On Eulerian Log-Gamma Integrals and Tornheim–Witten Zeta Functions

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Abstract

Stimulated by earlier work by Moll and his coworkers [1], we evaluate various basic log Gamma integrals in terms of partial derivatives of Tornheim–Witten zeta functions and their extensions arising from evaluations of Fourier series. In particular, we fully evaluate

$$\mathcal{LG}_n = \int_0^1 \log^n \Gamma(x)\,\mathrm{d}x$$

for $1 \leq n \leq 4$ and make some comments regarding the general case. The subsidiary computational challenges are substantial, interesting and significant in their own right.

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1 Euler's integral and some preliminaries

Throughout this paper, we make extensive of use of Γ, the classical gamma function [2, 5, 17] defined for $\operatorname{Re} x > 0$ by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t.$$

It will also help to recall that the logsin integrals are defined, for $0 \le \tau \le 2\pi$, $n = 1, 2, 3, \ldots$, by

$$\operatorname{Ls}_n(\tau) := -\int_0^\tau \log^{n-1}\left(2\sin\frac{\theta}{2}\right) \mathrm{d}\theta. \tag{1}$$

It is useful to know that $\operatorname{Ls}_n(2\pi) = 2\operatorname{Ls}_n(\pi)$ and that we have the exponential generating function [9]

$$-\frac{1}{\pi}\sum_{m=0}^\infty \operatorname{Ls}_{m+1}(\pi)\frac{\lambda^m}{m!} = \frac{\Gamma(1+\lambda)}{\Gamma^2\left(1+\frac{\lambda}{2}\right)} = \binom{\lambda}{\frac{\lambda}{2}}. \tag{2}$$

This implies the recurrence

$$\frac{(-1)^n}{n!}\operatorname{Ls}_{n+2}(\pi) = \pi\,\alpha(n+1) + \sum_{k=1}^{n-2}\frac{(-1)^k}{(k+1)!}\alpha(n-k)\operatorname{Ls}_{k+2}(\pi), \tag{3}$$

where

$$\alpha(m) := \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^m} = \left(1-2^{1-m}\right)\zeta(m),$$

and resolves all such integrals.

Victor Moll in [5] and with coauthors in [1] considers the integrals

$$\mathcal{LG}_n := \int_0^1 \log^n \Gamma(x) \, \mathrm{d}x, \tag{4}$$

for $n = 1, 2, 3, \ldots$. For $n = 1$ there is a classical evaluation due to Euler, see [5, p. 186] or [16, (7) p. 203] (where it is ascribed to Raabe). We establish it as follows:

Theorem 1 (Euler)**.**

$$\int_0^1 \log\Gamma(x)\,\mathrm{d}x = \log\sqrt{2\pi}. \tag{5}$$

Proof. Since $\mathcal{LG}_1 = \int_0^1 \log\Gamma(1-x)\,\mathrm{d}x$, we have

$$\begin{aligned} 2\,\mathcal{LG}_1 &= \int_0^1 \log\left(\Gamma(x)\Gamma(1-x)\right)\,\mathrm{d}x, \\ &= \int_0^1 \log(2\pi)\,\mathrm{d}x - \int_0^1 \log\left(2\sin(\pi x)\right)\,\mathrm{d}x \\ &= \log(2\pi), \end{aligned}$$

as required, since the first integral is constant and the second *logsin integral* [9] is zero: $\int_0^1 \log\left(2\sin(\pi x)\right)\,\mathrm{d}x = \mathrm{Ls}_2\,(2\pi) = 0$. Above, as in [5] we have used the classic product formula

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

for x in $(0,1)$. $\square$

A beautiful 1840 extension, see [1], in which $0\log 0 := 0$, is

Theorem 2 (Raabe)**.** *For* $t \geq 0$

$$\int_0^1 \log\Gamma(x+t)\,\mathrm{d}x = \log\sqrt{2\pi} + t\log t - t. \tag{6}$$

Proof. We rewrite (6) as

$$\int_t^{1+t} \log\Gamma(x)\,\mathrm{d}x \stackrel{?}{=} \log\sqrt{2\pi} + t\log t - t. \tag{7}$$

We check that both sides of (7) have derivative of $\log t$. Hence, by the Fundamental theorem of calculus, the result follows from Theorem 1. $\square$

In the same fashion, for $t \geq 0$

$$\int_0^1 \log^2\Gamma(x+t)\,\mathrm{d}x = \mathcal{LG}_2 + \log^2 t - 2t\log t + 2t + 2\int_0^t \Gamma(s)\log s\,\mathrm{d}s. \tag{8}$$

Example 1 (Average value of Γ)**.** Combining Raabe's result (6) for $t = 0, 1, \ldots n-1$ produces

$$\frac{1}{n}\int_0^n \log\Gamma\left(x\right)\,\mathrm{d}x = \mathcal{LG}_1 - \frac{n-1}{2} + \frac{1}{n}\sum_{k=1}^{n-1} k\log k. \tag{9}$$

for $n = 1, 2, 3, \ldots$. More exotic variants can be derived by applying the Gauss multiplication formula for the Gamma function [2, 17]. $\Diamond$

In [1] the more general integral

$$\mathcal{LG}_{a,b} := \int_0^1 \log^a \Gamma(x) \log^b \Gamma(1-x)\,\mathrm{d}x, \tag{10}$$

for nonnegative integers a, b is studied. We note that $\mathcal{LG}_{a,b} = \mathcal{LG}_{b,a}$ and $\mathcal{LG}_a = \mathcal{LG}_{0,a}$. These objects prove very helpful for our study.

The rest of the paper is organized as follows. In Section 2 we recover (see Theorem 3) a slightly more efficient form the evaluation of $\mathcal{LG}_2$ as given in [5, 1]. (We give the argument in entirety as parts of it are used in the new results.) This is followed in Section 3 by a study of partial derivatives of Witten zeta-functions. These objects which appear implicitly in $\mathcal{LG}_2$ make it possible to fully evaluate $\mathcal{LG}_3$ in Theorem 6 of Section 4. The precise result is

$$\begin{aligned}\mathcal{LG}_3 = {} & \frac{3}{4}\left(\frac{\zeta\,(3)}{\pi^2} + \frac{1}{3}\,\mathcal{LG}_1\right)\mathcal{A}^2 - \frac{3}{2}\left(\frac{\zeta'(2,1)}{\pi^2} + 2\,\mathcal{LG}_1\,\frac{\zeta'\,(2)}{\pi^2}\right)\mathcal{A} + \frac{3}{2}\,\mathcal{LG}_1\,\frac{\zeta''\,(2)}{\pi^2} \\ & + \left(\mathcal{LG}_1^3 + \frac{1}{16}\,\mathcal{LG}_1\,\pi^2 + \frac{3}{16}\,\zeta\,(3)\right) - \frac{3}{8}\,\frac{\omega_{1,1,0}(1,1,1) - 2\,\omega_{1,0,1}(1,1,1)}{\pi^2},\end{aligned} \tag{11}$$

where $\omega_{1,1,0}(1,1,1)$ and $2\omega_{1,0,1}(1,1,1)$ are examples of our new Witten values. Likewise $\mathcal{LG}_4$ is obtained in Theorem 7 and Theorem 8 of Section 5. Finally, in Section 6 we make various comments regarding $\mathcal{LG}_5$ and the general case of $\mathcal{LG}_n$. The growing complexity of terms in the evaluation of $\mathcal{LG}_4$ makes it clear that further progress awaits a deeper understanding — theoretical and algorithmic of partial derivatives of Witten sums. We return to this matter in the conclusion.

2 The integral $\mathcal{LG}_2$

In [5, 1] the following evaluation is given:

$$\mathcal{LG}_2 = \frac{1}{12}\gamma^2 + \frac{1}{48}\pi^2 + \frac{1}{6}\gamma\,\log(2\,\pi) + \frac{1}{3}\log^2(2\,\pi) - (\gamma + \log(2\,\pi))\frac{\zeta'(2)}{\pi^2} + \frac{1}{2}\frac{\zeta''(2)}{\pi^2}. \tag{12}$$

Note that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty}\frac{\log k}{k^2} = -\zeta'(2), \qquad \sum_{k=1}^{\infty}\frac{\log^2 k}{k^2} = \zeta''(2),$$

are classical and known to *Maple*. Likewise $\zeta(0) = -\frac{1}{2}, \zeta'(0) = -\mathcal{LG}_1$ and $\Psi(1) = \Gamma'(1) = -\gamma, \Gamma''(1) = \gamma^2 + \zeta(2)$. Here $\gamma := \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k} - \log n$ is the *Euler-Mascheroni* constant, also known as the first Stieltjes constant γ_0.

In the same vein as above we may determine that

$$\begin{aligned}\mathcal{LG}_2 + \mathcal{LG}_{1,1} &= \frac{1}{2}\log^2(2\pi) - \frac{1}{4\pi^2}\,\mathrm{Ls}_3(2\pi)\\ &= \frac{1}{2}\log^2(2\pi) + \frac{\pi^2}{24}. \end{aligned}\tag{13}$$

It remains either to find another relation between $\int_0^1 \log\Gamma(x)\log\Gamma(1-x)\,\mathrm{d}x$ and $\mathcal{LG}_2$, or to otherwise try and improve the result given in [5].

Generally, we have a more convenient version of a result in [1]:

Proposition 1. *For $n = 1, 2, \ldots$ we have*

$$\sum_{a+b=n}\binom{n}{a}\mathcal{LG}_{a,b} = \frac{1}{2\pi}\sum_{a+b=n}\binom{n}{a}(-1)^{a+1}\,\mathrm{Ls}_{a+1}(2\pi)\log(2\pi)^b. \tag{14}$$

Proof. We write

$$\log(\Gamma(x)) + \log(\Gamma(1-x)) = \log(2\pi) - \log(2\sin\pi x) \tag{15}$$

and integrate term by term after using the binomial theorem on both sides. □

When $n = 1$ this recovers Theorem 1 since $\mathrm{Ls}_1(2\pi) = 2\pi, \mathrm{Ls}_2(2\pi) = 0$. We recall that the exact structure of $\mathrm{Ls}_{a+1}(2\pi) = 2\,\mathrm{Ls}_{a+1}(\pi)$ is well understood [9] and that each such integral has a closed form as in (3).

2.1 Fourier series related to $\log\Gamma$

We will have reason to use

$$-\log(2\sin(\pi x)) = \sum_{n=1}^{\infty}\frac{1}{n}\cos(2n\pi x). \tag{16}$$

for $0 < x < 1$ [16, (1) p. 202].

Indeed, as in [1] we shall rely on Kummer's 1847 Fourier series, see [2, p. 28], or [16, (15) p. 201], giving

$$\begin{aligned}\log\Gamma(x) - \frac{1}{2}\log(2\pi) = &-\frac{1}{2}\log(2\sin(\pi x)) + \frac{1}{2}(1-2x)(\gamma + \log(2\pi))\\ &+ \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi k x) \end{aligned}\tag{17}$$

for $0 < x < 1$. Note that direct integration of the right-side of (17) to zero now verifies Theorem 1. Making the change of variable $x \mapsto 1 - x$, and considering the symmetry of $\sin(\pi k x)$, we obtain

$$\log \Gamma\left(1-x\right) - \frac{1}{2}\,\log(2\pi) = -\,\frac{1}{2}\,\log\left(2\sin\left(\pi\,x\right)\right) - \frac{1}{2}\,\left(1-2\,x\right)\left(\gamma + \log(2\pi)\right) \\ -\,\frac{1}{\pi}\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi k x) \tag{18}$$

We may now apply Parseval's theorem to obtain an expression for $\mathcal{LG}_2$. This is best done by adding (easy and equivalent to (13)) and subtracting (more interestingly) (17) and (18).

We write:

$$\log\left(\Gamma(x)\Gamma(1-x)\right) = \log(2\pi) - \log\left(2\sin(\pi x)\right), \tag{19}$$

$$\log\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = \left(1-2x\right)\left(\gamma + \log(2\pi)\right) + \frac{2}{\pi}\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi k x). \tag{20}$$

It follows from (19) that

$$\mathcal{I}_1 := \int_0^1 \log^2\left(\Gamma\left(x\right)\Gamma\left(1-x\right)\right)\,\mathrm{d}x = \log^2(2\pi) - \frac{1}{2\pi}\,\mathrm{Ls}_3\left(2\pi\right) \\ = \log^2(2\pi) + \frac{\pi^3}{12}, \tag{21}$$

and from (20) that

$$\mathcal{I}_2 := \int_0^1 \log^2\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)\,\mathrm{d}x = \frac{1}{3}\left(\gamma + \log(2\pi)\right)^2 \\ -\,\frac{8}{\pi}\left(\gamma + \log(2\pi)\right)\sum_{k=2}^{\infty}\frac{\log k}{k}\int_0^1 x\sin(2\pi k x)\,\mathrm{d}x \\ +\,\frac{4}{\pi^2}\int_0^1\left(\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi k x)\right)^2\,\mathrm{d}x, \tag{22}$$

so that

$$\mathcal{I}_2 = \frac{1}{3}\left(\gamma + \log\left(2\,\pi\right)\right)^2 + \frac{4}{\pi^2}\left(\gamma + \log(2\pi)\right)\sum_{k=2}^{\infty}\frac{\log k}{k^2} + \frac{2}{\pi^2}\sum_{k=2}^{\infty}\frac{\log^2 k}{k^2} \\ = \frac{1}{3}\left(\gamma + \log\left(2\,\pi\right)\right)^2 - \frac{4}{\pi^2}\left(\gamma + \log(2\pi)\right)\zeta'(2) + \frac{2}{\pi^2}\zeta''(2). \tag{23}$$

Now $\mathcal{LG}_{1,1} = \frac{1}{4}(\mathcal{I}_1 - \mathcal{I}_2)$, and $\mathcal{LG}_2 = \frac{1}{2}\,\log^2(2\pi) + \frac{\pi^2}{24} - \frac{1}{4}\mathcal{I}_1 + \frac{1}{4}\mathcal{I}_2$ from (13). Hence from (21) and (23) we have:

Theorem 3 (Evaluation of $\mathcal{LG}_2$)**.**

$$\mathcal{LG}_2 = \frac{1}{4}\log^2(2\pi) + \frac{1}{48}\pi^2 + \frac{1}{12}\left(\gamma + \log(2\pi)\right)^2 - \frac{1}{\pi^2}\left(\gamma + \log(2\pi)\right)\zeta'(2) + \frac{1}{2\pi^2}\zeta''(2), \tag{24}$$

$$\mathcal{LG}_{1,1} = \frac{1}{4}\log^2(2\pi) + \frac{1}{48}\pi^2 - \frac{1}{12}\left(\gamma + \log(2\pi)\right)^2 + \frac{1}{\pi^2}\left(\gamma + \log(2\pi)\right)\zeta'(2) - \frac{1}{2\pi^2}\zeta''(2), \tag{25}$$

Moll's evaluation in (12) agrees with (24).

Before continuing, it is helpful to identify two well-known classes of multi-zeta values.

3 Tornheim–Witten zeta functions and Euler sums

The *Witten*, *Tornheim–Witten* or *Mordell–Tornheim–Witten zeta* [4, 13, 14] function is defined by

$$\omega(r,s,t) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r\, m^s\, (n+m)^t} \qquad (r,s,t \geq 0). \tag{26}$$

The double sum clearly converges for $r > 1$ and $s > 1$ and is well defined for $r + t > 1, s + t > 1, r + s + t > 2$. Correspondingly

$$\zeta(t,s) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^s\,(n+m)^t} = \sum_{n>m>0} \frac{1}{n^t\, m^s}$$

is an *Euler double sum*. The central evaluation $\zeta(2,1) = \zeta(3)$ is described in detail in [6].

There is a simple algebraic relation

$$\omega(r,s,t) = \omega(r-1,s,t+1) + \omega(r,s-1,t+1). \tag{27}$$

This is based on writing

$$\frac{m+n}{(m+n)^{t+1}} = \frac{m}{(m+n)^{t+1}} + \frac{n}{(m+n)^{t+1}}.$$

Clearly

$$\omega(r,s,t) = \omega(s,r,t), \tag{28}$$

and it is straight-forward to check that

$$\omega(r,s,0) = \zeta(r)\,\zeta(s), \qquad \omega(r,0,t) = \zeta(t,r). \tag{29}$$

Hence, $\omega(s,s,t) = 2\,\omega(s,s-1,t+1)$, so

$$\omega(1,1,1) = 2\,\omega(1,0,2) = 2\,\zeta(2,1) = 2\,\zeta(3).$$

We observe that the analogue to (27), $\zeta(s,t)+\zeta(t,s) = \zeta(s)\,\zeta(t) - \zeta(s+t)$, shows that

$$\omega(s,0,s) = 2\,\zeta(s,s) = \zeta^2(s) - \zeta(2s).$$

In particular, $\omega(2,0,2) = 2\,\zeta(2,2) = \pi^4/36 - \pi^4/90 = \pi^4/72$.

Indeed, if δ denotes any derivative wrt order of a Witten sum then linearity of the differentiation operator means that the partial fraction argument leads to:

$$\delta(r,s,t) = \delta(r-1,s,t+1) + \delta(r,s-1,t+1), \tag{30}$$

and if δ is symmetric in the first two variables then $\delta(s,s,t) = 2\,\delta(s,s-1,t+1)$.

Remark 1 (Alternating Tornheim sums)**.** If we define

$$\nu(r,s,t) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{n^r\,m^s\,(n+m)^t} \qquad (r,s,t>0), \tag{31}$$

then, just as above,

$$-\,\nu(r,s,t) = \nu(r-1,s,t+1) + \nu(r,s-1,t+1). \tag{32}$$

Thence, in the language of MZV's [7, 8] we have

$$-\nu(s,s,t) = \zeta(\overline{t+1},s) := \sum_{n>m>0} \frac{(-1)^n}{n^{t+1}} \sum \frac{1}{m^s}.$$

In particular, $-\nu(1,1,1) = \zeta(3)/4$, $-\nu(2,1,1) = 5\zeta(4)/16$ while

$$-\nu(1,1,2) = 2\nu(1,0,3) = 2\,\zeta(\overline{3},1) = 4\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{6}\,\log^4 2 - \frac{1}{6}\,\pi^2\log^2 2 + \frac{7}{2}\,\zeta\,(3)\log 2 - \frac{1}{24}\pi^4,$$

see [10]. Likewise

$$\nu(2,0,2) = -4\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1}{6}\,\log^4 2 + \frac{1}{6}\,\pi^2\log^2 2 - \frac{7}{2}\,\zeta\,(3)\log 2 + \frac{13}{288}\pi^4.$$

and all other $\nu(a,b,c)$ with $a+b+c=4$ are similarly expressible. For example, $\nu(3,0,1) = 3\,\zeta(3)\log(2)/4 - 5\,\zeta(4)/16$. $\Diamond$

The general form of the reduction for integers r, s, and t is due to Tornheim, and expresses $\omega(r,s,t)$ in terms of $\zeta(a,b)$ with *weight* $a+b = N := r+s+t$:

Theorem 4 (Reduction to Euler sums)**.** *For positive integers r, s, and t*

$$\omega(r,s,t) = \sum_{i=1}^{\max\{r,s\}} \left\{ \binom{r+s-i-1}{s-1} + \binom{r+s-i-1}{r-1} \right\} \zeta\,(i, N-i)\,. \tag{33}$$

The same argument allows us more generally to reduce partial derivatives:

Theorem 5 (Reduction of Derivatives)**.** *Let nonnegative integers a, b, c and r, s, t be given. Then for $\delta := \omega_{a,b,c}$ we have*

$$\delta(r,s,t) = \sum_{i=1}^{r} \binom{r+s-i-1}{s-1} \delta\,(i,0,N-i) + \sum_{i=1}^{s} \binom{r+s-i-1}{r-1} \delta\,(0,i,N-i)\,. \tag{34}$$

Proof. For non-negative integers r, s, t, v, with $r+s+t=v$, and v fixed, we induct on s. Both sides satisfy the same recursion (30):

$$d(r,s,t) = d(r-1,s,t+1) + d(r,s-1,t+1)$$

and the same initial conditions $(r+s=1)$. $\square$

Of course (34) holds for any δ satisfying recursion (30) (without being restricted to partial derivatives).

Example 2 (Values of δ)**.** Richard Crandall using the techniques in [11] provides:

$\omega_{1,1,0}(1,0,3) = 0.07233828360935031113948057244763953352659776102642...$

$\omega_{1,1,0}(2,0,2) = 0.29482179736664239559157187114891977101838854886937848122804...$

$\omega_{1,1,0}(1,1,2) = 0.14467656721870062227896114489527906705319552205284127904072...$

while

$\omega_{1,0,1}(1,0,3) = 0.14042163138773371925054281123123563768136197000104827665935...$

$\omega_{1,0,1}(2,0,2) = 0.40696928390140268694035563517591371639834128770661373815447...$

$\omega_{1,0,1}(1,1,2) = 0.43097253394888316942248176511038963971077201581912157523 09...$

and

$\omega_{0,1,1}(2,1,1) = 3.00297121355668005079211509351534225995879828374320045987 9...$

We note that $\omega_{1,1,0}(1,1,2) = 2\,\omega_{1,1,0}(1,0,3)$ and $\omega_{1,0,1}(1,0,3)+\omega_{1,0,1}(0,1,3)-\omega_{1,0,1}(1,1,2)$

$= 0.14042163138773371924\overline{7} + 0.29055090256114945012 - 0.43097253394888316942$
$= 0.00000000000000000000...,$

both in accord with (30). We note also that PSLQ predicts

$$\zeta''(4) \stackrel{?}{=} 2\omega_{1,0,1}(1,0,3) - 2\omega_{1,1,0}(2,0,2) + 4\omega_{1,0,1}(2,0,2),$$

which also validates Crandall's computational accuracy. $\diamondsuit$

For computational and related reasons, the following formula is quite useful (a more subtle algorithm is given in [4, 11]):

$$\omega(r,s,t) = \frac{1}{\Gamma(t)}\int_0^1 \mathrm{Li}_r(\sigma)\,\mathrm{Li}_s(\sigma)\,\frac{(-\log\sigma)^{t-1}}{\sigma}\mathrm{d}\sigma. \tag{35}$$

Here the *polylogarithm* is defined by $\mathrm{Li}_s(x) := \sum_{n>0} x^n/n^s$ for $|x|<1$ and real $s>1$, continued analytically. Equation (35) has the feature that it can be differentiated symbolically wrt t. Thus, the first two partials wrt to t can be written as

$$\omega_{0,0,1}(r,s,t) = \frac{1}{\Gamma(t)}\int_0^1 \frac{(-\log x)^{t-1}\,\mathrm{Li}_r\,(x)\,\mathrm{Li}_s\,(x)\,(\log\,(-\log x) - \Psi\,(t))}{x}\mathrm{d}x \tag{36}$$

and

$$\omega_{0,0,2}(r,s,t) =$$
$$\frac{1}{\Gamma(t)}\int_0^1 \frac{(-\log x)^{t-1}\,\mathrm{Li_r}\,(x)\,\mathrm{Li}_s\,(x)\left(\log^2(-\log x) - 2\Psi\,(t)\log\,(-\log x) - \Psi'\,(t) + \Psi^2\,(t)\right)}{x}\mathrm{d}x$$

We shall also meet higher-dimensional sums such as the *generalized Tornheim sum*

$$\omega(a_1,a_2,\ldots,a_n,d) := \sum_{m_i>0} \frac{1}{m_1^{a_1}\,m_2^{a_2}\cdots m_n^{a_n}(m_1+m_2+\ldots+m_n)^d} \tag{37}$$

$$= \frac{1}{\Gamma(d)}\int_0^1 \left(\prod_{i=1}^n \mathrm{Li}_{a_i}(\sigma)\right)\frac{(-\log\sigma)^{d-1}}{\sigma}\,\mathrm{d}\sigma. \tag{38}$$

Finally, we introduce the following two-parameter or *double Witten zeta-function*: with M, N positive integers with $M \le N$:

$$\omega(s_1, \dots, s_N \mid t_1, \dots, t_M) := \sum_{n_i>0, m_j>0} \sum_{\sum_{i=1}^N n_i = \sum_{j=1}^M m_j} \prod_{i=1}^{N} \frac{1}{{n_i}^{s_i}} \prod_{j=1}^{M} \frac{1}{{m_j}^{t_j}} \tag{39}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \prod_{i=1}^{N} \mathrm{Li}_{s_i}\left(e^{i\theta}\right) \prod_{j=1}^{M} \mathrm{Li}_{t_j}\left(e^{-i\theta}\right) \mathrm{d}\theta. \tag{40}$$

If $M = 1$ this devolves to the previous form. In this notation (87) below can be written as $\omega_{1,1,1,1}(1,1 \mid 1,1)$. The two terms correspond to the possible number of positive and negative terms in any cosine expression originating in the Fourier analysis. Thus, for $\mathcal{LG}_6$ we have $6 = 5+1 = 4+2 = 3+3$.

3.1 Further computation of $\omega(r,s,t)$ and its partials

While there is a substantial literature regarding special values of Tornheim sums [1, 13, 14, 15], we have not found it to be of much help for current computational purposes — with the striking exception of the new paper [11].

Our goal is to access effective high-precision algorithms for sums such as:

$$\omega_{a,b,c}(r,s,t) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\log^a m \, \log^b n \, \log^c (m+n)}{n^r \, m^s \, (n+m)^t} \tag{41}$$

with non-negative integers a, b, c, r, s, t and $r+s+t > 2$ etc. so as to assure convergence. Recall that the Witten–Tornheim sums are:

$$\omega(r,s,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^r \, m^s \, (n+m)^t} \qquad (r,s,t>0, r+s+t>2)$$

$$= \frac{1}{\Gamma(t)} \int_0^1 \mathrm{Li}_r(\sigma) \, \mathrm{Li}_s(\sigma) \, \frac{(-\log \sigma)^{t-1}}{\sigma} \, d\sigma.. \tag{42}$$

For example, $\omega_{1,1,0}(1,1,1)$ and $\omega_{1,0,1}(1,1,1) = \omega_{0,1,1}(1,1,1)$ will occur in our evaluation of $\mathcal{LG}_3$ and $\mathcal{LG}_{2,1}$. Each sum in (41) is clearly a partial derivative of above-defined the Tornheim sum. Special cases then provide algorithms for partial derivatives of $\zeta(s), \zeta(r,s)$. Indeed, we shall see in the last section that for present purposes we need only evaluate first partials of ω with $r = s = t = 1$ and similar functions.

We also record a *Bose-Einstein* formula for the derivative of $\mathrm{Li}_s(x)$ with respect to s:

$$\frac{\partial \mathrm{Li}_s\left(x\right)}{\partial s} = \frac{x}{2\,\Gamma\left(s\right)}\int_0^\infty \mathrm{e}^{-t}t^{s-1}\left(\log\left(t\right) - \Psi\left(s\right)\right)\coth\left(\frac{t-\log\left(x\right)}{2}\right)\,\mathrm{d}t. \tag{43}$$

More sophisticated computational formulae for the *polylogarithm* of order s — $\mathrm{Li}_s(z) = \sum_{m\geq 1} z^m/m^s$ — include the following pair due to Erdélyi et al [12] and recorded in [18, 11]. If $s = n$ is a positive integer,

$$\mathrm{Li}_n(z) = {\sum_{m=0}^{\infty}}' \zeta(n-1-m)\frac{\log^m z}{m!} + \frac{\log^{n-1} z}{(n-1)!}\left(H_{n-1} - \log(-\log z)\right), \tag{44}$$

valid for $|\log z| < 2\pi$, where $H_n := 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. The primed sum $\sum'$ means we avoid the singularity at $\zeta(1)$. Otherwise — for any complex s not a positive integer —

$$\mathrm{Li}_s(z) = \sum_{m\geq 0}\zeta(s-m)\frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}.$$

One of Crandall's contributions in [11] is to adapt these formulae to allow effective computation of derivatives with respect to the order when s is integral.

Example 3 (Efficient computation of derivative sums, I)**.** Crandall [11, Eqn. (43)] provides formulae such as

$$L_1'(z) = \sum_{n=1}^{\infty}\zeta'\left(1-n\right)\frac{\log^n z}{n!} - \gamma_1 - \frac{1}{12}\pi^2 - \frac{1}{2}\left(\gamma + \log\left(-\log z\right)\right)^2, \tag{45}$$

for $|\log z| < 2\pi$. Here γ_1 is the second Stieltjes constant, and is known to *Maple* and *Mathematica*. So to resolve integrals such as (86) below we may use the Taylor series for $L_1'(z)^n/z$ on $(0, 1/e)$ and (45) on $(1/e, 1)$. For details, see [11, Eqns. (39–46)].

A key component is that

$$U_{m,n} := \int_{1/e}^{1}\frac{\left(\log z\ \right)^m\left(\log(-\log z)\right)^n}{z}\mathrm{d}z = \frac{\left(-1\right)^{m+n} n!}{\left(m+1\right)^{n+1}}$$

is pre-computed for nonnegative integers m and n and the integral is reduced to an accelerated double sum. Combining these ideas, after differentiating the integral (42) with respect to r and s, we obtain:

$$\begin{aligned}
\omega_{1,1,0}(1,1,1) &= \sum_{m,n\geq 1} \frac{\log m \, \log n}{mn(m+n)} e^{-m-n} + \int_{1/e}^{1} \frac{\mathrm{Li}_1'(t)^2}{t} \, \mathrm{d}t && (46) \\
&= \sum_{m,n\geq 1} \frac{\log m \, \log n}{mn(m+n)} e^{-m-n} + \beta && (47) \\
&+ \sum_{m,n\geq 1} \frac{\zeta^{(1)}(1-m)}{m!} \frac{\zeta^{(1)}(1-n)}{n!} U_{m+n,0} \\
&- \sum_{m\geq 1} \frac{\zeta^{(1)}(1-m)}{m!} \left(2\alpha \, U_{m,0} + 2\gamma \, U_{m,1} + U_{m,2}\right),
\end{aligned}$$

where

$$\alpha \;:=\; \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12}, \qquad \beta := 6 + 2\alpha + \alpha^2 - 6\gamma - 2\alpha\gamma + 2\gamma^2.$$

Taking 375 terms of every summation index, Crandall obtains a 140-digit value

$$\begin{aligned}
\omega_{1,1,0}(1,1,1) \approx\ & 4.30244762034222643331985179818698942752019447430436225 5\\
& 697498337340689645860570842383422066958966013290632081 4\\
& 848672664367230398478820315706236223995256611060718260 4\ .
\end{aligned}$$

To compute $\omega_{1,0,1}(1,1,1)$ we apply the same ideas to a special case of (36):

$$\omega_{1,0,1}(1,1,1) = -\int_0^1 \frac{\mathrm{Li}_1'(x) \, \mathrm{Li}_1(x)}{x} \left(\log\left(-\log x\right) + \gamma\right) \mathrm{d}x. \tag{48}$$

and we use (44) as well. $\Diamond$

In conjunction with (44) and the techniques of (45) and Example 3, it would appear that (40) provides a good entree to the numerical computation of two-parameter Witten zeta functions — as we shall see in Section 5.1.

Remark 2. Note that

$$\zeta(3) = \zeta(2,1), \quad \zeta''(3) = \sum_{n\geq 1} \frac{\log^2 n}{n^3}, \quad \sum_{n>m>0} \frac{\log n}{n^2 m} = -\zeta'(2,1)$$

where the derivative is with respect to the outer parameter. Moreover, it follows from

$$\zeta(s,1) = -\frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} \frac{\log(1-x)}{1-x} \, \mathrm{d}x$$

that

$$\zeta^{'}(2,1) = (2-\gamma)\,\zeta\,(3) + \frac{1}{2}\int_0^\infty \log^2\left(1-\mathrm{e}^{-t}\right)\log\left(t\right)\,\mathrm{d}t.$$

or

$$\begin{aligned}\zeta^{'}(2,1) &= 2\,\zeta^{'}(3) + (2-\gamma)\zeta(3) - \int_0^\infty \frac{\log\left(\mathrm{e}^t-1\right)}{\mathrm{e}^t-1}\,t\log\left(t\right)\mathrm{d}t\\ &\approx -2.42573972340456234746086319943.\end{aligned}$$

Likewise

$$\zeta^{'}(s,1) = \left(\gamma - \sum_{k=1}^{s-1}\frac{1}{k}\right)\zeta(s,1) - \frac{1}{\Gamma(s)}\int_0^1 \log(-\log x)\frac{(-\log x)^{s-1}\log(1-x)}{1-x}\,\mathrm{d}x.$$

These allow adequate manipulation of many quantities like $\zeta^{'}(2,1)$. $\Diamond$

4 The integral $\mathcal{LG}_3$

In light of Proposition 1, to obtain $\mathcal{LG}_3$ it is sufficient to generate one more relation engaging $\mathcal{LG}_3$ and $\mathcal{LG}_{2,1}$ since we have evaluated

$$\mathcal{LG}_3 + 3\mathcal{LG}_{2,1} = \frac{16\,\mathcal{LG}_1^3 + \mathcal{LG}_1\pi^2 + 3\,\zeta(3)}{4}. \tag{49}$$

A good candidate is:

$$\begin{aligned}\mathcal{LG}_3 - \mathcal{LG}_{2,1} &= \int_0^1 \log^2\Gamma(x)\log\frac{\Gamma(x)}{\Gamma(1-x)}\,\mathrm{d}x\\ &= \mathcal{LG}_1\int_0^1 \log^2\frac{\Gamma(x)}{\Gamma(1-x)}\,\mathrm{d}x - \frac{1}{2}\int_0^1 \log\left(2\sin(\pi x)\right)\log^2\frac{\Gamma(x)}{\Gamma(1-x)}\,\mathrm{d}x\\ &= \mathcal{LG}_1\,\mathcal{I}_2 - \mathcal{I}_3.\end{aligned} \tag{50}$$

Here, on setting $\mathcal{A} := \gamma + \log(2\pi)$ and using (23),

$$\mathcal{I}_2 = \frac{1}{3}\mathcal{A}^2 - \frac{4}{\pi^2}\mathcal{A}\,\zeta^{'}(2) + \frac{2}{\pi^2}\zeta^{''}(2). \tag{51}$$

Likewise from (20)

$$\begin{aligned}\mathcal{I}_3 &:= \frac{1}{2}\int_0^1 \log\left(2\sin(\pi x)\right)\left((1-2x)\mathcal{A} + \frac{2}{\pi}\sum_{k=2}^\infty \frac{\log k}{k}\sin(2\pi k x)\right)^2\mathrm{d}x\\ &= \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6,\end{aligned} \tag{52}$$

where $I_3 \approx -2.5625370020565243683203216778 4$ and

$$\mathcal{I}_4 := \frac{\mathcal{A}^2}{4\,\pi^3} \int_0^{2\pi} (\pi-\theta)^2 \log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}\theta$$
$$= -\mathcal{A}^2\,\frac{\zeta(3)}{\pi^2}, \tag{53}$$
$$\mathcal{I}_5 := \frac{\mathcal{A}}{\pi^3} \int_0^{2\pi} (\pi-\theta) \log\left(2\sin\frac{\theta}{2}\right) \left(\sum_{k=2}^{\infty} \frac{\log k}{k} \sin(k\theta)\right) \mathrm{d}\theta$$
$$= -2\,\frac{\mathcal{A}}{\pi^2} \sum_{n>0} \frac{H_{n-1}\log n}{n^2}. \tag{54}$$

Thus, it remains to resolve $\mathcal{I}_6$. Now

$$\mathcal{I}_6 := \frac{1}{\pi^3} \int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right) \left(\sum_{k=2}^{\infty} \frac{\log k}{k} \sin(k\theta)\right)^2 \mathrm{d}\theta$$
$$= \frac{1}{\pi^3} \sum_{j,k=1}^{\infty} \frac{\log j}{j}\frac{\log k}{k} \int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right) \sin(j\theta)\sin(k\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{4\pi^2} \sum_{n\ge 1} \frac{\log^2 n}{n^3} - \frac{1}{\pi^2} \sum_{m>n\ge 1} \frac{\log n \log m}{mn} \left(\frac{1}{m+n} - \frac{1}{m-n}\right), \tag{55}$$

on separating the terms with $k = j$. Here we rely on the formulas

$$\int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right) \cos(k\theta)\,\mathrm{d}\theta = -\frac{\pi}{k},$$

$$\int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right) \sin^2(k\theta)\,\mathrm{d}\theta = \frac{\pi}{4k},$$

for $k = 1, 2, \ldots$, and the double angle formula.

Hence

$$\mathcal{I}_6 = \frac{1}{2\pi^2}\zeta''(3) - \frac{1}{2\pi^2} \sum_{n,m\ge 1} \frac{\log n \log m}{mn} \frac{1}{m+n} + \frac{1}{\pi^2} \sum_{m>n\ge 1} \frac{\log n \log m}{mn} \frac{1}{m-n}. \tag{56}$$

It is nearly immediate from the definition of ω and (56), that

$$\mathcal{I}_6 := \frac{1}{2\pi^2}\left(\omega_a - 2\,\omega_b\right) \approx -0.66499615732508809836685856421 3, \tag{57}$$

where

$$\omega 1 := \omega_{1,1,0}(1,1,1),$$

and the subscript also denotes a partial derivative with respect to that parameter, while

$$\omega 2 := \sum_{m>n\geq 1} \frac{\log n \log m}{mn} \frac{1}{m-n} = \omega_{1,0,1}(1,1,1).$$

To see this write

$$\omega 2 := \sum_{m>n\geq 1} \frac{\log n \log m}{mn} \frac{1}{m-n} = \sum_{k,n\geq 1} \frac{\log n \log(n+k)}{(n+k)n} \frac{1}{k}.$$

Thus, all the new quantities are partial derivatives of Tornheim–Witten zeta functions with weight $r+s+t=3$.

For example

$$\zeta^{'}(2,1) = \omega_{0,0,1}(1,0,2),$$

and

$$\zeta^{''}(2,1) = \omega_{0,0,2}(1,0,2).$$

In light of (35) we have

$$\begin{aligned}\omega_a &= \int_0^1 \frac{\mathrm{Li}_1^{(1)}(\sigma)^2}{\sigma}\, d\sigma \\ &\approx 4.302447620342226433319851798186989427520194474304362255697498337340, \qquad (58)\\ \omega_b &= -\int_0^1 \frac{\log(1-\sigma)\,\mathrm{Li}_1^{(1)}(\sigma)}{\sigma} \left(\log(-\log(\sigma)) + \gamma\right) \mathrm{d}\sigma \\ &\approx 8.714472811214314238163007854462420651620135416423321505237309933615. \qquad (59)\end{aligned}$$

We note that Example 3 has indicated how to obtain much higher accuracy for such ω values. Here again $\mathrm{Li}_1^{(1)}(\sigma) = -\sum_{n\geq 1} \frac{\log n}{n} \sigma^n$.

So combining results, with

$$\mathcal{A} := \gamma + \log(2\pi)$$

we have

$$\begin{aligned}
\mathcal{LG}_3 + 3\mathcal{LG}_{2,1} &= \frac{16\,\mathcal{LG}_1^3 + \mathcal{LG}_1\pi^2 + 3\,\zeta(3)}{4} \quad (49) \\
\mathcal{LG}_3 - \mathcal{LG}_{2,1} &= \mathcal{LG}_1\,\mathcal{I}_2 - (\mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6) \quad (50) \\
\mathcal{I}_2 &= \frac{1}{3}\mathcal{A}^2 - \frac{4}{\pi^2}\mathcal{A}\,\zeta'(2) + \frac{2}{\pi^2}\zeta''(2) \quad (51) \\
\mathcal{I}_4 &= -\mathcal{A}^2\,\frac{\zeta(3)}{\pi^2} \quad (53), \quad \mathcal{I}_5 = 2\,\mathcal{A}\,\frac{\zeta'(2,1)}{2\pi^2} \quad (54) \\
\mathcal{I}_6 &= \frac{1}{2\pi^2}\left(\omega_{1,1,0}(1,1,1) - 2\omega_{1,0,1}(1,1,1)\right) \quad (57).
\end{aligned} \tag{60}$$

Putting all the terms in (60) together produces:

Theorem 6 (Evaluation of $\mathcal{LG}_3$)**.**

$$\begin{aligned}
\mathcal{LG}_3 = {}& \frac{3}{4}\left(\frac{\zeta\,(3)}{\pi^2} + \frac{1}{3}\,\mathcal{LG}_1\right)\mathcal{A}^2 - \frac{3}{2}\left(\frac{\zeta'(2,1)}{\pi^2} + 2\,\mathcal{LG}_1\,\frac{\zeta'\,(2)}{\pi^2}\right)\mathcal{A} \\
& + \left(\mathcal{LG}_1^3 + \frac{1}{16}\,\mathcal{LG}_1\,\pi^2 + \frac{3}{16}\,\zeta\,(3)\right) + \frac{3}{2}\,\mathcal{LG}_1\,\frac{\zeta''\,(2)}{\pi^2} \\
& - \frac{3}{8}\frac{\omega_{1,1,0}(1,1,1) - 2\,\omega_{1,0,1}(1,1,1)}{\pi^2}.
\end{aligned} \tag{61}$$

$$\begin{aligned}
-\mathcal{LG}_{2,1} = {}& \frac{1}{4}\left(\frac{\zeta\,(3)}{\pi^2} + \frac{1}{3}\,\mathcal{LG}_1\right)\mathcal{A}^2 - \frac{1}{2}\left(\frac{\zeta'(2,1)}{\pi^2} + 2\,\mathcal{LG}_1\,\frac{\zeta'\,(2)}{\pi^2}\right)\mathcal{A} \\
& + \left(\mathcal{LG}_1^3 + \frac{1}{16}\,\mathcal{LG}_1\,\pi^2 + \frac{3}{16}\,\zeta\,(3)\right) + \frac{1}{2}\,\mathcal{LG}_1\,\frac{\zeta''\,(2)}{\pi^2} \\
& - \frac{1}{8}\frac{\omega_{1,1,0}(1,1,1) - 2\,\omega_{1,0,1}(1,1,1)}{\pi^2}.
\end{aligned} \tag{62}$$

This has resolved all terms to "order three" zeta-type derivatives when one sets $\mathcal{A} = \log(2\pi) + \gamma = 2\mathcal{LG}_1 + \gamma$ uses the order-counting method in [1] — differentiation increases the order by one — and is a direct extension of the formulas (24) and (25).

Note that (30) implies that the choice of omega partials is not fully determined in (62) or in Theorem 8 below. Determining appropriate bases is a topic for future study.

Remark 3. We also record for possible future use that we can prove that

$$\omega_{1,1,0}(1,1,1) = 2\,\gamma\left(\zeta'\,(2,1) + \zeta'\,(3)\right) - 2\,\gamma^2\zeta\,(3) - \int_0^1 z\left(\int_0^1 \frac{\log\left(-\log\left(t\right)\right)}{1-zt}\,\mathrm{d}t\right)^2\mathrm{d}z, \tag{63}$$

while

$$\int_0^1 \frac{\log(-\log(t))}{1-zt}\,\mathrm{d}t = \sum_{n=1}^{\infty} \frac{z^{n-1}(\gamma+\log n)}{n}$$
$$= -\frac{\mathrm{Li}_1'(z)+\gamma\log(1-z)}{z}.$$

Likewise, writing the final integral, say T, in (63) as a triple integral and integrating w.r.t. to z first, leads to

$$\begin{aligned} T &= -\int_0^1\int_0^1 \log(-\log t)\log(-\log s)\,\frac{s\log(1-t)-t\log(1-s)}{s(t-s)t}\,\mathrm{d}t\,\mathrm{d}s \\ &= \sum_{n=2}^{\infty}\frac{1}{n}\int_0^1\int_0^1 \log(-\log t)\log(-\log s)\,\frac{t^{n-1}-s^{n-1}}{t-s}\,\mathrm{d}t\,\mathrm{d}s \\ &= \sum_{n=2}^{\infty}\frac{1}{n}\sum_{j+k=n}\frac{(\gamma+\log j)(\gamma+\log k)}{j\,k} \qquad (64) \\ &\approx 8.1325183431514868017969743982392235319381843270460. \end{aligned}$$

So

$$\omega_{1,1,0}(1,1,1) = 2\,\gamma\left(\zeta'(2,1)+\zeta'(3)\right) - 2\,\gamma^2\zeta(3) - \sum_{n=2}^{\infty}\frac{1}{n}\sum_{j+k=n}\frac{(\gamma+\log j)(\gamma+\log k)}{j\,k}. \qquad (65)$$

Which is quite an attractive expression. ◊

5 The integral $\mathcal{LG}_4$.

We give the following result as a partial counterpart to Proposition 1. When combined with (20) it provides another identity involving the $\mathcal{LG}_{a,b}$ integrals.

Proposition 2. *For $n = 2, 4, 6, \ldots$ we have*

$$\sum_{a+b=n}(-1)^b\binom{n}{a}\mathcal{LG}_{a,b} = \int_0^1 \log^n\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)\mathrm{d}x. \qquad (66)$$

One strategy is to obtain the Fourier series for $\log^2\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)$ and the other terms on the right of (67) and then to use

$$4\,\log^2(\Gamma(x)) = \left(\log\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) + \log\left(\Gamma(x)\Gamma(1-x)\right)\right)^2 \tag{67}$$

$$= \log^2\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) + 2\log\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)\log\left(\frac{2\pi}{2\sin(\pi x)}\right) + \log\left(\frac{2\pi}{2\sin(\pi x)}\right)^2.$$

Recall from (16) and (20) that

$$\log\left(\frac{2\pi}{2\sin(\pi x)}\right) = \log(2\pi) + \sum_{k=1}^{\infty}\frac{1}{k}\cos(2k\pi x). \tag{68}$$

$$\log\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = (1-2x)\left(\gamma + \log(2\pi)\right) + \frac{2}{\pi}\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi kx). \tag{69}$$

for $0 < x < 1$. At this point, in principle results are accessible from Parseval's formula.

With hindsight, however, it is more efficient to organize as follows. Let

$$C_0(x) := \sum_{n=1}^{\infty}\frac{1}{n}\cos(2n\pi x), \quad S_1(x) := \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\log n}{n}\sin(2n\pi x). \tag{70}$$

Note that S_1 is closely related to the derivative of Clausen's function with respect to its order.

(a.) Using Proposition 1, this time with $a + b = 4$,

$$2\,\mathcal{LG}_4 + 8\,\mathcal{LG}_{3,1} + 6\,\mathcal{LG}_{2,2} = \int_0^1 \log^4\left(\frac{2\pi}{2\sin(\pi x)}\right)\,\mathrm{d}x$$

$$= \frac{19}{240}\,\pi^4 + 6\,\log(2\,\pi)\,\zeta(3) + \frac{1}{2}\,\log^2(2\,\pi)\,\pi^2 + \log^4(2\,\pi)\,. \tag{71}$$

(b.) The alternating Proposition 2 yields

$$2\,\mathcal{LG}_4 - 8\,\mathcal{LG}_{3,1} + 6\,\mathcal{LG}_{2,2} = \int_0^1 \log^4\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)\,\mathrm{d}x$$

$$= \sum_{a=0}^{4}\binom{4}{a}\mathcal{A}^{4-a}\int_0^1 S_1(x)^a\,(1-2x)^{4-a}\,\mathrm{d}x$$

$$= \sum_{a=0}^{4}\binom{4}{a}\mathcal{A}^{4-a}\,\mathcal{M}_4(a)\,, \tag{72}$$

where we have denoted

$$\mathcal{M}_4(a) := \int_0^1 S_1(x)^a \, (1-2x)^{4-a} \; \mathrm{d}x.$$

Note that $\mathcal{M}_4(0) = 1/5$.

(c.) Finally,

$$\begin{aligned}
2\,\mathcal{LG}_4 - 2\,\mathcal{LG}_{2,2} &= \int_0^1 \log^2\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) \log^2\left(\frac{2\pi}{2\sin(\pi x)}\right) \mathrm{d}x \\
&= 2\,\mathcal{I}_3 \log(2\pi) - \int_0^1 \log^2\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) C_0^2(x)\,\mathrm{d}x \\
&= 2\,\mathcal{I}_3 \log(2\pi) - \mathcal{A}^2 \int_0^1 (1-2x)^2\, C_0^2(x)\,\mathrm{d}x - \int_0^1 S_1^2(x)\, C_0^2(x)\,\mathrm{d}x \\
&= 2\,\mathcal{I}_3 \log(2\pi) - \mathcal{A}^2\,\frac{11}{90}\,\pi^2 - \mathcal{CS}_{2,2}\,, \qquad (73)
\end{aligned}$$

where in terms of (70) we set

$$\mathcal{CS}_{a,b} := \int_0^1 C_0^a(x)\, S_1^b\,(x)\,\mathrm{d}x. \qquad (74)$$

We have obtained the following theorem:

Theorem 7 (Evaluation of $\mathcal{LG}_4$)**.** *$\mathcal{LG}_4, \mathcal{LG}_{3,1}, \mathcal{LG}_{2,2}$ are determined by the system of equations*

$$2\,\mathcal{LG}_4 + 8\,\mathcal{LG}_{3,1} + 6\,\mathcal{LG}_{2,2} = \frac{19}{240}\,\pi^4 + 6\,\log(2\,\pi)\,\zeta(3) + \frac{1}{2}\,\log^2(2\,\pi)\,\pi^2 + \log^4(2\,\pi) \qquad (75)$$

$$2\,\mathcal{LG}_4 - 8\,\mathcal{LG}_{3,1} + 6\,\mathcal{LG}_{2,2} = \sum_{a=0}^{4} \binom{4}{a} \mathcal{A}^{4-a}\,\mathcal{M}_4(a) \qquad (76)$$

$$2\,\mathcal{LG}_4 - 2\,\mathcal{LG}_{2,2} = 2\,\mathcal{I}_3 \log(2\pi) - \mathcal{A}^2\,\frac{11}{90}\,\pi^2 - \mathcal{CS}_{2,2} \qquad (77)$$

where $\mathcal{A} = \log(2\pi) + \gamma$*,*

$$\mathcal{I}_3 = -\mathcal{A}^2\,\frac{\zeta(3)}{\pi^2} + 2\,\mathcal{A}\,\frac{\zeta'(2,1)}{2\pi^2} + \frac{1}{2\pi^2}\left(\omega_{1,1,0}(1,1,1) - 2\omega_{1,0,1}(1,1,1)\right),$$

while evaluations of $\mathcal{M}_4(a)$ for $a = 0\ldots4$, and $\mathcal{CS}_{2,2}$ in terms of Witten zeta-values are given in Theorem 8.

Theorem 8 ($M_4(a)$ and $\mathcal{CS}_{2,2}$ as Mordell–Tornheim–Witten type sums)**.**

$$\mathcal{M}_4(0) = \frac{1}{5} \tag{78}$$

$$\mathcal{M}_4(1) = 12\,\frac{\zeta'(4)}{\pi^4} - 2\,\frac{\zeta'(2)}{\pi^2} \tag{79}$$

$$\mathcal{M}_4(2) = \frac{8}{\pi^4}\,\omega_{0,1,1}(2,1,1) - \frac{4}{\pi^4}\,\omega_{1,1,0}(1,1,2) + \frac{2}{3}\,\frac{\zeta''(2)}{\pi^2} \tag{80}$$

$$\mathcal{M}_4(3) = \frac{6}{\pi^2}\left(\omega_{1,1,0,1}(1,1,1,1) - \omega_{1,1,0,1}(1,1 \mid 1,1)\right) + \frac{2}{\pi^2}\,\omega_{1,1,1,0}(1,1,1,1) \tag{81}$$

$$\mathcal{M}_4(4) = \frac{6}{\pi^4}\,\omega_{1,1,1,1}(1,1 \mid 1,1) - \frac{8}{\pi^4}\,\omega_{1,1,1,1}(1,1,1,1) \tag{82}$$

$$\begin{aligned}\mathcal{CS}_{2,2} = {} & \frac{1}{\pi^2}\,\omega_{1,0,0,1}(1,1,1,1) + \frac{1}{\pi^2}\,\omega_{1,0,0,1}(1,1 \mid 1,1) \\ & - \frac{1}{\pi^2}\,\omega_{1,1,0,0}(1,1,1,1) - \frac{1}{2\pi^2}\,\omega_{1,1,0,0}(1,1 \mid 1,1).\end{aligned} \tag{83}$$

Proof. We outline two paths to these evaluations in the next section. □

5.1 The integrals $\mathcal{M}_4(a)$ for $a = 0 \ldots 4$, and $\mathcal{CS}_{2,2}$

It remains to explain the evaluation of the five Fourier integrals in the section title. Recall that

$$\begin{aligned}\mathcal{M}_4(a) &= \left(\frac{2}{\pi}\right)^a \int_0^1 \left(\sum_{n=1}^\infty \frac{\log n}{n}\,\sin(2n\pi x)\right)^a (1-2x)^{4-a}\,\mathrm{d}x \\ &= \frac{2^{a-1}}{\pi^5} \int_{-\pi}^{\pi} \left(\sum_{n=1}^\infty \frac{(-1)^n \log n}{n}\,\sin(n\theta)\right)^a \theta^{4-a}\,\mathrm{d}\theta.\end{aligned} \tag{84}$$

We have already noted $\mathcal{M}_4(0) = 1/5$. Now careful computation of the Fourier coefficients in $\mathcal{M}_4(1), \ldots \mathcal{M}_4(4)$, and exploiting symmetries yields evaluations as multiple sums many of which can be resolved entirely in terms of generalized Witten sums ((37) and (40)).

For numerical validation we record — in addition to the values in Example 2 — that to 30 places or more:

$$\mathcal{M}_4(1) = 0.181497695704128118600461629915$$

$$\mathcal{M}_4(2) = 0.375057423854310553199190528120$$

$$\mathcal{M}_4(3) = 1.1096002880578327311041574759 3$$

$$\mathcal{M}_4(4) = 4.388144647233686375385406620187148605898795723273997473496 05$$

and

$$\mathcal{CS}_{2,2} = 5.743023573467591588737452691 46.$$

In each case it is easiest to compute these integrals by using (68) and (69) but they can also be computed by the methods of [11].

To fully evaluate $\mathcal{LG}_4$ we must resolve $\mathcal{CS}_{2,2}$ and refine the values of $\mathcal{M}_4(a)$. Observe that as defined, all $\mathcal{M}_4(a)$ have order a and involve only a-th derivatives.

For $a = 2$**.** We obtain

$$\mathcal{M}_4(2) = \frac{32}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{\log n \log m}{\left(n^2 - m^2\right)^2} - 4\, \frac{\zeta''(4)}{\pi^4} + \frac{2}{3}\, \frac{\zeta''(2)}{\pi^2}.$$

Now the partial fraction decomposition

$$\frac{1}{\left(m^2 - n^2\right)^2} = \frac{1}{4}\, \frac{1}{n\, m\, (n-m)^2} - \frac{1}{4}\, \frac{1}{n\, m\, (n+m)^2}$$

leads to

$$\begin{aligned}
\mathcal{M}_4(2) &= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{\log n \log m}{m\, n\, (n-m)^2} - \frac{8}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{\log n \log m}{m\, n\, (n+m)^2} - 4\, \frac{\zeta''(4)}{\pi^4} + \frac{2}{3}\, \frac{\zeta''(2)}{\pi^2} \\
&= \frac{8}{\pi^4}\, \omega_{0,1,1}(2,1,1) - \frac{4}{\pi^4}\, \omega_{1,1,0}(1,1,2) + \frac{2}{3}\, \frac{\zeta''(2)}{\pi^2}.
\end{aligned}$$

This is as listed in (80) and is confirmed by the δ values in Example 2.

For $a = 3$**.** From the integral representation for $\mathcal{M}_4(3)$, one may similarly symbolically integrate the three-sine product, to arrive at

$$\mathcal{M}_4(3) = \frac{6}{\pi^2} \sideset{}{'}\sum_{m,n,p \geq 1} \frac{\log m \log n \log p}{mnp(m+n-p)} + \frac{2}{\pi^2}\, \omega_{1,1,1,0}(1,1,1,1), \tag{85}$$

where $\omega_{1,1,1,0}(1,1,1,1) = -58.836831434069077678716676752793744963 3\ldots$ and the other sum S is

$$\begin{aligned}
S &:= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Li}_1'(e^{it})^2 \mathrm{Li}_1'(e^{-it}) \left(\mathrm{Li}_1(e^{it}) - \mathrm{Li}_1(e^{-it})\right) \mathrm{d}t \\
&= 37.626469723153151879426555102165480233 4\ldots.
\end{aligned}$$

For verification, we note that with these numerics,

$$\frac{6}{\pi^2}\,S+\frac{2}{\pi^2}\,\omega_{1,1,1,0}(1,1,1,1)-\mathcal{M}_4(3) = -0.0000000000000000000000000000000000000001\ldots$$

This then reduces to the value give in (81) in terms of double Witten zeta function derivatives.

As Crandall notes: *To achieve such numerics, one may use either series methods, or careful quadrature, or a combination of these — sometimes a combination is best in practice.*

For $a = 4$**.** We write the product of four sin terms in the Fourier expansion as a sum of eight single $\cos(n \pm m \pm j \pm k)$ terms: those with $j + k + m = n$ (four symmetries) yield the Tornheim partials

$$\begin{aligned}\omega_{1,1,1,1}(1,1,1,1) &= \int_0^1 \frac{\mathrm{Li}_1'(\sigma)^3}{\sigma}\left(\log\left(-\log\sigma\right)+\gamma\right)\,d\sigma, \\ &= 393.9564419029741769026955454796027719391267774734182602...\end{aligned} \tag{86}$$

since much as with ω_a above, we obtain partials of the (generalized) Tornheim sum.

Those with $j+k = m+n$ (three symmetries) lead to the second sum not directly expressible as a ω derivative:

$$\begin{aligned}&\sum_N \left(\sum_{m+n=N} \frac{\log n\,\log m}{n\,m}\right)^2 = \frac{1}{\pi}\int_0^\pi \left|\mathrm{Li}_1'\left(e^{i\theta}\right)\right|^4\,\mathrm{d}\theta \\ &= 596.5161194394250137631544371515880910084673922558488607068...\end{aligned} \tag{87}$$

where the integral (see also (40)) is a consequence of $\int_{-\pi}^{\pi}\exp(nt\,i)\,\mathrm{d}t = 2\pi\delta_{n,0}$, and is easily computed via (45). Over-counting in the terms with two minus signs leads to boundary terms with all four integers equal say n, n, n, n, or two pairs say n, n, m, m. The first sum is again expressible in terms of the double Witten-zeta function. Thus, we arrive at the form

$$\mathcal{M}_4(4) = \frac{6}{\pi^4}\,\omega_{1,1,1,1}(1,1 \mid 1,1) - \frac{8}{\pi^4}\,\omega_{1,1,1,1}(1,1,1,1) + \frac{a}{\pi^4}\zeta^{(4)}(4) + \frac{b}{\pi^4}\left(\zeta^{(2)}(2)\right)^2. \tag{88}$$

Here a, b are rational. Ultimately, we determine that these boundary terms contribute zero and we simplify to (82).

For $\mathcal{CS}_{2,2}$. Using (70) we have a version of $a = 4$ with only two partial derivatives. This can be cast in terms of the double Witten-zeta function much as was $\mathcal{M}_4(4)$

in equation (82). This leads to (83) as required. An alternative route is shown in Example 4 below. $\square$

For $\mathcal{CS}_{2,2}$ Parseval's equation is also useful. Indeed

$$\begin{aligned}\pi\, C_0(x)\, S_1(x) &= \sum_{N=1}^{\infty} \sin(2\pi N\, x) \sum_{n=1}^{N-1} \frac{\log n}{(N-n)\, n} + \sum_{N=1}^{\infty}\sum_{n=1}^{N-1} \frac{\log n \sin(2\pi\, x(2n-N))}{(N-n)\, n} \\ &= \sum_{N=1}^{\infty} (a_N + b_N)\sin(2\pi N x), \end{aligned} \tag{89}$$

where $b_N = -\sum_{n=N+1}^{\infty} \frac{\log n - \log(n-N)}{(N-n)n}$. Hence we derive

$$\mathcal{CS}_{2,2} = \frac{1}{\pi^2} \sum_{N=1}^{\infty} \left(\sum_{n=1}^{N-1} \frac{\log n}{(N-n)\, n} - \sum_{n=N+1}^{\infty} \frac{\log n - \log(n-N)}{(N-n)\, n} \right)^2 . \tag{90}$$

Example 4 (Efficient computation of derivative sums, II)**.** In hindsight, there is a universal approach to evaluating any of the $\mathcal{M}_4(a)$ as well as the $\mathcal{CS}_{2,2}$ integral in terms of our ω-sums. Moreover, in this scenario one never need manipulate any explicit sums! Namely, observe these polylogarithmic representations implicit in (40):

$$\begin{aligned}\omega(r,s,t,u) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Li}_r(e^{iz})\mathrm{Li}_s(e^{iz})\mathrm{Li}_t(e^{iz})\mathrm{Li}_u(e^{-iz})\; dz, \\ \omega(r,s\,|\,t,u) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Li}_r(e^{iz})\mathrm{Li}_s(e^{iz})\mathrm{Li}_t(e^{-iz})\mathrm{Li}_u(e^{-iz})\; dz, \\ \mathcal{CS}_{2,2} &= \frac{1}{8\pi^3}\int_0^{2\pi} \left(\mathrm{Li}_1(e^{iz}) + \mathrm{Li}_1(e^{-iz})\right)^2 \left(\mathrm{Li}_1'(e^{iz}) - \mathrm{Li}_1'(e^{-iz})\right)^2\; dz,\end{aligned}$$

and for $z \in (0, 2\pi)$,

$$\frac{1}{2}(\pi - z) = \frac{1}{2i}\left(\mathrm{Li}_1(e^{iz}) - \mathrm{Li}_1(e^{-iz})\right).$$

With these relations — and the partial derivatives wrt order of the ω expressions— in hand, one only need inspect the integrands for the various $\mathcal{M}(a)$, with $a = 0, 1, 2, 3, 4$, and of $\mathcal{CS}_{2,2}$ to resolve each case as a superposition of ω-sums and their derivatives.

For instance, upon expanding out the integrand for $\mathcal{CS}_{2,2}$ we quickly recover that

$$\mathcal{CS}_{2,2} \;=\; \frac{1}{\pi^2}\left(-\omega_{1,1,0,0}(1,1,1,1)+\omega_{1,0,0,1}(1,1,1,1)-\frac{1}{2}\,\omega_{1,1,0,0}(1,1\,|\,1,1)+\omega_{1,0,0,1}(1,1\,|\,1,1)\right).$$

For numerical purposes, we record

$$\begin{aligned}
\omega_{1,1,0,0}(1,1\,|\,1,1) &= 43.787258840104772624652570891898 16\ldots,\\
\omega_{1,0,0,1}(1,1\,|\,1,1) &= 45.667564445279853285404215833073 40\ldots,\\
\omega_{1,0,0,1}(1,1,1,1) &= 54.945741698299297391060048257220 775\ldots,\\
\omega_{1,1,0,0}(1,1,1,1) &= 22.038305987271086934605246590844 36\ldots\ .
\end{aligned}$$

Each value is correct to the precision shown. ♢

6 The integral $\mathcal{LG}_5$ and higher order analogues

The process used for $\mathcal{LG}_4$ can be performed likewise for $\mathcal{LG}_5$. Here again three relations are needed. One comes from Propositions 1 but Proposition 2 is of no use when N is odd. Nonetheless, we may obtain additional relations of the kind we obtained when $N = 3$. One may — at least in outline — provide evaluations of all $\mathcal{LG}_N$ in terms of such Witten derivative values.

Example 5 (The general structure)**.** Fix positive integers a and b with $a \geq b$, and set $a + b = N$. Beginning again with (19) and (20):

$$\mathcal{U} := \log\left(\Gamma(x)\Gamma(1-x)\right) = \log(2\pi) - \log\left(2\sin(\pi x)\right), \tag{91}$$

$$\mathcal{V} := \log\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = (1-2x)\left(\gamma+\log(2\pi)\right) + \frac{2}{\pi}\sum_{k=2}^{\infty}\frac{\log k}{k}\sin(2\pi k x). \tag{92}$$

we may obtain a multinomial type expansion for $\mathcal{LG}_{a,b}$. Note that

$$\mathcal{LG}_{a,b} = \frac{1}{2^N}\int_0^1 \left(\mathcal{U}^2-\mathcal{V}^2\right)^b \left(\mathcal{U}+\mathcal{V}\right)^{a-b}\,\mathrm{d}x \tag{93}$$

After expanding (93), complete evaluation now relies on addressing terms of the form $\int_0^1 \mathcal{U}^j\,\mathcal{V}^k\,\mathrm{d}x$, and on using (91) and (92) this leads to terms $\mathcal{CS}_{n,m,p} :=$

$$\left(\frac{2}{\pi}\right)^{m+1}\int_0^{2\pi}\left(\mathrm{Li}_1(e^{iz})+\mathrm{Li}_1(e^{-iz})\right)^n\left(\mathrm{Li}_1'(e^{iz})-\mathrm{Li}_1'(e^{-iz})\right)^m\left(\frac{1}{2i}\left(\mathrm{Li}_1(e^{iz})-\mathrm{Li}_1(e^{-iz})\right)\right)^p\,\mathrm{d}z.$$

All such terms are susceptible to extensions of the treatment afforded $\mathcal{CS}_{2,2,0}$ in Example 4. Moreover, this shows that the basis elements always occur with a very specific form only engaging Li_1 and $\mathrm{Li}_1^{'}$, via (39) and (40), for which we Crandall has provided excellent computational tools as described in Example 3. Precisely we need only study n-fold derivatives of the form

$$\omega_{\delta_1,\cdots,\delta_n}(1,\cdots 1 \mid 1,1,\cdots 1)$$

with each $\delta_i = 0$ or 1. $\diamondsuit$

For $N \geq 6$ it appears fruitless to follow our current strategy in full, and further progress requires a better understanding of derivatives of Witten-like zeta functions. This will reveal what the appropriate bases and are.

Given the computational tools it should also be able to use PSLQ to explore and reveal the nascent structures! But a full study awaits careful implementation of the necessary extensions of the algorithms developed in [11]. As in many other settings, the interplay between theory driving the need for better computational tools and improved computational techniques unleashing new theory beckons.

Much of this technology is now available in [3] in which theory and computation have been sufficiently developed to fully express all $\mathcal{LG}_n$ in terms of appropriate extensions of our Mordell-Tornheim-Witten (MTW) sums. We finish by remarking that—as suggested by previous work on such integrals— can be seen as a form of degree n, where the degree of each real number appearing in the formula which had previously been heuristically assigned is now the well-defined *weight* of a MTW sum [3]. A fascinating byproduct is

Theorem 9. *For $n = 1, 2, \ldots$*

$$\mathcal{LG}_n \;=\; \sum_{m_1,\ldots,m_n \geq 1} \frac{\zeta^*(m_1)\,\zeta^*(m_2)\cdots\zeta^*(m_n)}{m_1 m_2 \cdots m_n(m_1+\ldots m_n+1)}, \tag{94}$$

where $\zeta^*(1) := \gamma$ and $\zeta^*(n) := \zeta(n)$ for $n \geq 2$.

In particular, the Euler's evaluation of $\mathcal{LG}_1$ leads to a rapidly convergent rational zeta-series:

$$\log\sqrt{2\pi} \;=\; \sum_{m\geq 1} \frac{\zeta^*(m)}{m(m+1)} = \frac{1}{2} \;+\; \gamma \;+\; \sum_{m\geq 2} \frac{\zeta(m)-1}{m(m+1)}.$$

We do not completely understand how the higher $\mathcal{LG}_n$ can be finite superpositions of *derivative* MTWs, as we now know, and yet as infinite sums require only the ζ-function convolutions as in Theorem 9 above.

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