and in Lemma 1 that we use the monotonicity of G+C.

If we re-examine Theorem 1 of Section 3 we see a very similar pattern.

In the second paper Browder (1965) proves the following theorem in which the interplay of monotonicity, coercivity and degree theory is again apparent.

<u>Theorem 2</u>: Let X be a separable reflexive Banach space. Let Y be a second Banach space such that the injection of X into Y is compact. Let $G: Y \times X \to X^*$ and denote by G_u the mapping from X to X* defined by $G_u v = G(u,v)$. If:

(a) For each positive integer N there is real valued continuous function $C_N(r)$ with $C_N(r) \to \infty$ as $r \to \infty$ and $C_N(r)$ is positive when r is such that

$$\mathrm{Re}(G_u(v)-G_u(w), v-w) \geq C_N(\|v-w\|_X)\,\|v-w\|_X$$

for u in Y with $\|u\|_Y \leq N$ and for all v,w in X.

(b) There is a continuous real valued function C(r) with $C(r) \to \infty$ as $r \to \infty$ such that for every $k \geq 1$

$$\mathrm{Re}(G_u(ku), u) \geq C(\|u\|_X)\,\|u\|_X.$$

(c) G_u is demicontinuous and the mapping from Y to X* given by $u \to G_u(v)$ is strongly continuous.

Then $F(u) = G_u(u)$ maps X onto X*.

<u>Lemma 5</u>: Let G be a demicontinuous mapping of X into X^* such that (α) $\text{Re}(G(u)-G(v),u-v) \geq C(\|u-v\|)\,\|u-v\|$ $(u,v \in X)$ for some continuous realvalued function $C(r)$ with $C(r)\to\infty$ as $r\to\infty$ and $C(r)$ positive when r is. Then:

(a) G^{-1} is defined on all of X^*, maintains boundedness and is strongly continuous.

(b) There is a realvalued monotone non-decreasing function $h(r)$ such that $h(o)=o$, h is continuous from the right, and

$$(\beta)\quad \| G^{-1}(w) - G^{-1}(w)\| \leq h(\| w-w_1 \|)$$

for all w and w_1 in X^*. h depends only on $C(r)$.

<u>Proof</u>: (α) gives immediately

$$\text{Re}\,(G(u),u) \geq (C(\| u\|) - \|G(o)\|)\,\|u\|.$$

We may conclude from Theorem 1 with $C(u)\equiv 0$ that G maps X onto X^*.

For each $R \geq o$, let

$$h(R)= \sup\{ r \mid C(r) \leq R\}.$$

Then as required and (β) holds, forcing G to be strongly continuous. Finally $\|G^{-1}(w)\| \leq h(\| w \|) + \|G^{-1}(o)\|$ and G^{-1} maps bounded sets in X^* into bounded sets in X.

<u>Proof of the Theorem</u>: Let u be an element of Y. Lemma 5 applied to G gives

$$(\gamma)\quad \| G_u^{-1}(w)-G_u^{-1}(w_1)\|_X \leq K_N(\|w - w_1\|_{X^*})$$

for u in Y with $\|u\|\leq N$. Let w be a fixed element of X^* and let $T:Y\to Y$ be given by $T(v)=G_v^{-1}(w)$. Then $G_u(u)=u$ if and only if $T(u)=u$.

We show the existence of a fixed point by applying Leray-Schauder theory to $T_t = tT + I$ for $o \leq t \leq 1$. To this end we must show

(1) T and hence tT is compact.

(2) For some $M > 0$ and $o \leq t \leq 1$ tT(v) has no fixed point with

$\|u\|_y = M.$

We then can assert that $A [o, \|u\| < M, T_t]$ will be constant throughout $[o,1]$. $A[o, \|u\| < M, T_o = I]$ is clearly 1 and we will know that $T(u) - u = o$ has solution.

(1) Follows somewhat tediously from (γ) and the continuity properties of G.

<u>Proof of (ii)</u>: Let t be a real number, $o \leq t \leq 1$, and suppose $tT(u) = u$ then

$$Gu^{-1}(u) = \frac{1}{t}u$$

and

$$Gu\left(\frac{1}{t}u\right) = u$$

Setting $\frac{1}{t} = R$, we have (from)b) of Theorem 2)

$$(Gu\ (ku),\ u) = (w,u) \geq C(\|u\|x)\|u\|x.$$

Hence $C(\|u\|x) \leq \|w\|x^*$. It follows that $\|u\|x \leq M1$ for some constant independent of T. By letting $M = 2M1$ and noting that if $t = o$, T_o has only O as a fixed point, we can satisfy (2).

<u>Remarks</u>: Although the projection method is not used directly, we have appeal to Theorem I in the proof of Lemma 5 and through Theorem I to the projection method. We note in passing that we have used the degree theory to find a fixed point of T rather than a solution of $T(u) = a$.

A General Theorem of Monotone Operators.

The next result indicates the degree to which one can generalize monotone operator theory to arbitrary convex spaces without having to modify either proof methods or hypothesis extensively.

Let (E_1, E_2) be a dual pair of Hausdorf convex spaces and let (u,v) denote the associated bilinear form. Within this framework the previous definition of monotonicity are still meaningful.

T is said to be <u>finitely continuous</u> if it is continuous considered as a mapping from any finite subspace E_1 into E_2.

C is said to be <u>completely continuous</u> with respect to (E_1, E_2) if C is continuous and the mapping $u \to (u, Cu)$ is continuous on compact subsets of E_1.

A subset S of E_1 is said to envelop $u \in K(S) - S$ if for every finite dimensional variety containing F the boundary of $K(S) \cap F$ is contained in $S \cap F$.

<u>Theorem 3 (Browder (1965))</u>: Let $(E_1 E_2)$ be dual and $T: E_1 \to E_2$ such that $T = To + C$ where To is finitely continuous and monotone and C is completely continuous with respect to (E_1, E_2). Let S be a subset and u_o a point of E, such that K(S) is compact and S envelops u_o. Suppose that for a given w in E_2 we have

$$(u - u_o , T(u) - w) \geq 0$$

For all u in S. Then there exists u_1 in K(S) with $Tu_1 = w$.

<u>Preliminary Remark</u>: There is no loss of generality in assuming that $u_o = 0$ and $w = 0$. To see this one sets $T_1(u) = T(u - u_o) - w$ and replaces S by $S_1 = S - u_o$ and verifies that the assertions of the theorem are invariant. It is apparent that the same remarks apply to the theorems of Section 3 with the proviso that the domain of T must be linear.

<u>Lemma 6</u>: Let T_0 be a finitely continuous map of E_1 into E_2, $u_1 \in E_1$, $u_2 \in E_2$. Suppose that (u_1, u_2) is M related to $(u, T_0(u))$ for all u in E_1. Then $u_2 = T_0(u_1)$.

<u>Proof</u>: This is a corollary of Lemma 1 of Section 3.

<u>Lemma 7</u>: The theorem holds if E_1 has finite dimension.

<u>Proof</u>: There is again no reason not to retopologise E_1, so that it is a Hilbert space H, $H = E_1 = E_2$. We now set $T_t = (1-t)T + tI$ and calculate the degree of T_t on D, the interior of K(S), with respect to 0.

Since S envelops 0 (the remark) $o \in D$ and hence T_1 has degree 1.

On D we have

$$(u, T_t(u)) = (1-t)(u, T(u)) + t\|u\|^2 \geq t\|u\|^2$$

for $u \in \partial D \subset \partial K(S) \subset S$. Thus, if $t > o$, $T_t(u)$ has no zero on ∂D. If $T_o u = o$ on D we are finished, otherwise the degree of T_o over D with respect to o is the same as the degree of T, which is 1. Hence there is a $u_1 \in D$ with $T(u_1) = 0$. To apply the degree theory we needed to know that T was continuous and that $T_t(K(S)) \subset K$, K compact for $o \leq t \leq 1$. Thus we need only the boundedness of S and the continuity of T as hypothesis.

<u>Proof of Theorem 3</u>: For fixed u in E_1, let

$$B(u) = K(S) \cap \{ v \mid (u-v, T_0(u) + C(v) \geq o \}$$

By the complete continuity of C, B(u) is a closed subset of the compact set K(S). We show that the B(u), $u \in E_1$, have the finite intersection property and it will follow that there is some v_0 in K(S) with

$$(u - v_0, T_0(u) + C(v_0)) \geqslant 0 \quad \text{for all } u \text{ in } E_1.$$

Lemma 6: Now gives the desired result. A standard projection argument establishes the finite intersection property It is in this proof that we use the monotonicity of T_0.

SECTION 6

Non Expansive Mappings and Fixed Point Theorems

Solutions of functional equations can often be found by using fixed-point theorems. This method was used in the proof of Theorem 2 of the previous section and we will encounter both the Brouwer and Poincaré fixed-point theorems in our discussion of variational inequalities. For the moment we restrict ourselves to some results for non expansive mappings in Banach space.

Lemma 1: If $U: X \to X^*$ is a non expansive mapping then $I-U$ is monotonic.

Proof:

$$(U(x)-Uy, x-y) \leq \|U(x)-U(y)\| \, \|x-y\| \leq \|x-y\|^2 = (x-y, x-y)$$

and hence

$$((x-U(x)) - (y-U(y)), x-y) \geqslant 0 .$$

The fundamental theorem proven independently by Browder (1965) and Kirk (1965) is

Theorem 1 (Browder): Let X be a uniformly convex Banach space, U a non expansive mapping of a given closed bounded convex subset C of X into C. Then U has a fixed point in C.

Proof: Let ϕ be the family of nonempty closed convex subsets of C which are invariant under U. ϕ has a minimal member C_0 since C is weakly compact. By minimality $C_0 = K(U(C_0))$.

Suppose x_1 and x_2 are distinct and belong to C . Let d_0 be the diameter of C and assume that $\|x_1 - x_2\| \geq d_0/2$. Let $x = \frac{x_1 + x_2}{2}$, then x belongs to C_o. By the uniform convexity of X there is a constant $q > o$ such that for any y in C_o

$$\| x - y \| \leq (1-q)\, d_o \leq d_1 \leq d_o$$

(since $\| x_1 - y) - (x_2 - y) \| \geq d_o/2)$. Now let

$$C_2 = \cap_{y \in C_o} \{ u : u \in C_o, \| u - y \| \leq d_1 \}.$$

C_1 is closed and convex and is non empty since x lies in C_2. C_2 is properly contained in C_o since $d_1 < d_o$. Finally $U(C_o) \subset C_2$. Suppose $u \in C_2$, $y \in C_o$. For any $\epsilon > o$ we can find a convex linear combination of $U(z_j)$, $z_j \in C_o$, such that

$$\| y - \sum \lambda_j U(z_j) \| < \epsilon \quad (\sum \lambda_j = 1 ; \ o \leq \lambda_j \leq 1)$$

Thus

$$\| U(u) - y \| \leq \| U(u) - \sum \lambda_j \ U(z_j) \| + \sum \lambda_j \| U(u) - U(z_j) \| + \epsilon$$

while

$$\| U(w) - U(z_j) \| \leq \| u - z_j \| \leq d_1$$

and

$\| U(u) - y \| \leq d_1$ and $U(u)$ lies in C_2. This contradicts the minimality of C_o which must, therefore, be a single point x_o and $U(x_o) = x_o$.

<u>Remarks</u>: (1) Uniformly convex spaces are reflexive and strictly convex and all L^p and l^p spaces $(1 < p < \infty)$ are uniformly convex.

(2) An example by Beals shows that the result is not true in a general Banach space. Let $X = C_o$, C be the unit ball in

the max.norm. Then $U(x) = (1, x_1, x_2 \ldots)$ is a non expansive map of C into itself with no fixed point.

(3) It would be interesting to know if the result is true in strictly convex spaces.

In what is essentially a corollary Browder proves a non linear extension of a theorem of Markov and Kakutani which states that if $\{U_\lambda\}$ is a commutative family of mappings satisfying Theorem 1, then the $\{U_\lambda\}$ have a common fixed point.

A paper by Edelstein (1964) contains a number of results valid in any reflexive Banach space. This generality is, however, only achieved by the imposition of a great many conditions on the mappings which appear so difficult to verify as to rule out application of the theorems. Kirk-Browder's theorem in contrast is easily utilized as is seen from the following four results due to Srinivascharyulu (1967). Similar results may be found in papers by Granas (1965) and Krasnoselskii (1964).

Let X be a Banach space and f: X X a non linear mapping; f is said to be <u>linearly upper bounded</u> if there exist numbers $\alpha, \gamma > 0$ such that $\|f(x)\| \leqslant \gamma \|x\|$ for $\|x\| \geqslant \alpha$. If

$$|f| = \inf_{0<\alpha<\infty} \left\{ \sup_{\|x\| \geqslant \alpha} \|f(x)\| / \|x\| \right\}$$

is finite, then f is linearly upper bounded, the number $|f|$ is called the <u>quasinorm</u> of f.

<u>Theorem 1</u>: Let f be a mapping of a uniformly convex Banach space X into X which is everywhere Gateaux differentiable. Let T be an invertible linear transformation on X and set $F = I - Tf$. Assume that $\| F'(x) \| \leq 1$ throughout X. If $|F| < 1$ then $f(x) = y$ has at least one solution x for each y in X.

<u>Proof</u>: Let w belong to X. $f(x) = (W)$ has solution if and only if $G(x) = F(x) + T(w)$ has a fixed point. By the Mean Value Theorem

$$\|G(x) - G(y)\| = \| F(x) - F(y)\| \leq \|F'(Z)\| \| x-y \|$$

and by hypothesis G is non expansive. It remains to find a closed, bounded convex set B invariant under G. Since $|F| < 1$, $\| F(x) \| < \epsilon \|x\|$ if $\|x\| > \delta_1$ and $\| G(x) \| \leq \|x\|$ if $\|x\| > \delta_2$.

If $\|x\| < \delta_2$ the Gateaux differentiability properties give

$$\| G(x) \| \leq \| T(y) \| + \| F(o) \| + \|x\| .$$

It is clear that if we take a sufficiently large disc B we will have $G(B) \subset B$ and we may apply the Browder-Kirk Theorem.

<u>Theorem 2</u>: Let T be a linear mapping with domain D a subset of X into X and with a bounded inverse. Let f be everywhere Gateaux differentiable and let $\lim_{x \to \infty} \| f(x) \| / \|x\| = 0$. If $\|T^{-1} f'(x)\| < 1$ for all x in X then $(T + f)\, x = y$ has solution for any y belonging to X.

<u>Proof</u>: Since f is asymptotically close to 0, $T^{-1}f'$ is also and we may proceed as in Theorem 1.

<u>Theorem 3</u>: Let $f: X \to X$ be an everywhere Gateaux differentiable mapping with $f(o) = 0$ and with f Fréchet differentiable at 0. Let $T: X \to X$ be a bounded linear mapping of X onto X having an inverse, and let $F = I - Tf$; assume further that $\|F'(o)\| < 1$ and that $\|F'(x)\| \leq 1$ for all x in X. Let $\epsilon > o$ be such that $\epsilon < 1 - \|F'(o)\|$; then there exists a $\delta > o$ such that for any y with $\|y\| \leq (1-d)\|T^{-1}\|$, where $d = \|F'(o)\| + \epsilon$, $f(x) = y$ has at least one solution in $B = \{x \mid \|x\| \leq \delta\}$.

<u>Proof</u>: As before F is non expansive and is Gateaux differentiable everywhere and Fréchet differentiable at 0. Hence

$$F(x) = F(o) + F'(o)x + g(o,x)$$

where $\|g(o,x)\| \leq \epsilon\|x\|$ when $\|x\| < \eta$ for some $\eta > o$. Thus

$\|F(x)\| < d\delta$ for $\|x\| < \delta$. Define $G: X \to X$ by $G(x) = F(x) + T(y)$ where y is a fixed element with $\|y\| \leq (1-d)\delta\|T^{-1}\|$

then

$$\|G(x)\| \leq \|F(x)\| + \|T(y)\| \leq \delta \quad \text{for } \|x\| < \delta,$$

G is non expansive and $G(B) \subset B$. By the theorem of Browder-Kirk we have a fixed point in B and a solution to $f(x) = y$ follows.

Finally we present the following result which uses the Schauder fixed point Theorem in lieu of Browder's result.

<u>Theorem 4</u>: Let f be a weakly continuous mapping of X into X. Under the same conditions as in Theorem 3, excepting that $F'(x)$ need not exist, we may still assert the conclusion.

<u>Remark</u>: If f is linear, then Gateaux differentiability implies weak continuity.

SECTION 7

VARIATIONAL INEQUALITIES

Having examined the solution of various non linear functional equalities, we turn now to variational inequalities and in particular, to monotone operator inequalities over convex sets. The theory thus developed has particular application to elliptic operator theory.

Let X be a reflexive Banach space with dual X*, A be a mapping from X to X*, f an element of X* and R a subset of X. We look for u in R such that

(1) $(A(u), v-u) \geqslant (f, v-u)$

for all v in R, and call (1) a <u>variational inequality</u>.

Lyons and Stampacchia (1965) have proved the follow existence theorem for A a bounded linear mapping in Hilbert space.

<u>Theorem 1</u>: If A is a bounded linear mapping in a real Hilbert space X satisfying

(2) $(A(v), v) \geqslant \alpha \|v\|^2$, $\alpha > 0$

and R is a closed convex subset of X, then there is a unique solution of (1). Moreover the mapping $f \to u$ is continuous X to X.

<u>Lemma1</u>: For $0 < p < 2\alpha / \|A\|^2$ there exists $\vartheta; 0 < \vartheta < 1$, with

$$\| u - pA(u) \| \leq \vartheta \|u\|.$$

<u>Proof</u>: $\| u - pA(u)\|^2 = \|u\|^2 + p^2 \|A\|^2 \|u\|^2 - 2\alpha p \|u\|^2 \leq \vartheta^2 \|u\|^2$

for $0 < p < 2\alpha / \|A\|^2$.

The following lemma relies on the standard projection theorem.

<u>Lemma 2:</u> The theorem holds for $A = I$.

<u>Proof:</u> It is necessary to find u with $(u-f, u-v) \geq 0$ for all v in R. Now

$$2(u-f, u-v) + \| f-u\|^2 = \|v-f\|^2 + \|u-v\|^2.$$

Hence

$$2(u-f, u-v) \geq \| v-f\|^2 - \|u - f\|^2.$$

If we set Uo to be the projection of f on R, we know from the projection theorem that $\| v-f \| \geq \|Uo-f\|$ throughout R and that Uo is uniquely so since R is closed and convex. Thus

$$(Uo-f, Uo-v) \geq 0 \quad \text{for all v in R}.$$

<u>Proof of the theorem:</u> With p as in Lemma 1, define $\phi(u)$ by $\phi(u) = u - p\,A(u) + pf$. Now

$$\| \phi(u) - \phi(Uo) \| = \| (u - Uo) - pA(u-Uo) \| \leq \vartheta \| u-Uo \|,$$

by Lemma 1. By Lemma 2, there is a unique W in R such that

$$(w, v-w) \geq (\phi(u), v-w)$$

and this is the projection of $\phi(u)$ on R. Call this w R(u).

Then

$$\| R(u) - R(Uo) \| \leq \|\phi(u) - \phi(Uo)\| \leq \vartheta\|u-Uo\| \quad (0 < \vartheta < 1)$$

By Poincaré's Theorem, R has a unique fixed point Uo and

$$(R(Uo), R(U_0) - v) = (Uo, U_0 - v) \geq (\phi(Uo), U_0 - v) =$$

$$(Uo, U_0-v) - p(A(Uo) - f, U_0-V),$$

since $p > 0$ we have the result. Continuity and uniqueness both follow easily.

Remarks: (1) If R is X then the inequality becomes equality and we again have the Lax-Milgram Lemma which is a special case of Theorem 1 of Section 5.

(2) It does not appear possible to weaken the condition (2) and to follow the same proof.

(3) In the proof of Lemma 1 it is essential that the space is real.

The Theorem can be extended to a more general result:

Theorem 2: Let A be as in Theorem 1. Let F mapping X into $[-\infty, \infty]$ be such that for all $\lambda > 0$ and all bounded linear functionals B on X there is a u in X with

$$(u,u) + \lambda F(u) + B(u) \leq (u,v) + \lambda F(v) + B(v) \quad , \forall v \in X ,$$

then there is a unique solution to

$$(A(u), v-u) \geq F(u) - F(v) \quad \text{for all } v \text{ in } X.$$

Proof: Condition (3) is specifically what is needed to prove an analogue to Lemma 1.

Remarks: (1) Lyons and Stampacchia show that if F is a convex lower semicontinuous mapping of X into $[-\infty, \infty]$ which is finite somewhere and never takes the value $-\infty$, then F will satisfy condition 3.

(2) Letting $F(v) = -(f,v)$ for v in C and equal $+\infty$ everywhere else, reduces Theorem 2 to Theorem 1.

Lyons and Stampacchia proceed to prove the existence by "elliptic regularization" of solutions when A satisfies only $(Au,u) \geq 0$ and where R is assumed bounded. They also produce a sufficiency condition for solutions to exist if R is not bounded. These results motivated the work of Browder (1965), (1965a), (1966b) with which we complete this section, and we will examine them within that context.

The next result generalizes Theorem 1 to a reflexive Banach space and to monotone functions. The proof method mirrors that of Theorem 1, Section 3 (which it includes in the case that $B = X$).

Theorem 3 (Browder) (1965): Let C be a closed convex subset of a reflexive Banach space X. Let T be monotone and hemicontinuous from X to X* and let $(T(u),u) \geq c(\|u\|)\|u\|$, $c(r) \to \infty$ as $r \to \infty$, then for all w_o in X*, we can find u_o in C such that for all v in C

(1) $(T(u_o) - w_o, v - u_o) \geq 0$

The next lemma, due to Minty, is used repeatedly in the theory of monotone inequalities.

Lemma 3: (1) is equivalent to $(T(v) - w_o, v - u_o) \geq 0$ for all u in C.

Proof: This is a straight forward application of the definition of monotonicity and of hemicontinuity.

Lemma 4: There is constant M depending only on w_o and $C(r)$ such that any solution, u_o, of (1) will have $\|u\| \leq M$.

Proof: This relies on the coercivity of T.

Lemma 5: If under the hypothesis of Theorem 3 θ is an internal point of C, then the set G defined by

$$G = \{ [u,w] \mid u \in C, w \in T(u)+z \text{ and } (z, u-v) \geq 0 \;\; \forall v \in C \}$$

is maximal monotone.

Proof: Monotonicity follows by calculation. Suppose $(u_0, w_0)M$ (u,w) for all (u,w) in G. If $u_0 \notin C$ we can find v_0 such that $(z_0, v_0 - v) \geq 0$ for all v in X where $u_0 = sv_0$, v_0 ..., $s > 1$ and $(z_0, v_0) > 0$. (We may apply the strict separation theorem since $0 \in C^0$)

Hence for all $\lambda > 0$ $[v_0, T(v_0) + \lambda z_0] \in G$ and.

$$0 \leq (w_0 - (T(v_0) + \lambda z_0), u_0 - v_0) = (s-1) \quad (w_0 - T(v_0) - \lambda z_0, v_0)$$

or for all $\lambda > 0$

$$\lambda^{-1} (w, v_0) - \lambda^{-1} (T(v_0), v_0) \geq (z_0, v_0) > 0$$

which is impossible. Thus $u_0 \in C$ and $(u_0, T(u_0)) \in G$. By hypothesis $(T(u) - w_0, u_0 - u)$ is non negative so, applying Lemma 1 and setting $z = w_0 - T(u_0)$, (u_0, w_0) belongs to G.

Lemma 6; If X has finite dimension Theorem 3 holds.

Proof: There is no loss of generality in assuming that X is a Hilbert space, that the dimension of C is the same as the dimension of X and that $0 \in C^0$. With G as in Lemma 3 we apply much the same method as in Theorem 1 of Section 3 to find a sequence $\{(u_n, w_n)\}$ of points with

$$u_n + nw_n = 0, \quad w_n = T(u_n) + z_n, \quad (z_n, u_n - v) \geq 0 \quad \text{for } v \text{ in } C$$

thus

$$(-\frac{1}{n} u_n, u_n - v) = (w_n, u_n - v) = (T(u_n), u_n - v) + (z_n, u_n - v)$$
$$\geq c(\|u_n\|)\|u_n\|.$$

Rearranging, estimating and applying Lemma 4, one deduces that $\{u_n\}$ is bounded. Since X has finite dimension we can assume that $\{u_n\}$ converges to $\{u_0\}$ and this means that $\{w_n\}$ tends to 0.

Now

$$(T(u)-w_n, u-u_n) = (T(u)-T(u_n), u-u_n) + (z_n, u_n-u) \geq 0$$

and taking limits, we have, for all u in C

$$(T(u), u-u_0) \geq 0.$$

Appeal to Lemma 1 produces the desired conclusion.

Proof of the theorem: The standard projection argument and Lemma 1 establish the general result.

Remark: Similar results have been established by Hartman and Stampacchia (1966).

It is most natural to consider monotone operator theory, "as an extension to non variational problems of the basic ideas of the direct method of the calculus of variations."[1] Perhaps the most basic application of the theory is to a class of non linear elliptic differential equations, which extends the class of equations. Moreover, we have the following Lemma due to Kacurouski (1960).

Lemma 7: If T is the Gateaux derivative of a real valued functional f on X then T is monotone if and only if f is convex.

This condition on f is essentially that upon which one bases the calculus of variations. For operators T as in the Lemma the existence of solution to $T(u) = 0$ is equivalent to the existence of critical points for the associated function f. So one sees that both with regard to application and theory, the calculus of variations and monotone operator theory are closely related.

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[1] Browder (1966a)

The following three Theorems of Browder (1966a) unify to a large extent the calculus of variations with the theory of monotone operator equations discussed in sections and with the theory of monotone operator inequalities over convex sets.

Theorem 4: Let T be a monotone hemicontinuous operator defined in a reflexive Banach space X with values in X*, f a lower semi-continuous function from X to $(-\infty,\infty]$ with $f(o)= o$. Suppose that for a given w in X*, there exists $R>o$ such that

(1) $(T(u)- w,u) + f(u) > 0$

for all u with $\|u\| = R$.

Then there exists u_o in $B_R = \{u \mid \|u\| \leq R\}$ such that

(2) $(T(u_o) - w, v-u_o) \geq f(u_o) - f(v) \qquad v \in X.$

Theorem 5: Under the hypothesis of Theorem 4, the set A(u) of solutions u_o of the system of equalities (2) is a closed convex subset of X. If T is strictly montone (i.e. $(T(u)- T(v), u-v) > 0$ for $u \neq v$; u, v in X) then A(w) is a simple point.

Theorem 6: If in addition to the hypothesis of Theorem 4, we add the condition that

(3) $(T(u),u) + f(u) = C(\|u\|)\|u\|$

where $c(r) \to \infty$ as $r \to \infty$, the A(w) is never empty.

Remarks: (1) if f(u) is identically 0, Theorem 4 becomes Theorem 1 of Section 3, in the case B = X.

(2) If f = o on C and $+\infty$ elsewhere, Theorem 4 becomes Theorem 3.

(3) if T = o, f has a minimum at u_o and Theorems 4,5,6 are results in the calculus of variations.

The next results generalizes Lemma 3.

<u>Lemma 8</u>: If $T:D(T) \subset X \to X^*$ is hemicontinuous, f a convex function from X to $(-\infty, \infty]$ which is somewhere finite, a sufficient condition that an element u_o of D(T) solve.

$$(T(u_o)-w, v - u_o) \geq f(u_o) - f(v), \quad v \in D(T),$$

is that u_o also solve

$$(T(v)-w, v-u_o) \geq f(u_o) - f(v), \quad v \in D(T).$$

If T is also monotone then the two systems of inequalities are equivalent.

<u>Proof</u>: Suppose the later inequality holds. Let x be any member of C and consider $v_t = (1-t)u_o + tx$, $o \leq t \leq 1$. If v_t is substituted for v in the inequality one finds

$$t(T(v_t)- w, \quad x - u_o) \geq t\left(f(u_o) - f(x)\right)$$

since f is convex. Cancelling t and letting $t \to o$ one has, since T is hemicontinuous, the previous inequality.

If T is monotone the converse follows easily.

It is now a simple matter to prove Theorem 5.

<u>Proof</u>: By Lemma 8 A(w) has the following explicit form.

$$A(w) = \bigcap_{v \in X} \{u \mid g_v(u) \leq 0\}$$

where $g_v(u)$ is the convex lower semicontinuous function defined by

$$g_v(u) = f(u) - f(v) - (T(v) - w, v-u)$$

Thus A(w) is closed and convex. If T is strictly monotone and u_1 and u_2 are elements in A(w) one has

$$(T(u_1)- w, u_2 - u_1) \geq f(u_1) - f(u_2)$$
$$(T(u_2)- w, u_1 - u_2) \geq f(u_2) - f(u_1)$$

Adding these equalities results in

$$(T(u_2) - T(u_1), u_2 - u_1) \leqslant 0$$

which is only possible if $u_1 = u_2$.

As a first step towards proving the finite dimensional case of Theorem 4, Browder proves the following very elegant Lemma which can be found in less elegant form in a paper by Hartmann and Stampacchia.

Lemma 9: Let X be a finite dimensional Banach space, T a continuous mapping from X to X*, f a lower semicontinuous function from X to $(-\infty,\infty)$ with $f(o) = 0$. Let $R > 0$ be given, $B_R(X) = \{u \mid u \in X, \|u\| \leq R\}$.

Then there exists u_o in $B_R(X)$ such that.

$$(T(u_o) - w, v-u_o) \geqslant f(u_o) - f(v), \quad v \in B_R(X).$$

Proof: There is no loss of generality in setting $w = o$. If the conclusion of the Lemma were false, then for each u in $B_R(X)$ there would exist a v in $B_R(X)$ such that

$$(T(u), v-u) < f(u) - f(v).$$

The set of u in $B_R(X)$ for which the strict inequality holds for a fixed v is open since f is lower semicontinuous. $B_R(X)$ is compact so one can find a finite set $\{v_1, - v_n\}$ in $B_R(X)$, such that the sets

$$U_j = \{u \mid u \in B_R(X), (T(u), v_j - u) < f(u) - f(u_j)\}$$

form a covering of $B_R(X)$. Let $\{B_1, \ldots, B_n\}$ be a partition of unity corresponding to the covering, such that $\sum B_j(u) = 1$, $0 \leq B_j(u) \leq 1$ for u in $B_R(X)$. Define

$$q(u) = \sum B_j(u)\, v_j \qquad u \in B_R(X) .$$

q is continuous mapping of $B_R(X)$ into itself since the B_j are continuous, and by Brouwer's fixed-point theorem there is a fixed point u_o with $q(u_o) = u_o$. Now

$$(T(u), q(u)-u) = \sum B_j(u)\,(T(u), v_j - u) < \sum B_j(u)\,(f(u) - f(v_j)) = f(u) - \sum B_j(u)\, f(v_j)$$

and by the convexity of f

$$f(q(u)) \leq \sum B_j(u)\, f(v_j) .$$

Hence for all u in $B_R(X)$

$$(T(u), q(u)-u) < f(u) - f(q(u))$$

which is patently impossible for u_o. Thus Lemma 9 must hold.

This argument neatly disguises, in the form of Brouwer's theorem, all the topological theory which is apparent in Hartman-Stampacchia's proof.

<u>Lemma 10</u>: Let T be a mapping from a convex domain D(T) in X to X*, f a convex function from X to (-oo,oo] with f(o) = o. Let $u_o \in B_R \cap D(T)$ be a solution to

$$(T(u_o) - w, v - u_o) \geq f(u_o) - f(v), \qquad v \in D(T) \cap B_R .$$

If, for u in D(T) with $\|u\| = R$.

$$(T(u) - w, u) + f(w) > o,$$

then $\|u_o\| < R$ and the inequality holds through out D(T).

Proof: By hypothesis

$$-(T(u_o)-w,\ v_o) \geqslant f(u_o) - f(0) = f(u_o)$$

which is only possible if $\|u_o\| < R$. Let v be any element of D(T). For some $t > 0$ $v_t = (1-t)\,u_o + tv$ lies in $B_R \cap D(T)$, the inequality of the hypothesis gives

$$t(T(u_o) - w,\ v-u_o) \geqslant f(u_o) - [(1-t)\,f(u_o) + t\,f(v)] =$$
$$t\,[f(u_o) - f(v)]$$

Cancelling $t > 0$, we obtain the desired inequality.

Proof of Theorem 4: Without loss of generality w may be assumed to equal 0. The proof again proceeds by using a projection agreement. In the finite dimensional case we appeal to Lemma 9 and then to Lemma 10. The sequence of potential solutions thus obtained is shown to be weakly convergent and the limit is verified to be a solution.

Proof of Theorem 6: For a given w in X*, condition (3) yields

$$(T(u) - w, u) + f(u) \geqslant (c(\|u\|) - \|w\|)\|u\| > 0$$

For $\|u\| > K$. By Theorem 4 $A(\omega) \neq \phi$ for each ω in X*.

It is apparent that Lemmas 8 and 10 are more general than is necessary for the proceeding theorems. They can be used to prove an extension of Theorem 4 to derisely defined monotone operators.

QUASIMONOTONE AND PSUEDO-MONOTONE OPERATORS:

Two wider classes of operators for which many of the theorems of monotone operator theory can be proved are the classes of Quasimonotone and Psuedomonotone Operators.

A mapping T from X into X* is said to be <u>quasimonotone</u> if there is a mapping S of X x X into X* such that T(u) = S(u,u) for u in X with

(a) For each fixed v S(· , v) is a hemicontinuous monotone mapping

(b) For each fixed u S(u,·) is a compact mapping

(c) If $\{u_j\}$ is a sequence converging weakly to u in X, and if (S (u_j,u_j) - S(u,u_j) u_j-u) $\to$ 0 then S(v,u) converges weakly to S(v,u) for each v in X.

Quasimonotone operators have the following important convergence property.

<u>Lemma 11:</u> Let X be reflexive Banach space, T a quasimonotone mapping of X into X*, f a lower semicontinuous convex function, from X to (-oo,oo). Let $\{w_j\}$ be a sequence in X* converging to w and let $u_j \in A(w_j)$. If the sequence $\{u_j\}$ converges to u then $u \in A(w)$.

<u>Proof:</u> The proof follows from the definitions and Lemma 8. For monotone operators (i.e. setting S(u,v) = T(u) it is an immediate consequence of that Lemma).

The Lemma enables one to prove the next theorem which generalizes theorems by Lyons and Stampacchia and introduces the process of elliptic regularization. The following definition of Browder's (1966b) is essential to the theorem.

<u>Definition:</u> X is said to be <u>improvable</u> if there exists a hemicontinuous monotone map M of X into X* with (M(u) u) > 0, u≠o, which maintains boundedness and such that

(1) $(M(u)\ u) \geq C(\|u\|)\ \|u\|$ where $C(r) \to \infty$ as $r \to \infty$, $u \in X$,

(2) For each u_1 in X and w_1 in X^*, $(M(u)-w_1, u-u_1) \to \infty$ as $\|u\| \to +\infty$,

(3) If $\{u_j\}$ is a bounded sequence in X, $u \in X$ and if $(M(u_j) - M(u), u_j - u) \to 0$ then u_j converges strongly to u in X.

Remarks: (1) In any Hilbert space we may set $M(u) = u$.

(2) Any uniformly convex space is improvable with respect to any mapping M obtained as a duality mapping of X into X^* for a strictly increasing continuous gauge function ϕ.

Theorem 7: Let X be an improvable reflexive Banach space with M the corresponding map of the improvability definition. Let T be a monotone operator from X to X^*, f a lower semicontinuous function from X to $(-\infty, \infty]$ with $f(0) = 0$. Let V_0 and ω belong to X^*. Suppose $A(w)$ is non empty. Then

(a) If, for $\epsilon > 0$, we set $T_\epsilon = T + \epsilon M$ the variational inequality

$$(T_\epsilon(u_\epsilon) - w,\ v - u_\epsilon) \geq f(u_\epsilon) - f(v),\ v \in X,$$

has exactly one solution u_ϵ, where $\omega_\epsilon = \omega + \epsilon V_0$.

(b) As $\epsilon \to 0$ u_ϵ converges strongly to $u_0 \in A(w)$. In $A(w)$, u_0 is characterizes as the unique solution of the variational inequality.

$$(M(u_0) - V_0,\ v - u_0) \geq 0,\ v \in A(w).$$

Proof: There is some constant k such that $f(v) \geq -k\|v\|$. Hence,

$$(T_\epsilon(u), u) + f(u) \geq \epsilon (M(u), u) + (T(u), u) + f(u) \geq \{\epsilon C(\|u\|) - \|T(0)\| - k\}\ \|u\|$$

which together with the strict monotonicity of T implies that there is exactly one solution u_ϵ to the inequality of (a) (Theorem 6).

Let u_1 be any element in A(w). One has

$$(T_\epsilon(u_1) - (w + \epsilon M(u_1), \; u_\epsilon - u_1) \geq f(u_1) - f(u_\epsilon)$$

$$(T_\epsilon(u_\epsilon) - w + \epsilon v_o, \; u_1 - u_\epsilon) \geq f(u_\epsilon) - f(u).$$

Adding and rearranging, one has

$$(1) \; \epsilon(M(u_\epsilon) - M(u_1), u_\epsilon - u_1) + (T(u_\epsilon) - T(u_1) \; u_\epsilon - u_1)$$
$$\leq \epsilon(M(u_1) - v_o, \; u_\epsilon - u_1).$$

Since T is monotone, and using property (2) of the improvability definition, the family $\{u_\epsilon\}$ is uniformly bounded. If u_o is the unique solution to the inequality in (b) (which exists because A(w) is convex and M satisfies Theorem 5) it is sufficient to show that any weakly convergent subsequence of the family $\{u_j\}$, $(\epsilon_j \to 0)$, is strongly convergent to u_o.

Let u_1 be the weak limit of u_j. For each u_j

$$(T(u_j) - [w + \epsilon_j v_o - \epsilon_j M(u_j) \; , \; v - u_j)] \geq f(u_j) - f(v), \quad v \in X.$$

Taking limits, and applying Lemma 11, yields $u_1 \in A(w)$. Also, inequality (1) with u_1 replaced by u_o, gives

$$(M(u_j) - M(u_o), u_j - u_o) \leq (M(u_o) - M(v_o), \; u_j - u_o) =$$
$$(M(u_o) - v_o, \; u_j - u_1) + (M(u_o) - v_o, \; u_1 - u_o).$$

The first summand on the right converges to 0 since u_j converges weakly to u_1, and the second summand is always negative by the characterizations of u_1 and u_o. Hence $(M(u_j) - M(u_o), \; u_j - u_o) \to 0$ and by property (3) of the improvability definition u_j converges strongly to u_0.

The following fixed point theorem is an immediate corollary of Theorem 7.

<u>Theorem 8</u>: Let U be a non expansive mapping in a Hilbert space H such that $U(C) \subset C$ for some closed bounded non empty convex subset of H. For each B with $0<B<1$ and a fixed v_0 in C let $U_B(u) = BU(u) + (1-B)v_0$. Then (a) For each $B<1$ U_B is a strict contraction mapping of C into itself and has a unique fixed point.

(b) As $B \to 1$ U_B converges strongly to a fixed point u_0 of U in C. u_0 is uniquely characterized as that fixed point of U in C nearest to v_0.

<u>Proof</u>: By Lemma 1 of Section 6, T = I - U is a monotone mapping. Theorem 7 applied to T produces Theorem 8.

<u>Remark</u>: The existence of a fixed point u for U in C is Theorem 1 of Section 6.

It is possible to prove necessary and sufficient conditions for A(w) to be non empty for T a quasimonotone mapping, and to extend Theorem 4 to such mappings. The generalization of Theorem 4 given in the linear case by Lyons and Stampacchia, is proved for a more restricted class of convex functions f.

We complete this section by defining another class of monotone type operators on which considerable work has been done. The definition is due to Brezis (1968).

Definition: A mapping T from a Banach space X to its dual space X* is said to be pseudo-monotone if for any sequence $\{u_j\}$ in X with $\{u_j\}$ converging weakly to u in X such that $\limsup (Tu_j; u-u_j) \leq 0$, it follows that for any v in X $\liminf(T(u_j), u_j-v) \geq (T(u), u-v)$.

As the following example shows, a monotone operator need not be psuedo-monotone.

Example: Let $T: R \to R$ be defined by $T(x) = 1$, $x > 0$; and $T(x) = 0$, $x \leq 0$. Let $u_n = 1/n$ for each positive integer, and let $v = 1$.

Then $\limsup (T(u_j), u-u_j) = 0$, but

$$(T(u_j), u_j - 1) = 1/n - 1$$

which converges to -1 while, since u_j converges to $u = 0$,

$$(T(u), u-1) = 0$$

Thus

$$-1 = \liminf (T(u_j), u_j-1) < (T(u), u-1) = 0$$

and T is monotone but not psuedo-monotone.

We do, however, have

Theorem 9: If T is weakly continuous, then if T is monotone T is psuedo-monotone.

Proof: Suppose T is monotone and that $\{u_j\}$ converges weakly to u. Then

$$\liminf \, (T(u_j) - T(u), u_j - v) \geqslant \liminf \, (T(u_j) - T(u) \; u_j - u)$$
$$+ \liminf \, (T(u_j) - T(u), u - v)$$

which is non negative since T is monotone and $\{T(u_j)\}$ converges weakly to T(u). Thus

$$\liminf \, (T(u_j), u_j - v) \geqslant \limsup (T(u) \;\; u_j - v), = (T(u) \; u - v)$$

since $\{uj\}$ converges weakly to u, and T is psuedo-monotone.

Remark: When T is monotone, limsup $(T(u_j), u - u_j) \leqslant 0$ for all weakly convergent sequences $\{u_j\}$ with limit u.

SECTION 8

AN EXAMPLE (Browder and Gupta)(1969)

A non linear integral equation of Hammerstein type is one of the form

$$(1)\quad u(x) + \int_G K(x, y)\ f(y, u(y))dy = 0$$

Where G is a σ-finite measure space and the unknown function $u(y)$ is defined on G. The problem of finding a solution u the equation (1) with u lying in a given Banach space Y of functions on G can be framed as

$$(2)\quad u + A\,N(u) = 0$$

with the linear mapping A given by

$$(3)\quad A\,v(x) = \int K(x,y)\ v(y)\ dy$$

and the non linear mapping N given by

$$(4)\quad N\,u(x) = f(x,u(x)).$$

Definition 1: If A is a bounded monotone linear mapping of X into X* then A is said to be angle-bounded with constant $c \geq 0$ if for all u,v in X.

$$|(A(u),v) - (A(v),u)| \leq 2c \left\{(A(u),u)\right\}^{\frac{1}{2}} \left\{(A(v),v)\right\}^{\frac{1}{2}}.$$

Definition 2: If A is a bounded linear mapping of X into X*, A is said to be Symmetric if for all u and v in X

$$(A(u),v) = (A(v),u).$$

The main result is

<u>Theorem 1:</u> Let X be a real Banach space with dual X^*. Let A be a continuous, monotone, angle bounded linear mapping, with constant of angle boundedness c, from X to X^*.

Let N be a demicontinuous mapping of X^* into X such that for a given constant $k \geq 0$

(5) $(v - v_1, \; N(v) - N(v_1)) \geq -k \| v - v_1 \|^2$

for all v and v_1, in X^*. Suppose finally there exists a constant R with $k(1 + c^2)R < 1$ such that for all u in X.

(6) $(A(u), u) \leq R \| u \|^2$.

Then there exists exactly one solution w, in X^*, of equation (2).

The next two lemmas allow the theorems of Section 3 to be employed.

<u>Lemma 1:</u> There exists a Hilbert space H, a bounded linear mapping S of X onto H with S^* injective and a bounded skew- symmetric linear mapping B of H into H such that $A = S^*(I+B)S$ and the following two inequalities hold:

(I) $\| B \| \leq C$, the constant of angle boundedness of A.

(II) $\| S \|^2 \leq R$ if and only if, for all u in X, $(A(u), u) \leq R \| u \|^2$.

<u>Lemma 2:</u> Let H be a given Hilbert space, B a skew-symmetric bounded linear mapping of H into H. Then the bounded linear mapping, I + B, is a bijective monotone mapping of H onto H. Further, for any u in H we have

$$((I + B)^{-1}(u), u) \geq \frac{1}{1 + \| B \|^2} \| u \|^2$$

The proof of Lemma 2 is straight forward while the proof of Lemma 1 relies on a fair amount of Hilbert space theory.

<u>Proof of Theorem 1:</u> Suppose w in X* solves equation (2). By Lemma 1 $A = S^* (1 + B) S$, and since S^* is injective, equation (2) has exactly one solution in X if and only if

$$u + (I + B) \; SNS^* \; (u)$$

has exactly one solution in H. By Lemma 2 this is equivalent to the existence of a solution to $T(u) = 0$ where $T = (I + B)^{-1} + SNS^*$.

For u,v in H

$$(T(u) - T(v), u-v) = ((I + B)^{-1} (u-v), u-v) + (SNS^* (u) - SNS^*(v), u-v).$$

Using the approximations in Lemmas 1 and 2 leads to

$$(T(u) - T(v), u-v) \geq (1/(1 + c^2) - kR) \| u-v \|^2 = c_1 \| u-v \|^2$$

and c_1 is positive ($k(1 + c^2) R < 1$ by hypothesis). Thus T is a monotone demicontinuous injective mapping of H into H. Moreover, for u in H, we have

$$(T(u),u) = (T(u) - T(o), u-o) + (T(o),u)$$
$$\geq (c_1\|u\| - \|T(o)\|) \|u\|$$

so that T is coercive and satisfies the hypothesis of Theorem 1 of Section 5. Hence we have a solution to $T(u) = o$, which is unique since T is injective and hence also a unique solution to $w + AN(w) = 0$.

SECTION 9

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