Thirty-two Goldbach Variations

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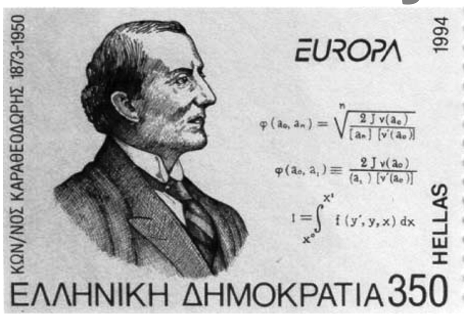


MAA 1936

"I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science."

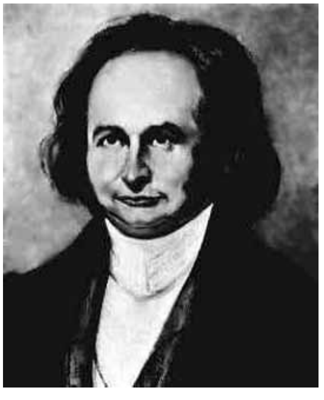
◀ Revised 12-03-10 ▶

Abstract

We give thirty-two diverse proofs of a small mathematical gem—the fundamental *Euler sum* identity

$$\boxed{\zeta(2,1) = \zeta(3) = 8\,\zeta(\overline{2},1).}$$

We also discuss various generalizations for multiple harmonic (Euler) sums and some of their many connections, thereby illustrating both the wide variety of techniques fruitfully used to study such sums and the attraction of their study.



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REFERENCE. J.M. Borwein and D.M. Bradley, "Thirty Two Goldbach Variations," *Int. J. Number Theory*, **2** 1 (2006), 65–103.

1. Introduction

There are several ways to introduce and make attractive a new or unfamiliar subject. We choose to do so by emulating Glenn Gould's passion for Bach's *Goldberg variations*.

We shall illustrate most of the techniques used to study Euler sums by focusing almost entirely on the identities of (2) and (5) below, viz

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{1}{n^3} = 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{m=1}^{n-1} \frac{1}{m}$$

and some of their many generalizations. In doing so we make a tour through a large variety of topics.

1.1 Euler, Goldbach and the birth of ζ

What follows is a transcription of correspondence between Euler and Goldbach that led to the origin of the zeta-function and multi-zeta values:

> **59. Goldbach an Euler, Moskau, 24. Dez. 1742.*** [...]*Als ich neulich die vermeinten summas der beiden letzteren serierum in meinem vorigen Schreiben wieder betrachtet, habe ich alsofort wahrgenommen, daß selbige aus einem bloßem Schreibfehler entstanden, von welchem es aber in der Tat heißet: Si non errasset, fecerat ille minus*.†

This is the letter in which Goldbach precisely formulates the series which sparked Euler's investigations into what became the zeta-function. These investigations were due to a serendipitous mistake.

The above translates:

*AAL: F.136, Op. 2, Nr.8, Blatt 54–55.

†Frei zitiert nach Marcus Valerius Martialis, I, 21,9.

*When I recently considered further the indicated sums of the last two series in my previous letter, I realized immediately that the same series arose due to a mere writing error, from which indeed the saying goes, "Had one not erred, one would have achieved less."** Goldbach continues...

Ich halte dafür, daß es ein problema problematum ist, die summam huius:

$$/54r/ 1+\frac{1}{2^n}\left(1+\frac{1}{2^m}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right) +\frac{1}{4^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}\right)+etc.$$

in den casibus zu finden, wo m *et* n *nicht numeri integri pares et sibi aequales sind, doch gibt es casus, da die summa angegeben werden kann, exempli gr[atia], si* $m=1$, $n=3$, *denn es ist*

$$1 \quad +\frac{1}{2^3}\left(1+\frac{1}{2}\right)+\frac{1}{3^3}\left(1+\frac{1}{2}+\frac{1}{3}\right) +\frac{1}{4^3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+etc.=\frac{\pi^4}{72}.$$

**Opera Omnia*, vol. IVA4, Birkhäuser Verlag.

1.2. The Modern Language of Euler Sums

For positive integers $s_1, \ldots, s_m$ and signs $\sigma_j = \pm 1$, consider the m-fold Euler sum

$$\zeta(s_1, \ldots, s_m; \sigma_1, \ldots, \sigma_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{\sigma_j^{k_j}}{k_j^{s_j}}.$$

We combine strings of exponents and signs by replacing s_j by $\overline{s}_j$ in the argument list if and only if $\sigma_j = -1$, and denote n repetitions of a substring S by $\{S\}^n$. Thus, for example, $\zeta(\overline{1}) = -\log 2$, $\zeta(\{2\}^3) = \zeta(2,2,2) = \pi^6/7!$ and

$$\zeta(s_1, \ldots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}. \qquad (1)$$

The identity

$$\zeta(2,1) = \zeta(3) \qquad (2)$$

goes back to Euler and has been repeatedly rediscovered. In this language Goldbach had found

$$\zeta(3,1) + \zeta(4) = \frac{\pi^4}{72}.$$

The more general formula

$$2\zeta(m,1) = m\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j), \quad (3)$$

for $m \geq 2$ is also due to Euler.

- Nielsen obtained (3) and related results based on partial fractions. Formula (3) has also been discovered many times.

Study of the multiple zeta function (1) led to the discovery of a new generalization of (2), involving nested sums of arbitrary depth:

$$\zeta(\{2,1\}^n) = \zeta(\{3\}^n), \qquad n \in \mathbb{Z}^+. \quad (4)$$

- Although numerous proofs of (2) and (3) are known (we give many), the *only* proof of (4) of which we are aware involves making a simple change of variable in a multiple iterated integral (see (31) below).

An alternating version of (2) is

$$8\zeta(\overline{2},1) = \zeta(3), \tag{5}$$

which has also resurfaced from time to time.

Equation (5) hints at the **generalization**:

$$8^n\zeta(\{\overline{2},1\}^n) \stackrel{?}{=} \zeta(\{3\}^n), \qquad n \in \mathbb{Z}^+, \tag{6}$$

we originally conjectured in 1996, and which remained open until 2008—despite abundant, even overwhelming, evidence. Zhao's 2008 proof (see also *Math by Experiment*, ed. 2) relies on *double shuffles* (below) and while very clever adds little insight.

- The first 85 instances of (6) were recently affirmed to 1000 decimal place accuracy by Petr Lisonek. He also checked the case $n = 163$, a calculation that required ten hours run time on a 2004-era computer.

1.3. Hilbert and Hardy Inequalities

Much of the early 20th century history - and philosophy - of the “ *‘bright’ and amusing*” subject of inequalities charmingly discussed in G.H. Hardy’s retirement lecture as London Mathematical Society Secretary.

He comments that *Harald Bohr is reported to have remarked “Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.”* Central to Hardy’s essay are:

Theorem 1 (Hilbert) *For non-negative sequences* (a_n) *and* (b_n)*, not both zero, and for* $1 \le p, q \le \infty$ *with* $1/p + 1/q = 1$ *one has*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n\, b_m}{n+m} < \pi \csc\left(\frac{\pi}{p}\right) \|a_n\|_p\, \|b_n\|_q. \tag{7}$$

Theorem 2 (Hardy) *For a non-negative sequence* (a_n) *and for* $p > 1$

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{8}$$

- We return to these inequalities in Section six.

Hardy remarks that his *"own theorem was discovered as a by-product of my own attempt to find a really simple and elementary proof of Hilbert's."*

He reproduces Elliott's proof of (8), writing "*it can hardly be possible to find a proof more concise or elegant*" and also "I *have given nine [proofs] in a lecture in Oxford, and more have been found since then.*"

We wish to emulate Hardy and to present proofs that are either elementary, bright and amusing, concise or elegant - ideally all at the same time!

1.4. Further Motivation and Intentions

In doing so we note that: $\zeta(3)$, while provably irrational, is still quite mysterious. Hence, exposing more relationships and approaches can only help.

- We certainly hope one of them will lead to a proof of conjecture (6).

Identities for $\zeta(3)$ are abundant and diverse. We give three each of which is the entry-point to a fascinating set:

A first favourite is a *binomial sum* that played a role in Apéry's 1976 proof of the irrationality of $\zeta(3)$:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}. \tag{9}$$

A second favourite is Broadhurst's *BBP formula*

$$\zeta(3) =$$

$$\frac{1}{672}\sum_{k=0}^{\infty}\frac{1}{2^{12k}}\Bigg[\frac{2048}{(24k+1)^3}-\frac{11264}{(24k+2)^3}-\frac{1024}{(24k+3)^3}+\frac{11776}{(24k+4)^3}$$
$$-\frac{512}{(24k+5)^3}+\frac{4096}{(24k+6)^3}+\frac{256}{(24k+7)^3}+\frac{3456}{(24k+8)^3}+\frac{128}{(24k+9)^3}$$
$$-\frac{704}{(24k+10)^3}-\frac{64}{(24k+11)^3}-\frac{128}{(24k+12)^3}-\frac{32}{(24k+13)^3}-\frac{176}{(24k+14)^3}$$
$$+\frac{16}{(24k+15)^3}+\frac{216}{(24k+16)^3}+\frac{8}{(24k+17)^3}+\frac{64}{(24k+18)^3}-\frac{4}{(24k+19)^3}$$
$$+\frac{46}{(24k+20)^3}-\frac{2}{(24k+21)^3}-\frac{11}{(24k+22)^3}+\frac{1}{(24k+23)^3}\Bigg].$$

- This discovery led Bailey & Crandall to their recent work on *normality* of BBP constants.

A third favourite due to Ramanujan is the *hyperbolic series* approximation

$$\zeta\,(3) = \frac{7\,\pi^3}{180} - 2\,\sum_{k=1}^{\infty}\frac{1}{k^3\left(e^{2\,\pi\,k}-1\right)},$$

with 'error' is $\zeta(3) - 7\,\pi^3/180 \approx -0.003742745$. To our knowledge this is the 'closest' one gets to writing $\zeta(3)$ as a rational multiple of π^3.

- Often results about $\zeta(3)$ are really results about $\zeta(2,1)$ or $\zeta(\overline{2},1)$, as we shall exhibit.

- Double and multiple sums are still under-studied and under-appreciated. We should like to partially redress that.

- One can now prove these seemingly analytic facts in an entirely finitary manner via words over alphabets, dispensing with notions of infinity and convergence.

- Many subjects are touched upon—from computer algebra, integer relation methods, generating functions and techniques of integration to polylogarithms, hypergeometric and special functions, non-commutative rings, combinatorial algebras and Stirling numbers.

- For example, there has been an explosive recent interest in q-analogs, see §, and in quantum field theory, algebraic K-theory and knot theory.

1.5. Further Notation

For positive N, we write

$$H_N := \sum_{n=1}^{N} 1/n.$$

We use $\psi = \Gamma'/\Gamma$ for *digamma*, the logarithmic derivative of Euler's Gamma. Then $\psi(N+1)+\gamma = H_N$, where $\gamma = 0.5772156649\ldots$ is *Euler's constant*. The *Pochhammer symbol* $(a)_n = a(a+1)\cdots(a+n-1)$ for complex a and integer $n > 0$, and the *Kronecker* $\delta_{m,n}$ is 1 if $m = n$ and 0 otherwise.

We organize proofs by technique, though this is somewhat arbitrary as many proofs fit well within more than one category. Broadly their sophistication increases as we move through the talk.
We invite additions to a collection which for us has all the beauty of Blake's grain of sand*:

"*To see a world in a grain of sand*
And a heaven in a wild flower,
Hold infinity in the palm of your hand
And eternity in an hour."

*William Blake from *Auguries of Innocence*.

2. Telescoping and Partial Fractions

2.1. A **first quick proof** of (2) considers

$$S := \sum_{n,k>0} \frac{1}{nk(n+k)} = \sum_{n,k>0} \frac{1}{n^2}\left(\frac{1}{k} - \frac{1}{n+k}\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3) + \zeta(2,1).$$

On the other hand,

$$S = \sum_{n,k>0} \left(\frac{1}{n} + \frac{1}{k}\right) \frac{1}{(n+k)^2}$$
$$= \sum_{n,k>0} \frac{1}{n(n+k)^2} + \sum_{n,k>0} \frac{1}{k(n+k)^2} = 2\zeta(2,1),$$

by symmetry.

©

- The above argument goes back at least to Steinberg and Klamkin.

2.2. A **second proof** runs as follows:

$$\begin{aligned}
&\zeta(\overline{2},\overline{1})+\zeta(3) && (10)\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{n}\frac{(-1)^k}{k}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k}-\frac{(-1)^{n+k}}{n+k}\right)\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{n+k-(-1)^n k}{k(n+k)}\right)\\
&=\sum_{n,k>0}\frac{(-1)^{n+k}}{nk(n+k)}+\sum_{n,k>0}\frac{(-1)^{n+k}}{n^2(n+k)}-\sum_{n,k>0}\frac{(-1)^k}{n^2(n+k)}\\
&=\sum_{n,k>0}\left(\frac{1}{n}+\frac{1}{k}\right)\frac{(-1)^{n+k}}{(n+k)^2}+\zeta(\overline{1},2)-\sum_{n,k>0}\frac{(-1)^n(-1)^{n+k}}{n^2(n+k)}\\
&=\sum_{n,k>0}\frac{(-1)^{n+k}}{n(n+k)^2}+\sum_{n,k>0}\frac{(-1)^{n+k}}{k(n+k)^2}+\zeta(\overline{1},2)-\zeta(\overline{1},\overline{2})\\
&=2\zeta(\overline{2},1)+\zeta(\overline{1},2)-\zeta(\overline{1},\overline{2}). && (11)
\end{aligned}$$

Similarly we write

$$
\begin{aligned}
&\zeta(2,\overline{1})+\zeta(\overline{3}) &&(12)\\
&=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{(-1)^k}{k}=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k}-\frac{(-1)^{n+k}}{n+k}\right)\\
&=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{n+k-(-1)^n k}{k(n+k)}\right)\\
&=\sum_{n,k>0}\frac{(-1)^k}{nk(n+k)}+\sum_{n,k>0}\frac{(-1)^k}{n^2(n+k)}-\sum_{n,k>0}\frac{(-1)^{n+k}}{n^2(n+k)}\\
&=\sum_{n,k>0}\left(\frac{1}{n}+\frac{1}{k}\right)\frac{(-1)^k}{(n+k)^2}+\sum_{n,k>0}\frac{(-1)^n(-1)^{n+k}}{n^2(n+k)}-\zeta(\overline{1},2)\\
&=\sum_{n,k>0}\frac{(-1)^n(-1)^{n+k}}{n(n+k)^2}+\sum_{n,k>0}\frac{(-1)^k}{k(n+k)^2}+\zeta(\overline{1},\overline{2})-\zeta(\overline{1},2)\\
&=\zeta(\overline{2},\overline{1})+\zeta(2,\overline{1})+\zeta(\overline{1},\overline{2})-\zeta(\overline{1},2). &&(13)
\end{aligned}
$$

Adding equations (11) and (13) now gives

$$2\zeta(\overline{2},1) = \zeta(3) + \zeta(\overline{3}), \tag{14}$$

That is,

$$8\zeta(\overline{2},1) = 4\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n^3} \tag{15}$$

$$= 4\sum_{m=1}^{\infty} \frac{2}{(2m)^3}$$

$$= \zeta(3), \tag{16}$$

which is (5). ©

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3. Finite Series Transformations

Lemma. For any positive integer N, we have

$$\sum_{n=1}^{N}\frac{1}{n^3}-\sum_{n=1}^{N}\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{k}=\sum_{n=1}^{N}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{N-k+1} \tag{17}$$

by induction. ©

Alternatively, consider

$$T:=\sum_{\substack{n,k=1\\k\neq n}}^{N}\frac{1}{nk(k-n)}=\sum_{\substack{n,k=1\\k\neq n}}^{N}\left(\frac{1}{n}-\frac{1}{k}\right)\frac{1}{(k-n)^2}=0.$$

E&R

On the other hand

$$T = \sum_{\substack{n,k=1 \\ k \neq n}}^{N} \frac{1}{n^2}\left(\frac{1}{k-n} - \frac{1}{k}\right)$$

$$= \sum_{n=1}^{N} \frac{1}{n^2}\left(\sum_{k=1}^{n-1} \frac{1}{k-n} + \sum_{k=n+1}^{N} \frac{1}{k-n} - \sum_{k=1}^{N} \frac{1}{k} + \frac{1}{n}\right)$$

$$= \sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{n-k}$$

$$+ \sum_{n=1}^{N} \frac{1}{n^2}\left(\sum_{k=n+1}^{N} \frac{1}{k-n} - \sum_{k=1}^{N} \frac{1}{k}\right).$$

H&L

Since $T = 0$, this implies that

$$\sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^{N} \frac{1}{n^2} \left(\sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N-n} \frac{1}{k} \right)$$

$$= \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{N-k+1},$$

which is (17). ©

Now the right hand side satisfies

$$\begin{aligned} \frac{H_N}{N} &= \sum_{n=1}^{N} \frac{1}{n^2} \cdot \frac{n}{N} \\ &\leq \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{N-k+1} \\ &\leq \sum_{n=1}^{N} \frac{1}{n^2} \cdot \frac{n}{N-n+1} \\ &= \frac{1}{N+1} \sum_{n=1}^{N} \left(\frac{1}{n} + \frac{1}{N-n+1} \right) = \frac{2H_N}{N+1}. \end{aligned}$$

Letting N grow without bound now gives (2), since

$$\lim_{N \to \infty} \frac{H_N}{N} = 0.$$

©

4. Geometric Series

4.1 Convolution of Geometric Series

Let $2 \le m \in \mathbb{Z}$, and consider

$$\sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{k=1}^{N} \sum_{j=1}^{m-2} \frac{1}{n^{j+1}} \frac{1}{k^{m-j}}$$

$$= \lim_{N\to\infty} \left\{ \sum_{\substack{n,k=1 \\ k\neq n}}^{N} \left(\frac{1}{n^{m-1}(k-n)k} - \frac{1}{n(k-n)k^{m-1}} \right) + \sum_{n=1}^{N} \frac{m-2}{n^{m+1}} \right\}$$

$$= (m-2)\zeta(m+1) + 2 \lim_{N\to\infty} \sum_{\substack{n,k=1 \\ k\neq n}}^{N} \frac{1}{n^{m-1}k(k-n)}.$$

- As suggested by Williams (also Bracken)

Thus, we find that

$$(m-2)\zeta(m+1)-\sum_{j=1}^{m-2}\zeta(j+1)\zeta(m-j)$$

$$= 2\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^m}\sum_{\substack{k=1\\k\neq n}}^{N}\left(\frac{1}{k}-\frac{1}{k-n}\right)$$
$$= 2\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^m}\left\{\sum_{k=1}^{n-1}\frac{1}{k}-\frac{1}{n}+\sum_{k=1}^{n}\frac{1}{N-k+1}\right\}$$
$$= 2\zeta(m,1)-2\zeta(m+1)$$
$$+ 2\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^m}\sum_{k=1}^{n}\frac{1}{N-k+1},$$

and hence

$$2\zeta(m,1)=m\zeta(m+1)-\sum_{j=1}^{m-2}\zeta(j+1)\zeta(m-j)$$
$$-2\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^m}\sum_{k=1}^{n}\frac{1}{N-k+1}.$$

But, in light of

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n^m}\sum_{k=1}^{n}\frac{1}{N-k+1} &\le \sum_{n=1}^{N}\frac{1}{n^m}\cdot\frac{n}{N-n+1}\\ &\le \frac{1}{N+1}\sum_{n=1}^{N}\left(\frac{1}{N-n+1}+\frac{1}{n}\right)\\ &= \frac{2H_N}{N+1},\end{aligned}$$

the identity (3) now follows. ©

4.2. A Sum Formula



- Equation (2) is the case $n = 3$ of the following summation.

Theorem 3 *If* $3 \le n \in \mathbb{Z}$ *then*

$$\zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j). \tag{18}$$

Proof. Summing the geometric series (in j) on the right hand side gives

$$\begin{aligned}
&\sum_{j=1}^{n-2} \sum_{h=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{h^j (h+m)^{n-j}} \\
&= \sum_{h,m=1}^{\infty} \left[\frac{1}{h^{n-2} m(h+m)} - \frac{1}{m(h+m)^{n-1}} \right] \\
&= \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{h+m} \right) - \zeta(n-1,1) \\
&= \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^{h} \frac{1}{k} - \zeta(n-1,1) \\
&= \sum_{h=1}^{\infty} \frac{1}{h^{n}} + \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^{n-1} \frac{1}{k} - \zeta(n-1,1) \\
&= \zeta(n).
\end{aligned}$$

©

4.3. A q-Analog

Following Zudilin, for $s > 1$, $0 < q < 1$ begin with

$$\frac{uv^2}{(1-v)(1-uv)^s}$$
$$= \frac{uv}{(1-u)(1-v)^s} - \sum_{j=1}^{s-1} \frac{uv^2}{(1-v)^{j+1}(1-uv)^{s-j}}.$$

Put $u = q^m$, $v = q^n$ and sum over $m, n > 0$. Thus,

$$\sum_{m,n>0} \frac{q^{m+n}}{(1-q^m)(1-q^{m+n})^s} + \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)(1-q^{m+n})^s}$$
$$= \sum_{m,n>0} \frac{q^{m+n}}{(1-q^m)(1-q^n)^s} - \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^s(1-q^{m+n})}$$
$$- \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}$$
$$= \sum_{m,n>0} \frac{q^n}{(1-q^n)^s}\left[\frac{q^m}{1-q^m} - \frac{q^{m+n}}{1-q^{m+n}}\right] - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}$$
$$= \sum_{n>0} \frac{q^n}{(1-q^n)^s} \sum_{m=1}^{n} \frac{q^m}{1-q^m} - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}$$
$$= \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}} + \sum_{n>m>0} \frac{q^{n+m}}{(1-q^n)^s(1-q^m)}$$
$$- \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}$$

Cancelling the second double sum on the left with the corresponding sum on the right and replacing $m+n$ by k in the remaining sums yields

$$\sum_{k>m>0} \frac{q^k}{(1-q^k)^s(1-q^m)} = \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}}$$
$$-\sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^m)^{j+1}(1-q^k)^{s-j}},$$

or equivalently, that

$$\sum_{k>0} \frac{q^{2k}}{(1-q^k)^{s+1}} = \sum_{k>m>0} \frac{q^k}{(1-q^k)^s(1-q^m)}$$
$$+ \sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^k)^{s-j}(1-q^m)^{j+1}}. \qquad (19)$$

Multiplying (19) through by $(1-q)^{s+1}$ and letting $q \to 1$ gives

$$\zeta(s+1) = \zeta(s,1) + \sum_{j=1}^{s-2} \zeta(s-j, j+1),$$

which is just a restatement of (18). Taking $s=2$ gives (2) again. ©

The q-analog of an integer $n \geq 0$ is

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q},$$

and the *multiple q-zeta function*

$$\zeta[s_1, \ldots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j - 1)k_j}}{[k_j]_q^{s_j}}, \qquad (20)$$

for real numbers with $s_1 > 1$ and $s_j \geq 1$, $2 \leq j \leq m$.

Multiplying (19) by $(1-q)^{s+1}$ with $s = 2$ gives $\zeta[2,1] = \zeta[3]$ — a q-analog of (2) (i.e., the latter follows as $q \to 1-$.) Also, $s = 3$ in (19) gives

$$\zeta[4] + (1-q)\zeta[3] = \zeta[3,1] + (1-q)\zeta[2,1] + \zeta[2,2],$$

so $\zeta[2,1] = \zeta[3]$ implies $\zeta[3,1] = \zeta[4] - \zeta[2,2]$.

Bradley shows $\zeta[2,2]$ reduces to depth 1 multiple q-zeta values. Indeed, by the q-stuffle multiplication rule, $\zeta[2]\zeta[2] = 2\zeta[2,2] + \zeta[4] + (1-q)\zeta[3]$. Thus,

$$\zeta[3,1] = \zeta[4] - \zeta[2,2] = \tfrac{3}{2}\zeta[4] - \tfrac{1}{2}(\zeta[2])^2 + \tfrac{1}{2}(1-q)\zeta[3],$$

which is a q-analog of the evaluation

$$\zeta(3,1) = \frac{\pi^4}{360}.$$

5. Integral Representations

5.1. Single Integrals I

We use the fact that

$$\int_0^1 u^{k-1}(-\log u)\,du = \frac{1}{k^2}, \qquad k > 0. \tag{21}$$

Thus

$$\begin{aligned}
\sum_{k>n>1} \frac{1}{k^2 n} &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k>n} \int_0^1 u^{k-1}(-\log u)\,du \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log u) \sum_{k>n} u^{k-1}\,du \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log u) \frac{u^n}{1-u}\,du \\
&= -\int_0^1 \frac{\log u}{1-u} \sum_{n=1}^{\infty} \frac{u^n}{n}\,du \\
&= \int_0^1 \frac{(-\log u)\ \log(1-u)}{1-u}\,du.
\end{aligned} \tag{22}$$

- Interchanges of sum and integral are justified by *Lebesgue's monotone convergence theorem*.

After making the change of variable $t = 1 - u$, we obtain

$$\sum_{k>n>1} \frac{1}{k^2 n} = \int_0^1 \log(1-t)\,(-\log t)\,\frac{dt}{t} \qquad (23)$$

$$= \int_0^1 (-\log t) \sum_{n=1}^{\infty} \frac{t^{n-1}}{n}\, dt.$$

- Again, since all terms of the series are positive, Lebesgue's monotone convergence theorem permits us to interchange the order of summation and integration.

Thus, invoking (21) again, we obtain

$$\sum_{k>n>1} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log t)\, t^{n-1}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which is (2). ©

5.2. Single Integrals II

The *Laplace transform*

$$\int_0^1 x^{r-1}(-\log x)^\sigma \, dx = \int_0^\infty e^{-ru}\, u^\sigma \, du = \frac{\Gamma(\sigma+1)}{r^{\sigma+1}}, \tag{24}$$

for $r > 0$, $\sigma > -1$ generalizes (21) and yields the representation

$$\zeta(m+1) = \frac{1}{m!}\sum_{r=1}^\infty \frac{\Gamma(m+1)}{r^{m+1}} = \frac{1}{m!}\sum_{r=1}^\infty \int_0^1 x^{r-1}(-\log x)^m \, dx$$
$$= \frac{(-1)^m}{m!}\int_0^1 \frac{\log^m x}{1-x} dx.$$

Interchange is valid if $m > 0$, so $x \mapsto 1-x$ yields

$$\zeta(m+1) = \frac{(-1)^m}{m!}\int_0^1 \log^m(1-x)\,\frac{dx}{x}, \qquad 1 \le m \in \mathbb{Z}. \tag{25}$$

Farnum used (24) with clever changes of variable and integration by parts, to prove the identity

$$k!\,\zeta(k+2) = \sum_{n_j=1}^\infty \frac{1}{n_1 n_2 \cdots n_k} \sum_{p=1+\sum n_j} \frac{1}{p^2}, \; 0 \le k \in \mathbb{Z}. \tag{26}$$

- The case $k = 1$ of (26) is again (2). We give a simpler proof of (26), dispensing with integration by parts.

From (24), $k!\,\zeta(k+2) =$

$$= \sum_{r=1}^{\infty} \frac{1}{r} \cdot \frac{\Gamma(k+1)}{r^{k+1}} = \sum_{r=1}^{\infty} \frac{1}{r} \int_0^1 x^{r-1} (-\log x)^k \, dx$$

$$= \int_0^1 (-\log x)^k \log(1-x)^{-1} \frac{dx}{x}$$

$$= \int_0^1 \log^k (1-x)^{-1} (-\log x) \frac{dx}{1-x}$$

$$= \sum \frac{1}{n_1 n_2 \cdots n_k} \int_0^1 \frac{x^{n_1+n_2+\cdots+n_k}}{1-x} (-\log x) \, dx$$

$$= \sum \frac{1}{n_1 n_2 \cdots n_k} \sum_{p > n_1+n_2+\cdots+n_k} \int_0^1 x^{p-1} (-\log x) \, dx$$

$$= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \sum_{p > n_1+n_2+\cdots+n_k} \frac{1}{p^2}.$$

©

5.3. Double Integrals I

Write

$$\zeta(2,1) = \sum_{k,m>0} \frac{1}{k(m+k)^2}$$
$$= \int_0^1 \int_0^1 \sum_{k>0} \frac{(xy)^k}{k} \sum_{m>0} (xy)^{m-1}\, dx\, dy$$
$$= -\int_0^1 \int_0^1 \frac{\log(1-xy)}{1-xy}\, dx\, dy.$$

Make the change of variable $u = xy$, $v = x/y$ with *Jacobian* $1/(2v)$, obtaining

$$\zeta(2,1) = -\frac{1}{2}\int_0^1 \frac{\log(1-u)}{1-u} \int_u^{1/u} \frac{dv}{v}\, du$$
$$= \int_0^1 \frac{(\log u)\log(1-u)}{1-u}\, du,$$

which is (22). Now continue as in §5.1. ©

5.4. Double Integrals II

Let $\varepsilon > 0$. Expand the integrand as a geometric series. One gets:

$$\sum_{n=1}^{\infty} \frac{1}{(n+\varepsilon)^2} = \int_0^1 \int_0^1 \frac{(xy)^\varepsilon}{1-xy}\, dx\, dy.$$

Differentiate wrt ε and let $\varepsilon = 0$:

$$\begin{aligned} \zeta(3) &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{\log(xy)}{1-xy}\, dx\, dy \\ &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{\log x + \log y}{1-xy}\, dx\, dy \\ &= -\int_0^1 \int_0^1 \frac{\log x}{1-xy}\, dx \end{aligned}$$

by symmetry. Now integrate with respect to y to get

$$\zeta(3) = \int_0^1 (\log x) \log(1-x) \frac{dx}{x}. \qquad (27)$$

Comparing (27) with (23) yields (2). ©

- This was reconstructed from a conversation with Krishna Alladi.

5.5. Integration by Parts

Start with (27) and integrate by parts to obtain

$$\begin{aligned}
2\zeta(3) &= 2\int_0^1 (\log x)\log(1-x)\frac{dx}{x} \\
&= \int_0^1 \frac{\log^2 x}{1-x}\,dx \\
&= \int_0^1 \log^2(1-x)\frac{dx}{x} \\
&= \sum_{n,k>0}\int_0^1 \frac{x^{n+k-1}}{nk}\,dx \\
&= \sum_{n,k>0}\frac{1}{nk(n+k)} = 2\,\zeta(2,1),
\end{aligned}$$

on appealing to the first telescoping result of §2. ©

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5.6. Triple Integrals I

Instead of (21) we use the identity

$$\frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} y_2^{k-n-1} \int_0^{y_2} y_3^{n-1}\, dy_3\, dy_2\, dy_1,$$

for $k > n > 0$. This yields

$$\sum_{k>n>0} \frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} (1-y_2)^{-1} \int_0^{y_2} (1-y_3)^{-1}\, dy_3 dy_2 dy_1. \tag{28}$$

The change of variables $y_i = 1 - x_i$ $(i = 1, 2, 3)$ gives

$$\sum_{k>n>0} \frac{1}{k^2 n} = \int_0^1 (1 - x_1)^{-1} \int_{x_1}^1 x_2^{-1} \int_{x_2}^1 x_3^{-1}\, dx_3\, dx_2\, dx_1$$

$$= \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} (1 - x_1)^{-1}\, dx_1\, dx_2\, dx_3.$$

Expand $(1-x_1)^{-1}$ and interchange sum and integral:

$$\sum_{k>n>0} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} x_1^{n-1}\, dx_1\, dx_2\, dx_3$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which is is (2) again. ©

- Additionally:

$$\begin{aligned}\zeta(s_1,\ldots,s_k) &= \sum_{n_1>\cdots>n_k>0}\prod_{j=1}^{k} n_j^{-s_j} \qquad (29)\\ &= \int \prod_{j=1}^{k}\left(\prod_{r=1}^{s_j-1}\frac{dt_r^{(j)}}{t_r^{(j)}}\right)\frac{dt_{s_j}^{(j)}}{1-t_{s_j}^{(j)}},\end{aligned}$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(k)} > \cdots > t_{s_k}^{(k)} > 0,$$

and is abbreviated by

$$\int_0^1 \prod_{j=1}^{k} a^{s_j-1}b, \qquad a = \frac{dt}{t}, \qquad b = \frac{dt}{1-t}. \qquad (30)$$

Using $t \mapsto 1-t$ at each level switches the forms a and b, yielding the *duality formula*

$$\begin{aligned}&\zeta(s_1+2,\{1\}^{r_1},\ldots,s_n+2,\{1\}^{r_n})\\ &= \zeta(r_n+2,\{1\}^{s_n},\ldots,r_1+2,\{1\}^{s_1}), \qquad (31)\end{aligned}$$

for all $s_i, r_i \in \mathbb{N}$. Then $s_1 = 0$, $r_1 = 1 = n$ is (2).

- More generally, (4) can be restated as

$$\boxed{\int_0^1 (ab^2)^n = \int_0^1 (a^2b)^n}$$

and (4) is recovered with $s_j \equiv 0, r_j \equiv 1$ in (31).

For alternations we add the differential form $c := -dt/(1+t)$ to obtain the generating function

$$\sum_{n=1}^{\infty} z^{3n}\zeta(\{\overline{2},1\}^n) =$$

$$\sum_{n=0}^{\infty}\left\{z^{6n+3}\int_0^1 (ac^2ab^2)^n ac^2 + z^{6n+6}\int_0^1 (ac^2ab^2)^{6n+6}\right\}.$$

A lengthy calculation shows the only changes of variables preserving $[0,1]$ and sending the non- commutative polynomial ring $\mathbb{Q}\langle a,b\rangle$ into $\mathbb{Q}\langle a,b,c\rangle$ are

$$S(a,b) = S(a,b), \qquad t \mapsto t, \tag{32}$$

$$S(a,b) = R(b,a), \qquad t \mapsto 1-t, \tag{33}$$

$$S(a,b) = S(2a, b+c), \qquad t \mapsto t^2, \tag{34}$$

$$S(a,b) = S(a+c, b-c), \qquad t \mapsto \frac{2t}{1+t}, \tag{35}$$

$$S(a,b) = S(a+2c, 2b-2c), \qquad t \mapsto \frac{4t}{(1+t)^2}, \tag{36}$$

and compositions thereof, such as $t \mapsto 1 - 2t/(1+t) = (1-t)/(1+t)$, etc.

- In (32)–(36), $S(a,b)$ denotes a non-commutative word on the alphabet $\{a,b\}$ and $R(b,a)$ denotes the word formed by switching a and b and then reversing the order of the letters.

Now view a, b and c as indeterminates. In light of the polynomial *identity*

$$\begin{aligned} ab^2 - 8ac^2 = {} & 2[ab^2 - 2a(b+c)^2] \\ & + 8[ab^2 - (a+c)(b-c)^2] \\ & + [(a+2c)(2b-2c)^2 - ab^2] \end{aligned}$$

in the non-commutative ring $\mathbb{Z}\langle a,b,c\rangle$ and the transformations (34), (35) and (36) above, each bracketed term vanishes after the identifications $a = dt/t$, $b = dt/(1-t)$, $c = -dt/(1+t)$ and perform the requisite iterated integrations.

Thus,

$$\boxed{\zeta(2,1) - 8\zeta(\overline{2},1) = \int_0^1 ab^2 - 8\int_0^1 ac^2 = 0}$$

which in light of (2) proves (5). ©

By expanding the integrands in geometric series and integrating term by term,

$$\zeta(2,1) = 8\int_0^1 \frac{dx}{x}\int_0^x \frac{y\,dy}{1-y^2}\int_0^y \frac{z\,dz}{1-z^2}.$$

Now make the change of variable

$$\frac{x\,dx}{1-x^2} = \frac{du}{1+u}, \qquad \frac{y\,dy}{1-y^2} = \frac{dv}{1+v}, \qquad \frac{z\,dz}{1-z^2} = \frac{dw}{1+w}$$

to obtain the equivalent integral

$$\zeta(2,1)$$

$$= 8\int_0^\infty \left(\frac{du}{2u} + \frac{du}{2(2+u)} - \frac{du}{1+u}\right)\int_0^u \frac{dv}{1+v}\int_0^v \frac{dw}{1+w}.$$

The two inner integrals can be directly performed, hence

$$\zeta(2,1) = 4\int_0^\infty \frac{\log^2(u+1)}{u(u+1)(u+2)}\,du.$$

Finally, the substitution $u+1 = 1/\sqrt{1-x}$ yields

$$\zeta(2,1) = \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{x}\,dx = \zeta(3),$$

by (25). ©

5.8. Complex Line Integrals I

Here we apply the *Mellin inversion formula*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^z \frac{dz}{z} = \begin{cases} 1, & y > 1 \\ 0, & y < 1 \\ \frac{1}{2}, & y = 1 \end{cases}$$

which is valid for fixed $c > 0$.

It follows that if $c > 0$ and $s - 1 > c > 1 - t$ then the *Perron-type formula*

$$\begin{aligned} \zeta(s,t) + \frac{1}{2}\zeta(s+t) & \qquad (37) \\ = \sum_{n=1}^{\infty} n^{-s} \sum_{k=1}^{\infty} k^{-t} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{n}{k}\right)^z \frac{dz}{z} & \\ = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s-z)\zeta(t+z)\,\frac{dz}{z} & \qquad (38) \end{aligned}$$

is valid.

- We have not yet found a way to exploit (38) in proving identities such as (2)

We do note that integrating around the rectangular contour with corners $(\pm c \pm iM)$ and letting $M \to +\infty$, establishes the *stuffle formula* in the form

$$\boxed{\zeta(s,t) + \frac{1}{2}\zeta(s+t) + \zeta(t,s) + \frac{1}{2}\zeta(t+s) = \zeta(s)\zeta(t)}$$

for $s, t > 1 + c$. The right hand side arises as the residue contribution of the integrand at $z = 0$.

- One may also use (38) to establish

$$\begin{aligned}\sum_{s=2}^{\infty}\left[\zeta(s,1) + \tfrac{1}{2}\zeta(s+1)\right] x^{s-1} &= \sum_{n>m>0} \frac{x}{mn(n-x)} \\ &+ \frac{1}{2}\sum_{n=1}^{\infty} \frac{x}{n(n-x)},\end{aligned}$$

but this is easy to prove directly.

5.9. Complex Line Integrals II

Let $\lambda(s) := \sum_{n>0} \lambda_n \, n^{-s}$ be a formal *Dirichlet series* with real coefficients λ_n, and set $s := \sigma + i\,\tau$ with $\sigma = \Re(s) > 0$. We consider the following integral:

$$\iota_\lambda(\sigma) := \int_0^\infty \left|\frac{\lambda(s)}{s}\right|^2 \, d\tau = \frac{1}{2}\int_{-\infty}^\infty \left|\frac{\lambda(s)}{s}\right|^2 \, d\tau, \quad (39)$$

as a function of λ.

A useful variant of the *Mellin inversion formula* is

$$\boxed{\int_{-\infty}^\infty \frac{\cos{(at)}}{t^2+u^2} dt = \frac{\pi}{u}\, e^{-au}} \qquad (40)$$

for $u, a > 0$, as follows by contour integration, from a computer algebra system, or otherwise.

- This leads to:

Theorem 4 *For* $\lambda(s) = \sum_{n=1}^{\infty} \lambda_n \, n^{-s}$ *and* $s = \sigma + i\,\tau$ *with fixed* $\sigma = \Re(s) > 0$ *such that the Dirichlet series is absolutely convergent it is true that*

$$\iota_\lambda(\sigma) = \int_0^\infty \left|\frac{\lambda(s)}{s}\right|^2 d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}} \quad (41)$$

where $\Lambda_n := \sum_{k=1}^{n} \lambda_k$ *and* $\Lambda_0 := 0$*. More generally, for absolutely convergent Dirichlet series* $\alpha(s) := \sum_{n=1}^{\infty} \alpha_n \, n^{-s}$*,* $\beta(s) := \sum_{n=1}^{\infty} \beta_n \, n^{-s}$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\alpha(s)\,\overline{\beta}(s)}{\sigma^2 + \tau^2}\, d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{A_n \overline{B_n} - A_{n-1}\overline{B_{n-1}}}{n^{2\sigma}} \quad (42)$$

in which $A_n = \sum_{k=1}^{n} \alpha_k$ *and* $B_n = \sum_{k=1}^{n} \beta_k$*.*

- Note that the right side of (41) is always a generalized Euler sum.

i. For the Riemann zeta function and for $\sigma > 1$, Theorem 4 applies and yields

$$\frac{\sigma}{\pi}\,\iota_\zeta(\sigma) = \zeta(2\sigma - 1) - \frac{1}{2}\,\zeta(2\sigma),$$

as $\lambda_n = 1$ and $\Lambda_n = n - 1/2$. By contrast it is known that on the critical line

$$\frac{1/2}{\pi}\,\iota_\zeta\left(\frac{1}{2}\right) = \log(\sqrt{2\,\pi}) - \frac{1}{2}\,\gamma.$$

- There are similar formulae for $s \mapsto \zeta(s-k)$ with k integral. For instance, applying (41) with $\zeta_1 := t \mapsto \zeta(t+1)$ yields

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{|\zeta(3/2 + i\tau)|^2}{1/4 + \tau^2}\,d\tau &= \frac{1}{\pi}\,\iota_{\zeta_1}\left(\frac{1}{2}\right)\\ &= 2\,\zeta(2,1) + \zeta(3)\\ &= 3\,\zeta(3),\end{aligned}$$

on using (2).

ii. For the alternating zeta function, $\alpha := s \mapsto (1-2^{1-s})\zeta(s)$, the same approach via (42) produces

$$\frac{1}{\pi}\int_0^\infty \frac{\alpha(3/2+i\tau)\,\overline{\alpha(3/2+i\tau)}}{1/4+\tau^2}\,d\tau \;=\; 2\,\zeta(\overline{2},\overline{1})+\zeta(3) = 3\,\zeta(2)\,\log(2) \;-\; \frac{9}{4}\zeta(3),$$

and

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\alpha(3/2+i\tau)\,\overline{\zeta(3/2+i\tau)}}{1/4+\tau^2}\,d\tau \;=\; \zeta(\overline{2},1)+\zeta(2,\overline{1})+\alpha(3) = \frac{9}{8}\zeta(2)\,\log(2) \;-\; \frac{3}{4}\zeta(3),$$

since as we have seen repeatedly $\zeta(\overline{2},1) = \zeta(3)/8$; while one can show

$$\zeta(2,\overline{1}) = \zeta(3) - \frac{3}{2}\zeta(2)\log(2)$$

and

$$\zeta(\overline{2},\overline{1}) = \frac{3}{2}\zeta(2)\log(2) - \frac{13}{8}\zeta(3).$$

- As in the previous subsection we have not been able to directly obtain (5) or even (2), but we have linked them to quite difficult line integrals.

5.10. Contour Integrals and Residues

Let $\mathcal{C}_n$ $(n \in \mathbb{Z}^+)$ be the square contour with vertices $(\pm 1 \pm i)(n + 1/2)$. Using the asymptotic expansion

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2r z^{2r}}, \qquad |\arg z| < \pi$$

in terms of the *Bernoulli numbers*

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} t^{2r}, \qquad |t| < 2\pi$$

and the identity

$$\psi(z) = \psi(-z) - \frac{1}{z} - \pi \cot \pi z,$$

we can show that for each integer $k \geq 2$,

$$\lim_{n \to \infty} \int_{\mathcal{C}_n} z^{-k} \, \psi^2(-z) \, dz = 0.$$

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Then by the *Cauchy residue theorem*, we obtain:

Theorem 5 *For every integer $k \geq 2$,*

$$2\sum_{n=1}^{\infty} \frac{\psi(n)}{n^k} = k\zeta(k+1) - 2\gamma\zeta(k) - \sum_{j=1}^{k-1} \zeta(j)\zeta(k-j+1),$$

where $\gamma = 0.577215664\ldots$ is Euler's constant.

In light of the identity

$$\psi(n) + \gamma = H_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}, \qquad n \in \mathbb{Z}^+,$$

Theorem 5 is equivalent to (3). The case $k = 2$ thus gives (2). ©

- Flajolet and Salvy develop the residue approach more systematically, and apply it to a number of other Euler sum identities in addition to (3).

6. Witten Zeta-functions

We recall that for $r, s > 1/2$:

$$\mathcal{W}(r,s,t) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^r\, m^s\, (n+m)^t}$$

is a *Witten* ζ*-function*.* Ours are also called *Tornheim double sums*. There is a simple algebraic relation

$$\mathcal{W}(r,s,t) = \mathcal{W}(r-1,s,t+1) + \mathcal{W}(r,s-1,t+1). \quad (43)$$

This is based on writing

$$\frac{m+n}{(m+n)^{t+1}} = \frac{m}{(m+n)^{t+1}} + \frac{n}{(m+n)^{t+1}}.$$

Also

$$\mathcal{W}(r,s,t) = \mathcal{W}(s,r,t), \quad (44)$$

and

$$\mathcal{W}(r,s,0) = \zeta(r)\,\zeta(s) \text{ while } \mathcal{W}(r,0,t) = \zeta(t,r). \quad (45)$$

Hence, $\mathcal{W}(s,s,t) = 2\,\mathcal{W}(s,s-1,t+1)$ and so

$$\boxed{\mathcal{W}(1,1,1) = 2\,\mathcal{W}(1,0,2) = 2\,\zeta(2,1) = 2\,\zeta(3)}$$

*Zagier describes the uses of more general Witten ζ-functions.

Note the analog to (43), viz.

$$\boxed{\zeta(s,t) + \zeta(t,s) = \zeta(s)\,\zeta(t) - \zeta(s+t)}$$

shows $\mathcal{W}(s,0,s) = 2\,\zeta(s,s) = \zeta^2(s) - \zeta(2s)$. Thus, $\mathcal{W}(2,0,2) = 2\,\zeta(2,2) = \pi^4/36 - \pi^4/90 = \pi^4/72$.

- More generally, recursive use of (43) and (44), along with initial conditions (45) shows that *all integer $\mathcal{W}(s,r,t)$ values are expressible in terms of double (and single) Euler sums.*

Again $\Gamma(s)/(m+n)^t = \int_0^1 (-\log\sigma)^{t-1}\,\sigma^{m+n-1}\,d\sigma$ gives

$$\mathcal{W}(r,s,t) = \frac{1}{\Gamma(t)} \int_0^1 \mathrm{Li}_r(\sigma)\,\mathrm{Li}_s(\sigma)\,\frac{(-\log\sigma)^{t-1}}{\sigma}\,d\sigma.$$

For example, we recover an analytic proof of

$$2\,\zeta(2,1) = \mathcal{W}(1,1,1) = \int_0^1 \frac{\ln^2(1-\sigma)}{\sigma}\,d\sigma = 2\,\zeta(3),$$

Indeed S in the telescoping proof of §2.1 is precisely $\mathcal{W}(1,1,1)$. ©

- We may now discover many analytic as opposed to algebraic relations.

For example, integration by parts yields

$$\begin{aligned} \mathcal{W}(r, s+1, 1) \;&+\; \mathcal{W}(r+1, s, 1) \qquad (46) \\ = \mathrm{Li}_{r+1}(1)\,\mathrm{Li}_{s+1}(1) \;&+\; \zeta(r+1)\,\zeta(s+1). \end{aligned}$$

So, in particular, $\mathcal{W}(s+1, s, 1) = \zeta^2(s+1)/2$.

Symbolically, *Maple* immediately evaluates

$$\mathcal{W}(2, 1, 1) = \frac{\pi^4}{72},$$

and while it fails directly with $\mathcal{W}(1, 1, 2)$, we know it must be a multiple of π^4 or equivalently $\zeta(4)$; and numerically obtain

$$\frac{\mathcal{W}(1, 1, 2)}{\zeta(4)} = .4999999999999999999998\ldots$$

6.1. The Hilbert Matrix

Letting $a_n := 1/n^r$ and $b_n := 1/n^s$, inequality (7) of Section 1.3 yields

$$\mathcal{W}(r,s,1) \le \pi \csc\left(\frac{\pi}{p}\right) \sqrt[p]{\zeta(pr)}\, \sqrt[q]{\zeta(qs)}. \qquad (47)$$

Indeed, the constant in (7) is best possible. Set

$$\boxed{\mathcal{R}_p(s) := \frac{\mathcal{W}((p-1)s, s, 1)}{\pi\, \zeta(ps)}}$$

and observe that with

$$\sigma_n^p(s) := \sum_{m=1}^{\infty} (n/m)^{-(p-1)s}/(n+m) \to \pi \csc\left(\frac{\pi}{q}\right),$$

we have

$$\begin{aligned}
\mathcal{L}_p : &= \lim_{s\to 1/p} (ps-1) \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n^{-s}\, m^{-(p-1)s}}{n+m} \\
&= \lim_{s\to 1/p} (ps-1) \sum_{n=1}^{\infty} \frac{1}{n^{ps}}\, \sigma_n^p(s) \\
&= \lim_{s\to 1/p} (ps-1) \sum_{n=1}^{\infty} \frac{\left\{\sigma_n^p(s) - \pi \csc(\pi/q))\right\}}{n^{ps}} \\
&+ \lim_{s\to 1/p} (2s-1)\zeta(ps)\, \pi \csc\left(\frac{\pi}{q}\right) = 0 + \pi \csc\left(\frac{\pi}{q}\right).
\end{aligned}$$

Setting $r := (p-1)s, s \to 1/p^+$ we check that

$$\zeta(ps)^{1/p}\zeta(qr)^{1/q} = \zeta(ps)$$

and hence the best constant in (47) is the one given.

- In terms of the celebrated infinite *Hilbert matrix*

$$\boxed{\mathcal{H}_0 := \{1/(m+n)\}_{m,n=1}^{\infty}}$$

we have actually recovered:

Theorem 6 *Let $1 < p, q < \infty$ have $1/p + 1/q = 1$. The Hilbert matrix $\mathcal{H}_0$ determines a bounded linear mapping on the sequence space ℓ^p with*

$$\|\mathcal{H}_0\|_{p,p} = \lim_{s \to 1/p} \frac{\mathcal{W}(s,(p-1)s,1)}{\zeta(ps)} = \pi \csc\left(\frac{\pi}{p}\right).$$

Proof. Appealing to the isometry between $(\ell^p)^*$ and ℓ^q, and the evaluation $\mathcal{L}_p$ above, we compute the operator norm of $\mathcal{H}_0$ as

$$\begin{aligned} \|\mathcal{H}_0\|_{p,p} &= \sup_{\|x\|_p=1} \|\mathcal{H}_0 x\|_p \\ &= \sup_{\|y\|_q=1} \sup_{\|x\|_p=1} \langle \mathcal{H}_0 x, y\rangle = \pi \csc\left(\frac{\pi}{p}\right). \end{aligned}$$

©

- A delightful operator-theoretic introduction to the Hilbert matrix $\mathcal{H}_0$ is given by Choi in a Chauvenet prize winning article.

One may also study the corresponding behaviour of Hardy's inequality (8).

For example, setting $a_n := 1/n$ in (8) and again denoting $H_n = \sum_{k=1}^n 1/k$ yields

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \zeta(p).$$

Application of the integral test and the evaluation

$$\int_1^{\infty} \left(\frac{\log x}{x}\right)^p dx = \frac{\Gamma(1+p)}{(p-1)^{p+1}},$$

for $p > 1$ easily shows the constant in (8) is again best possible.

7. A Stirling Number Generating Function

- Following Butzer, we begin with integral (25).

In terms of (unsigned) *Stirling numbers of the first kind* (also called Stirling cycle numbers)

$$\boxed{\frac{(-1)^m}{m!}\log^m(1-x) = \sum_{n=0}^{\infty} u(n,m)\frac{x^n}{n!}}$$

for integer $m \geq 0$. This implies

$$\zeta(m+1) = \int_0^1 \left\{\sum_{n=1}^{\infty} u(n,m)\frac{x^n}{n!}\right\}\frac{dx}{x} = \sum_{n=1}^{\infty} \frac{u(n,m)}{n!\,n},$$

for $m > 0$. Telescoping the known recurrence

$$u(n,m) = u(n-1,m-1) + (n-1)u(n-1,m), \quad (48)$$

for $1 \leq m \leq n$ yields

$$u(n,m) = (n-1)!\left\{\delta_{m,1} + \sum_{j=1}^{n-1} \frac{u(j,m-1)}{j!}\right\}. \quad (49)$$

Iterating this gives the representation

$$\boxed{\zeta(m+1) = \zeta(2,\{1\}^{m-1})}$$

for $m \geq 1$, the $m = 2$ case of which is (2). ©

The alternating case begins by writing the recurrence (48) in the form

$$u(n+1,k)+(j-n)u(n,k)=u(n,k-1)+j\,u(n,k).$$

Multiply both sides by $(-1)^{n+k+1}j^{k-m-1}/(j-n)_n$, where $1\le n\le j-1$ and $k,m\in\mathbb{Z}^+$, yielding

$$(-1)^k\left\{\frac{(-1)^{n+1}\,u(n+1,k)}{(j-n)_n}-\frac{(-1)^n\,u(n,k)}{(j-n+1)_{n-1}}\right\}j^{k-m-1}$$
$$=\frac{(-1)^n}{(j-n)_n}\left\{(-1)^{k-1}\,u(n,k-1)j^{k-m-1}-(-1)^k\,u(n,k)j^{k-m}\right\}.$$

Now sum on $1\le k\le m$ and $1\le n\le j-1$, obtaining

$$\sum_{k=1}^{m}\frac{(-1)^{k+j}\,u(j,k)}{j!\,j^{m-k}}-\frac{1}{j^m}=\frac{(-1)^{m+1}}{(j-1)!}\sum_{n=m}^{j-1}(-1)^n(j-n-1)!\,u(n,m).$$

Finally, sum on $j\in\mathbb{Z}^+$ to obtain

$$\zeta(m)=\sum_{k=1}^{m}\sum_{j=k}^{\infty}\frac{(-1)^{k+j}\,u(j,k)}{j!\,j^{m-k}}$$
$$=\sum_{n=m}^{\infty}(-1)^{n+m}\,u(n,m)\sum_{j=n+1}^{\infty}\frac{(j-1-n)!}{(j-1)!}.$$

Noting that

$$\sum_{j=n+1}^{\infty} \frac{(j-1-n)!}{(j-1)!} = \sum_{k=0}^{\infty} \frac{k!}{(k+n)!}$$

$$= \frac{1}{n!}\, {}_2F_1(1,1;n+1;1) = \frac{1}{(n-1)!\,(n-1)},$$

we find that

$$\zeta(m) = \sum_{k=1}^{m} \sum_{j=k}^{\infty} \frac{(-1)^{j+k}\, u(j,k)}{j!\, j^{m-k}} + \sum_{n=m}^{\infty} \frac{(-1)^{n+m}\, u(n,m)}{(n-1)!\,(n-1)}.$$

Now employ the recurrence (48) again to get

$$\begin{aligned}
\zeta(m) &= \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k}\, u(j,k)}{j!\, j^{m-k}} \\
&\quad + \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1}\, u(j,m-1)}{j!\, j} + \sum_{j=m}^{\infty} \frac{(-1)^{j+m}\, u(j,m)}{j!} \\
&\quad + \sum_{n=m}^{\infty} \frac{(-1)^{n+m}\, u(n-1,m)}{(n-1)!} \qquad (50) \\
&\quad + \sum_{n=m}^{\infty} \frac{(-1)^{n+m}\, u(n-1,m-1)}{(n-1)!\,(n-1)} \\
&= \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k}\, u(j,k)}{j!\, j^{m-k}} + 2 \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1}\, u(j,m-1)}{j!\, j}.
\end{aligned}$$

Using (49) again, we find that the case $m = 3$ gives

$$\begin{aligned}
\zeta(3) &= \sum_{j=1}^{\infty} \frac{(-1)^j\, u(j,1)}{j!\, j^2} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j\, u(j,2)}{j!\, j} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j}{j!\, j}(j-1)! \sum_{k=1}^{j-1} \frac{u(k,1)}{k!} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j}{j^2} \sum_{k=1}^{j-1} \frac{1}{k} \\
&= 2\zeta(\overline{2},1) - \zeta(\overline{3}),
\end{aligned}$$

which easily rearranges to give (14), shown by telescoping to be trivially equivalent to (5). ©

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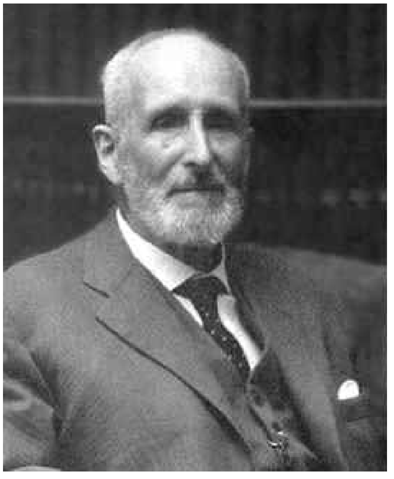

8. Polylogarithm Identities

8.1. Dilogarithm and Trilogarithm

Consider the power series

$$J(x) := \zeta_x(2,1) = \sum_{n>k>0} \frac{x^n}{n^2 k}, \qquad 0 \le x \le 1.$$

In light of (72), we have

$$J(x) = \int_0^x \frac{dt}{t} \int_0^t \frac{du}{1-u} \int_0^v \frac{dv}{1-v} = \int_0^x \frac{\log^2(1-t)}{2t}\, dt.$$

Maple readily evaluates

$$\int_0^x \frac{\log^2(1-t)}{2t}\, dt = \zeta(3) + \frac{1}{2}\log^2(1-x)\log(x) + \log(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x) \tag{51}$$

where $\mathrm{Li}_s(x) := \sum_{n=1}^\infty x^n/n^s$ is the classical *polylogarithm.*

- One can also verify (51) by differentiating both sides by hand, and checking that (51) holds as $x \to 0+$.

Thus,

$$J(x) = \zeta(3) + \frac{1}{2}\log^2(1-x)\log(x) + \log(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x).$$

Letting $x \to 1-$ gives (2) again. ©

In *Ramanujan's Notebooks*, we also find that

$$J(-z) + J(-1/z) = -\tfrac{1}{6}\log^3 z - \mathrm{Li}_2(-z)\log z + \mathrm{Li}_3(-z) + \zeta(3) \tag{52}$$

and

$$J(1-z) = \tfrac{1}{2}\log^2 z\log(z-1) - \tfrac{1}{3}\log^3 z - \mathrm{Li}_2(1/z)\log z - \mathrm{Li}_3(1/z) + \zeta(3). \tag{53}$$

Putting $z = 1$ in (52) and employing the well-known *dilogarithm evaluation*

$$\mathrm{Li}_2(-1) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

gives (5). ©

Putting $z = 2$ in (53) and employing Euler's *dilogarithm evaluation*

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2\, 2^n} = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$$

along with Landen's *trilogarithm evaluation* (see Lewin)

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3\, 2^n} = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3 2$$

gives (5) again. ©

- These evaluations follow from

  $$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)-\log(x)\log(1-x) \equiv C = \mathrm{Li}_2(1)$$

  and a similar trilogarithmic identity.

Alternatively, differentiation shows that

$$J(-x) = -J(x) + \frac{1}{4}J(x^2) + J\left(\frac{2x}{x+1}\right) - \frac{1}{8}J\left(\frac{4x}{(x+1)^2}\right). \tag{54}$$

Putting $x = 1$ gives $8J(-1) = J(1)$ immediately, i.e. (5). ©

- Once the component functions in (54) are known, the coefficients can be deduced by computing each term to high precision with a common transcendental value of x and employing a linear relations finding algorithm.

We note next a harder (now guided) but more human method for arriving at (54). First, as in the section on alternating iterated integrals one determines the fundamental transformations (32)–(36).

This is not so difficult but is lengthy. Performing these transformations on the function $J(x)$, one finds that

$$J(x) = \int_0^x ab^2, \quad J\left(\frac{2x}{1+x}\right) = \int_0^x (a+c)(b-c)^2,$$
$$J(-x) = \int_0^x ac^2, \quad J(x^2) = \int_0^x 2a(b+c)^2,$$
$$J\left(\frac{4x}{(1+x)^2}\right) = \int_0^x (a+2c)4(b-c)^2.$$

We seek rationals r_1, r_2, r_3 and r_4 such that

$$ac^2 = r_1 ab^2 + 2r_2\, a(b+c)^2 + r_3(a+c)(b-c)^2 + r_4(a+2c)4(b-c)^2$$

is an identity in the non-commutative ring $\mathbb{Q}\langle a, b, c\rangle$.

Finding such rationals reduces to solving a finite set of linear equations: comparing coefficients of ab^2 tells us that $r_1 + 2r_2 + r_3 + 4r_4 = 0$.

Coefficients of other monomials give us additional equations, and we readily find that $r_1 = -1$, $r_2 = 1/4$, $r_3 = 1$ and $r_4 = -1/8$, thus proving (54) as expected. ©

8.2. Convolution of Polylogarithms

For real $0 < x < 1$ and integers s and t, consider

$$T_{s,t}(x) := \sum_{\substack{m,n=1\\m\neq n}}^{\infty} \frac{x^{n+m}}{n^s\, m^t (m-n)} = \sum_{\substack{m,n=1\\m\neq n}}^{\infty} \frac{x^{n+m}(m-n+n)}{n^s\, m^{t+1}(m-n)}$$

$$= \sum_{\substack{m,n=1\\m\neq n}}^{\infty} \frac{x^{n+m}}{n^s\, m^{t+1}} + \sum_{\substack{m,n=1\\m\neq n}}^{\infty} \frac{x^{n+m}}{n^{s-1}\, m^{t+1}(m-n)}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m=1}^{\infty} \left(\frac{x^m}{m^{t+1}} - \frac{x^n}{n^{t+1}} \right) + T_{s-1,t+1}(x)$$

$$= \mathrm{Li}_s(x)\mathrm{Li}_{t+1}(x) - \mathrm{Li}_{s+t+1}(x^2) + T_{s-1,t+1}(x).$$

Telescoping this gives

$$T_{s,t}(x) = T_{0,s+t}(x) - s\,\mathrm{Li}_{s+t+1}(x^2) + \sum_{j=1}^{s} \mathrm{Li}_j(x)\mathrm{Li}_{s+t+1-j}(x)$$

for $s \in \mathbb{N}$. With $t = 0$, this becomes

$$T_{s,0}(x) = T_{0,s}(x) - s\,\mathrm{Li}_{s+1}(x^2) + \sum_{j=1}^{s} \mathrm{Li}_j(x)\mathrm{Li}_{s+1-j}(x).$$

But for any integers s and t, there holds

$$T_{s,t}(x) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^{n+m}}{n^t m^s (m-n)}$$

$$= -\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^{n+m}}{m^s n^t (n-m)} = -T_{s,t}(x).$$

Therefore,

$$T_{s,0}(x) = \frac{1}{2}\sum_{j=1}^{s} \mathrm{Li}_j(x)\mathrm{Li}_{s+1-j}(x) - \frac{s}{2}\mathrm{Li}_{s+1}(x^2). \tag{55}$$

On the other hand,

$$T_{s,0}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{\substack{m=1\\ m\neq n}}^{\infty} \frac{x^m}{m-n}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{n^s} \sum_{m=n+1}^{\infty} \frac{x^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m=1}^{n-1} \frac{x^m}{n-m}$$

$$= \mathrm{Li}_s(x^2)\mathrm{Li}_1(x) - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j}. \tag{56}$$

- Comparing (56) with (55) gives

$$\sum_{n=1}^{\infty}\frac{x^n}{n^s}\sum_{j=1}^{n-1}\frac{x^{n-j}}{j}=\frac{s}{2}\,\mathrm{Li}_{s+1}(x^2)$$

$$-\left[\mathrm{Li}_s(x)-\mathrm{Li}_s(x^2)\right]\mathrm{Li}_1(x)-\frac{1}{2}\sum_{j=2}^{s-1}\mathrm{Li}_j(x)\mathrm{Li}_{s+1-j}(x), \tag{57}$$

where we now require $s > 2$ because the terms $j = 1$ and $j = s$ in (55) were separated, and assumed to be distinct.

Now for $n \in \mathbb{N}$ and $0 < x < 1$, $1-x^n = (1-x)\sum_{j=0}^{n-1} x^j < (1-x)n$. Thus, if $2 \le s \in \mathbb{N}$ and $0 < x < 1$, then

$$0 < \left[\mathrm{Li}_s(x)-\mathrm{Li}_s(x^2)\right]\mathrm{Li}_1(x) = \mathrm{Li}_1(x)\sum_{n=1}^{\infty}\frac{x^n(1-x^n)}{n^s}$$

$$< (1-x)\mathrm{Li}_1(x)\sum_{n=1}^{\infty}\frac{x^n}{n^{s-1}} < (1-x)\log^2(1-x) \to 0,$$

as $x \to 1-$, so the limit in (57) gives

$$\zeta(s,1)=\frac{1}{2}\,s\,\zeta(s+1)-\frac{1}{2}\sum_{j=1}^{s-2}\zeta(j+1)\zeta(s-j),$$

which is (3). ©

9. Fourier Series

The Fourier expansions

$$\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = \frac{\pi - t}{2} \text{ and } \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} = -\log|2\sin(t/2)|$$

are both valid in the open interval $0 < t < 2\pi$. Multiplying these together, simplifying, and using a partial fraction decomposition gives

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \sum_{k=1}^{n-1} \frac{1}{k} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \sum_{k=1}^{n-1} \left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= \frac{1}{2} \sum_{n>k>0} \frac{\sin(nt)}{k(n-k)} \\
&= \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\sin(m+n)t}{mn} \\
&= \sum_{m,n=1}^{\infty} \frac{\sin(mt)\cos(nt)}{mn} \\
&= -\frac{\pi - t}{2} \log|2\sin(t/2)| \qquad (58)
\end{aligned}$$

again for $0 < t < 2\pi$.

- Integrating (58) term by term yields

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \zeta(2,1) + \frac{1}{2}\int_0^{\theta} (\pi - t) \log|2\sin(t/2)|\, dt \tag{59}$$

valid for $0 \le \theta \le 2\pi$. Likewise for $0 \le \theta \le 2\pi$,

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^3} = \zeta(3) + \int_0^{\theta} (\theta - t) \log|2\sin(t/2)|\ dt. \tag{60}$$

Setting $\theta = \pi$ in (59) and (60) produces

$$\begin{aligned} \zeta(2,1) - \zeta(\overline{2},1) &= -\frac{1}{2}\int_0^{\pi} (\pi - t) \log|2\sin(t/2)|\ dt \\ &= \frac{\zeta(3) - \zeta(\overline{3})}{2}. \end{aligned}$$

In light of (2), this implies

$$\begin{aligned} \zeta(\overline{2},1) = \frac{\zeta(3) + \zeta(\overline{3})}{2} &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^3} \\ &= \sum_{m=1}^{\infty} \frac{1}{(2m)^3} = \frac{1}{8}\zeta(3), \end{aligned}$$

which is (5). ©

- Much more follows. For example:

Applying Parseval's equation to (58) leads to the integral evaluation

$$\frac{1}{4\pi}\int_0^{2\pi}(\pi - t)^2 \log^2(2\sin(t/2))\,dt = \sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2} = \frac{11}{4}\zeta(4).$$

- A reason for valuing such integral representations is that they are frequently easier to use numerically than the corresponding sums.

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10. Further Generating Functions

10.1. Hypergeometric Functions

First, $\zeta(2,1)$ is the coefficient of xy^2 in

$$G(x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^{m+1}y^{n+1}\zeta(m+2,\{1\}^n)$$
$$= y\sum_{m=0}^{\infty} x^{m+1}\sum_{k=1}^{\infty}\frac{1}{k^{m+2}}\prod_{j=1}^{k-1}\left(1+\frac{y}{j}\right) \quad (61)$$

Thus,

$$\frac{y}{k}\prod_{j=1}^{k-1}\left(1+\frac{y}{j}\right) = \frac{(y)_k}{k!}$$

where $(y)_k := y(y+1)\cdots(y+k-1)$ is the *rising factorial*. Substituting this into (61), interchanging order of summation, and summing the resulting geometric series yields the *hypergeometric series*

$$G(x,y) = \sum_{k=1}^{\infty}\frac{(y)_k}{k!}\left(\frac{x}{k-x}\right)$$
$$= -\sum_{k=1}^{\infty}\frac{(y)_k(-x)_k}{k!(1-x)_k} = 1 - {}_2F_1\left(\begin{matrix} y, -x \\ 1-x \end{matrix}\,\middle|\, 1\right).$$

Now *Gauss's summation theorem* for the hypergeometric function and the series expansion for the logarithmic derivative of the gamma function yield

$$
{}_2F_1\left(\begin{matrix} y, -x \\ 1-x \end{matrix} \middle| 1\right) = \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}
$$

$$
= \exp\left\{\sum_{k=2}^{\infty}\left(x^k + y^k - (x+y)^k\right)\frac{\zeta(k)}{k}\right\}.
$$

So, we obtain the generating function equality

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x^{m+1}y^{n+1}\zeta(m+2,\{1\}^n) \tag{62}
$$

$$
= 1 - \exp\left\{\sum_{k=2}^{\infty}\left(x^k + y^k - (x+y)^k\right)\frac{\zeta(k)}{k}\right\}.
$$

Extracting coefficients of xy^2 from both sides of (62) yields (2). ©

Generalization (3) can be similarly derived: extract the coefficient of $x^{m-1}y^2$ in (62). Also, the coefficients of xy^2 in *Kummer's summation theorem*

$$
\boxed{{}_2F_1\left(\begin{matrix} x, y \\ 1+x-y \end{matrix} \middle| -1\right) = \frac{\Gamma(1+x/2)\Gamma(1+x-y)}{\Gamma(1+x)\Gamma(1+x/2-y)}}
$$

yields (5). ©

10.2. A Generating Function for Sums

Identity (2) also follows by setting $x = 0$ in the next unlikely identity:

Theorem 7 *If* x *is any complex number not equal to a positive integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m-x} = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}.$$

Proof. Fix $x \in \mathbb{C} \setminus \mathbb{Z}^+$. Let S_x denote the left hand side. By partial fractions,

$$\begin{aligned} S_x &= \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \left(\frac{1}{n(n-m)(m-x)} - \frac{1}{n(n-m)(n-x)} \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m-x} \sum_{n=m+1}^{\infty} \frac{1}{n(n-m)} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{n-m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=m+1}^{\infty} \left(\frac{1}{n-m} - \frac{1}{n} \right) - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}. \end{aligned}$$

Now for fixed $m \in \mathbb{Z}^+$,

$$\sum_{n=m+1}^{\infty}\left(\frac{1}{n-m}-\frac{1}{n}\right) = \lim_{N\to\infty}\sum_{n=m+1}^{N}\left(\frac{1}{n-m}-\frac{1}{n}\right)$$
$$= \sum_{n=1}^{m}\frac{1}{n} - \lim_{N\to\infty}\sum_{n=1}^{m}\frac{1}{N-n+1}$$
$$= \sum_{n=1}^{m}\frac{1}{n},$$

since m is fixed. Therefore, we have

$$S_x = \sum_{m=1}^{\infty}\frac{1}{m(m-x)}\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n(n-x)}\sum_{m=1}^{n-1}\frac{1}{m}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n(n-x)}\left(\sum_{m=1}^{n}\frac{1}{m} - \sum_{m=1}^{n-1}\frac{1}{m}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2(n-x)}.$$

©

Theorem 7 is equivalent to the *Ohno-Granville sum formula*

$$\sum_{\substack{\sum a_i = s \\ a_i \ge 0}} \zeta(a_1+2, a_2+1, \ldots, a_r+1) = \zeta(r+s+1), \quad (63)$$

valid for all integers $s \ge 0$, $r \ge 1$.

10.3. An Alternating Generating Function

An alternating counterpart to Theorem 7 is:

Theorem 8 *For all non-integer* x

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-x^2}\left\{H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2-x^2)}\right\}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-x^2}\left\{\psi(n)-\psi(x)-\frac{\pi}{2}\cot(\pi x)-\frac{1}{2x}\right\}$$

$$= \sum_{o>0 \text{ odd}}^{\infty} \frac{1}{o\,(o^2-x^2)} + \sum_{n=1}^{\infty} \frac{(-1)^n\, n}{(n^2-x^2)^2}$$

$$= \sum_{e>0 \text{ even}}^{\infty} \frac{e}{(x^2-e^2)^2} - x^2 \sum_{o>0 \text{ odd}}^{\infty} \frac{1}{o\,(x^2-o^2)^2}.$$

Setting $x = 0$ reproduces (5) in the form

$$\zeta(\overline{2},1) = \sum_{n>0}^{\infty} \frac{1}{(2n)^3}.$$

We record that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} = \frac{1}{2x^2} - \frac{\pi}{2x \sin(\pi x)},$$

while

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ \psi(n) - \psi(x) - \frac{\pi}{2} \cot(\pi x) - \frac{1}{2x} \right\}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 - x^2)} \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)((2n-1)^2 - x^2)} + \sum_{n=1}^{\infty} \frac{n(-1)^n}{(n^2 - x^2)^2}.$$

D&R

10.4. The Digamma Function

Define an auxiliary function Λ by

$$x\Lambda(x) := \tfrac{1}{2}\psi'(1-x) - \tfrac{1}{2}\left(\psi(1-x)+\gamma\right)^2 - \tfrac{1}{2}\zeta(2). \tag{64}$$

It is easy to verify that

$$\begin{aligned} \psi(1-x)+\gamma &= \sum_{n=1}^{\infty} \frac{x}{n(x-n)}, \\ \psi'(1-x) - \zeta(2) &= \sum_{n=1}^{\infty} \left(\frac{1}{(x-n)^2} - \frac{1}{n^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{2nx - x^2}{n^2(n-x)^2}, \end{aligned} \tag{65}$$

and

$$\sum_{n=0}^{\infty} \zeta(n+2,1)x^n = \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}.$$

Hence,

$$\Lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m(m-x)}.$$

Now,

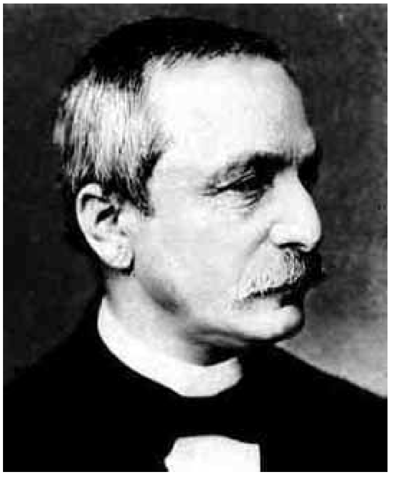
$$\sum_{n=1}^{\infty} \frac{1}{n^2(n-x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m(m-x)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}$$

is directly equivalent to Theorem 7, and we have proven

$$\Lambda(x) = \sum_{n=0}^{\infty} \zeta(n+2,1)\, x^n,$$

so that comparing coefficients yields yet another proof of Euler's reduction (3).

In particular, setting $x = 0$ again produces (2). ©



K&K

10.5. The Beta Function

The *beta function* is defined for positive real x and y by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We begin with the following easily obtained generating function:

$$\boxed{\sum_{n=1}^{\infty} t^n H_n = -\frac{\log(1-t)}{1-t}}$$

For $m \geq 2$, the Laplace integral (24) now gives

$$\begin{aligned}\zeta(m,1) &= \frac{(-1)^m}{(m-1)!}\int_0^1 \frac{\log^{m-1}(t)\log(1-t)}{1-t}dt \\ &= \frac{(m-1)(-1)^m}{2(m-1)!}\int_0^1 \log^{m-2}(t)\log^2(1-t)\,\frac{dt}{t} \\ &= \frac{(-1)^m}{2(m-2)!}\,b_1^{(m-2)}(0), \qquad (66)\end{aligned}$$

where in terms of (64)

$$b_1(x) := \frac{\partial^2}{\partial y^2}B(x,y)\bigg|_{y=1} = 2\Lambda(-x).$$

Since

$$\frac{\partial^2}{\partial y^2}B(x,y) = B(x,y) \times \left[(\psi(y)-\psi(x+y))^2+\psi'(y)-\psi'(x+y)\right],$$

we derive

$$b_1(x) = \frac{(\psi(1)-\psi(x+1))^2+\psi'(1)-\psi'(x+1)}{x}.$$

Now observe that from (66),

$$\zeta(2,1) = \frac{1}{2}b_1(0) = \lim_{x\downarrow 0}\frac{(-\gamma-\psi(x+1))^2}{2x} - \lim_{x\downarrow 0}\frac{\psi'(x+1)-\psi'(1)}{2x} = -\frac{1}{2}\psi''(1) = \zeta(3).$$

©

- (66) is well suited to symbolic computation

We also note the pleasing identity

$$\boxed{\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \psi^2(x)} \tag{67}$$

In some informal sense (67) generates (3), but we have been unable to make this sense precise.

Continuing, from the following two identities, cognate to (65)

$$(-\gamma-\psi(x+1))^2 = \left(\sum_{m=1}^{\infty}(-1)^m\zeta(m+1)\,x^m\right)^2$$
$$= \sum_{m=1}^{\infty}(-1)^m\sum_{k=1}^{m-1}\zeta(k+1)\zeta(m-k+1)\,x^m,$$
$$\zeta(2)-\psi'(x+1) = \sum_{m=1}^{\infty}(-1)^{m+1}(m+1)\zeta(m+2)\,x^m,$$

we get

$$b_1(x) = 2\sum_{m=2}^{\infty}(-1)^m\zeta(m,1)\,x^{m-2}$$
$$= \sum_{m=2}^{\infty}\frac{b_1^{(m-2)}(0)}{(m-2)!}\,x^{m-2} = \sum_{m=1}^{\infty}(-1)^{m-1}$$
$$\left((m+1)\zeta(m+2)-\sum_{k=1}^{m-1}\zeta(k+1)\zeta(m-k+1)\right)x^{m-1},$$

from which Euler's reduction (3) follows—indeed this is close to Euler's original path.

11. A Decomposition Formula of Euler

For positive integers s and t and distinct non-zero real numbers α and x, the partial fraction expansion

$$\frac{1}{x^s(x-\alpha)^t} = (-1)^t \sum_{r=0}^{s-1} \binom{t+r-1}{t-1} \frac{1}{x^{s-r}\alpha^{t+r}} + \sum_{r=0}^{t-1} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}} \qquad (68)$$

implies *Euler's decomposition formula*

$$\zeta(s,t) = (-1)^t \sum_{r=0}^{s-2} \binom{t+r-1}{t-1} \zeta(s-r,t+r) + \sum_{r=0}^{t-2} (-1)^r \binom{s+r-1}{s-1} \zeta(t-r)\zeta(s+r) - (-1)^t \binom{s+t-2}{s-1} \Big\{ \zeta(s+t) + \zeta(s+t-1,1) \Big\}. \qquad (69)$$

The depth-2 sum formula (18) is obtained by setting $t=1$ in (69).

If we also set $s=2$, the identity (2) results. ©

To derive (69) **from** (68), separate the last terms of sums on the right side of (68), obtaining

$$\frac{1}{x^s(x-\alpha)^t} = (-1)^t \sum_{r=0}^{s-2} \binom{t+r-1}{t-1} \frac{1}{x^{s-r}\alpha^{t+r}} + \sum_{r=0}^{t-2} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}} - (-1)^t \binom{s+t-2}{s-1} \frac{1}{\alpha^{s+t-1}} \left(\frac{1}{x-\alpha} - \frac{1}{x} \right).$$

Now sum over all integers $0 < \alpha < x < \infty$. ©

- Nielsen states (68) which Markett proves by induction. One may directly expand the left side into partial fractions using residue calculus.

Alternatively, apply the partial derivative operator

$$\frac{1}{(s-1)!}\left(-\frac{\partial}{\partial x}\right)^{s-1} \frac{1}{(t-1)!}\left(-\frac{\partial}{\partial y}\right)^{t-1}$$

to the identity

$$\frac{1}{xy} = \frac{1}{(x+y)x} + \frac{1}{(x+y)y},$$

and then set $y = \alpha - x$ to obtain (68).

12. Equating Shuffles and Stuffles

We begin with an informal argument. By the *stuffle multiplication* rule ('sum')

$$\zeta(2)\zeta(1) = \zeta(2,1) + \zeta(1,2) + \zeta(3). \tag{70}$$

On the other hand, the *shuffle multiplication* rule ('integral') gives $ab \sqcup b = 2abb + bab$, whence

$$\zeta(2)\zeta(1) = 2\zeta(2,1) + \zeta(1,2). \tag{71}$$

The identity (2) now follows immediately on subtracting (70) from (71). ©

- This needs help—we cancelled divergent series.

To make the argument rigorous, we introduce the *multiple polylogarithm*. For $0 \le x \le 1$ and positive integers $s_1, \ldots, s_k$ with $x = s_1 = 1$ excluded for convergence, define

$$\begin{aligned} \zeta_x(s_1, \ldots, s_k) \;&:=\; \sum_{n_1 > \cdots > n_k > 0} x^{n_1} \prod_{j=1}^{k} n_j^{-s_j} \qquad (72) \\ &= \int \prod_{j=1}^{k} \left(\prod_{r=1}^{s_j - 1} \frac{dt_r^{(j)}}{t_r^{(j)}} \right) \frac{dt_{s_j}^{(j)}}{1 - t_{s_j}^{(j)}}, \end{aligned}$$

where the integral is over the simplex

$$x > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(k)} > \cdots > t_{s_k}^{(k)} > 0,$$

and is abbreviated by

$$\int_0^x \prod_{j=1}^{k} a^{s_j - 1} b, \qquad a = \frac{dt}{t}, \quad b = \frac{dt}{1-t}. \tag{73}$$

Then

$$\begin{aligned} \zeta(2)\zeta_x(1) &= \sum_{n>0} \frac{1}{n^2} \sum_{k>0} \frac{x^k}{k} \\ &= \sum_{n>k>0} \frac{x^k}{n^2 k} + \sum_{k>n>0} \frac{x^k}{kn^2} + \sum_{k>0} \frac{x^k}{k^3}, \end{aligned}$$

and

$$\zeta_x(2)\zeta_x(1) = \int_0^x ab \int_0^x b = \int_0^x (2abb + bab)$$

$$= 2\zeta_x(2,1) + \zeta_x(1,2).$$

Subtracting the two equations gives

$$\Big[\zeta(2) - \zeta_x(2)\Big]\zeta_x(1) = \zeta_x(3) - \zeta_x(2,1) + \sum_{n>k>0} \frac{x^k - x^n}{n^2 k}.$$

We now take the limit as $x \to 1-$. Uniform convergence implies the right hand side tends to $\zeta(3) - \zeta(2,1)$. That the left hand side tends to zero follows immediately from the inequalities

$$\begin{aligned}
0 &\le x\Big[\zeta(2) - \zeta_x(2)\Big]\zeta_x(1) \\
&= x\int_x^1 \log(1-t)\log(1-x)\frac{dt}{t} \\
&\le \int_x^1 \log^2(1-t)\,dt \\
&= (1-x)\left\{1 + (1-\log(1-x))^2\right\}.
\end{aligned}$$

©

The alternating case (5) is actually easier in this approach, since the divergent sum $\zeta(1)$ is replaced by the conditionally convergent sum $\zeta(\overline{1}) = -\log 2$. By the stuffle multiplication rule,

$$\zeta(\overline{2})\zeta(\overline{1}) = \zeta(\overline{2},\overline{1}) + \zeta(\overline{1},\overline{2}) + \zeta(3), \tag{74}$$

$$\zeta(2)\zeta(\overline{1}) = \zeta(2,\overline{1}) + \zeta(\overline{1},2) + \zeta(\overline{3}). \tag{75}$$

On the other hand, the shuffle multiplication rule gives $ac ⧢ c = 2ac^2 + cac$ and $ab ⧢ c = abc + acb + cab$, whence

$$\zeta(\overline{2})\zeta(\overline{1}) = 2\zeta(\overline{2}, 1) + \zeta(\overline{1}, 2), \tag{76}$$

$$\zeta(2)\zeta(\overline{1}) = \zeta(2, \overline{1}) + \zeta(\overline{2}, \overline{1}) + \zeta(\overline{1}, \overline{2}). \tag{77}$$

Comparing (74) with (76) and (75) with (77) yields the two equations

$$\zeta(\overline{2}, \overline{1}) = \zeta(\overline{1}, 2) + 2\zeta(\overline{2}, 1) - \zeta(\overline{1}, \overline{2}) - \zeta(3),$$
$$\zeta(\overline{2}, \overline{1}) = \zeta(\overline{1}, 2) - \zeta(\overline{1}, \overline{2}) + \zeta(\overline{3}).$$

Subtracting the latter two equations yields

$$2\zeta(\overline{2}, 1) = \zeta(3) + \zeta(\overline{3}),$$

i.e. (14), which was shown to be trivially equivalent to (5) in by telescoping. ©

13. Conclusion

There are doubtless other roads to Rome, and as indicated in the introduction we should like to learn of them.

We finish with the three open questions we are most desirous of answers to.

- A truly combinatorial proof of $\zeta(2,1) = \zeta(3)$.

- A direct proof that the appropriate line integrals in Sections 5.8 and 5.9 evaluate to the appropriate multiples of $\zeta(3)$.

- An analytic or combinatoric proof of (6), or of at least some additional cases of it, say $n = 2, 3$ of:

$$8^n \zeta(\{\overline{2}, 1\}^n) \stackrel{?}{=} \zeta(\{3\}^n), \qquad n \in \mathbb{Z}^+. \tag{78}$$

D. ZAGIER'S CONJECTURE

For $r \geq 1$ and $n_1, \ldots, n_r \geq 1$, consider:

$$L(n_1, \ldots, n_r; x) := \sum_{0<m_r<\ldots<m_1} \frac{x^{m_1}}{m_1^{n_1} \ldots m_r^{n_r}} .$$

Thus

$$L(n; x) = \frac{x}{1^n} + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \cdots$$

is the classical *polylogarithm*, while

$$\begin{aligned} L(n, m; x) &= \frac{1}{1^m} \frac{x^2}{2^n} + (\frac{1}{1^m} + \frac{1}{2^m}) \frac{x^3}{3^n} + (\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m}) \frac{x^4}{4^n} \\ &+ \cdots , \\ L(n, m, l; x) &= \frac{1}{1^l} \frac{1}{2^m} \frac{x^3}{3^n} + (\frac{1}{1^l} \frac{1}{2^m} + \frac{1}{1^l} \frac{1}{3^m} + \frac{1}{2^l} \frac{1}{3^m}) \frac{x^4}{4^n} + \cdots . \end{aligned}$$

- The series converge absolutely for $|x| < 1$ and conditionally on $|x| = 1$ unless $n_1 = x = 1$.

These polylogarithms

$$L(n_r, \ldots, n_1; x) = \sum_{0<m_1<\ldots<m_r} \frac{x^{m_r}}{m_r^{n_r} \ldots m_1^{n_1}} ,$$

are *determined uniquely* by the **differential equations**

$$\frac{d}{dx} L(\mathbf{n_r}, \ldots, n_1; x) = \frac{1}{x} L(\mathbf{n_r - 1}, \ldots, n_2, n_1; x)$$

if $n_r \geq 2$ and

$$\frac{d}{dx} L(\mathbf{n_r}, \ldots, n_2, n_1; x) = \frac{1}{1-x} L(\mathbf{n_{r-1}}, \ldots, n_1; x)$$

if $n_r = 1$ with the *initial conditions*

$$L(n_r, \ldots, n_1; 0) = 0$$

for $r \geq 1$ and

$$L(\emptyset; x) \equiv 1.$$

Set $\overline{s} := (s_1, s_2, \ldots, s_N)$. Let $\{\overline{s}\}_n$ denotes concatenation, and $w := \sum s_i$.

Then every *periodic* polylogarithm leads to a function

$$L_{\overline{s}}(x,t) := \sum_n L(\{\overline{s}\}_n; x) t^{wn}$$

which solves an algebraic ordinary differential equation in x, and leads to nice *recurrences*.

A. In the simplest case, with $N = 1$, the ODE is $\mathbf{D_s F = t^s F}$ where

$$D_s := \left((1-x)\frac{d}{dx}\right)^1 \left(x\frac{d}{dx}\right)^{s-1}$$

and the solution (by series) is a generalized hypergeometric function:

$$L_{\overline{s}}(x,t) = 1 + \sum_{n \geq 1} x^n \frac{t^s}{n^s} \prod_{k=1}^{n-1} \left(1 + \frac{t^s}{k^s}\right),$$

as follows from considering $D_s(x^n)$.

B. Similarly, for $N = 1$ and negative integers

$$L_{\overline{-s}}(x,t) := 1 + \sum_{n\geq 1} (-x)^n \frac{t^s}{n^s} \prod_{k=1}^{n-1} \left(1 + (-1)^k \frac{t^s}{k^s}\right),$$

and $L_{\overline{-1}}(2x-1,t)$ solves a hypergeometric ODE.

▶ Indeed

$$L_{\overline{-1}}(1,t) = \frac{1}{\beta(1+\frac{t}{2}, \frac{1}{2} - \frac{t}{2})}.$$

C. We may obtain ODEs for eventually periodic Euler sums. Thus, $L_{-2,1}(x,t)$ is a solution of

$$\begin{aligned} t^6\,F &= x^2(x-1)^2(x+1)^2\,D^6F \\ &+ x(x-1)(x+1)(15x^2-6x-7)\,D^5F \\ &+ (x-1)(65x^3+14x^2-41x-8)\,D^4F \\ &+ (x-1)(90x^2-11x-27)\,D^3F \\ &+ (x-1)(31x-10)\,D^2F + (x-1)\,DF. \end{aligned}$$

- This leads to a four-term recursion for $F = \sum c_n(t)x^n$ with initial values $c_0 = 1, c_1 = 0, c_2 = t^3/4, c_3 = -t^3/6$, and the ODE can be simplified.

We are now ready to prove Zagier's conjecture. Let $F(a, b; c; x)$ denote the *hypergeometric function*. Then:

Theorem 1 (BBGL) *For* $|x|, |t| < 1$ *and integer* $n \geq 1$

$$\begin{aligned} \sum_{n=0}^{\infty} & L(\underbrace{3, 1, 3, 1, \ldots, 3, 1}_{n-fold}; x)\, t^{4n} \\ = & F\left(\frac{t(1+i)}{2}, \frac{-t(1+i)}{2}; 1; x\right) \\ \times & F\left(\frac{t(1-i)}{2}, \frac{-t(1-i)}{2}; 1; x\right). \end{aligned} \tag{17}$$

Proof. Both sides of the putative identity start

$$1+\frac{t^4}{8}x^2+\frac{t^4}{18}x^3+\frac{t^8+44t^4}{1536}x^4+\cdots$$

and are *annihilated* by the differential operator

$$D_{31} := \left((1-x)\,\frac{d}{dx}\right)^2\left(x\,\frac{d}{dx}\right)^2 - t^4\,.$$

QED

- Once discovered — and it was discovered after much computational evidence — this can be checked variously in Mathematica or Maple (e.g., in the package *gfun*)!

Corollary 2 *(***Zagier Conjecture***)*

$$\zeta(\underbrace{3,1,3,1,\ldots,3,1}_{n-fold}) = \frac{2\,\pi^{4n}}{(4n+2)!} \tag{18}$$

Proof. We have

$$F(a,-a;1;1)=\frac{1}{\Gamma(1-a)\Gamma(1+a)}=\frac{\sin\pi a}{\pi a}$$

where the first equality comes from Gauss's evaluation of $F(a,b;c;1)$.

Hence, setting $x=1$, in (17) produces

$$F\left(\frac{t(1+i)}{2},\frac{-t(1+i)}{2};1;1\right)F\left(\frac{t(1-i)}{2},\frac{-t(1-i)}{2};1;1\right)$$

$$=\frac{2}{\pi^2t^2}\sin\left(\frac{1+i}{2}\pi t\right)\sin\left(\frac{1-i}{2}\pi t\right)$$

$$=\frac{\cosh\pi t-\cos\pi t}{\pi^2t^2}=\sum_{n=0}^{\infty}\frac{2\pi^{4n}t^{4n}}{(4n+2)!}$$

on using the Taylor series of cos and cosh. Comparing coefficients in (17) ends the proof. **QED**

- ▶ What other deep Clausen-like hypergeometric factorizations lurk within?

- • If one suspects that (2) holds, once one can compute these sums well, it is easy to verify many cases numerically and be entirely convinced.

- ♠ This is the *unique* non-commutative analogue of Euler's evaluation of $\zeta(2n)$.