An Empirical Approach to the Normality of π

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Abstract

Using the results of several extremely large recent computations [17] we tested positively the normality of a prefix of roughly four trillion hexadecimal digits of π. This result was used by a Poisson process model of normality of π: in this model, it is extraordinarily unlikely that π is not asymptotically normal base 16, given the normality of its initial segment.

1 Introduction

The question of whether (and why) the digits of well-known constants of mathematics are statistically random in some sense has long fascinated mathematicians. Indeed,

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Table 1: Digit counts in the first trillion hexadecimal (base-16) digits of π. Note that deviations from the average value 62,500,000,000 occur only after the first six digits, as expected from the central limit theorem.

Hex Digits	Occurrences	Hex Digits	Occurrences
0	62499881108	8	62500216752
1	62500212206	9	62500120671
2	62499924780	A	62500266095
3	62500188844	B	62499955595
4	62499807368	C	62500188610
5	62500007205	D	62499613666
6	62499925426	E	62499875079
7	62499878794	F	62499937801
Total	1000000000000		

one prime motivation in computing and analyzing digits of π is to explore the age-old question of whether and why these digits appear "random." The first computation on ENIAC in 1949 of π to 2037 decimal places was proposed by John von Neumann to shed some light on the distribution of π (and of e) [3, pp. 277–281].

Since then, numerous computer-based statistical checks of the digits of π, for instance, so far have failed to disclose any deviation from reasonable statistical norms. See, for instance, Table 1, which presents the counts of individual hexadecimal digits among the first trillion hex digits, as obtained by Yasumasa Kanada. By contrast, the early computations did reveal provable abnormalities in the behavior of e [5, §11.2]. Figure 2 shows π as a random walk drawn as we describe below.

We use the normality for strings introduced and studied in [6]: a sequence whose prefixes are normal is normal, but the converse is not true. Using the results of several extremely large recent computations [17], we tested positively the normality of a prefix of roughly four trillion hexadecimal digits of π. This result was used by a Poisson process model of normality of π: in this model, it is extraordinarily unlikely that π is not asymptotically normal base 16, given the normality of its initial segment.

2 Normality of real numbers

In the pictures in Figures 2 through 5, a digit string for a given number is used to determine the angle of unit steps (multiples of 120 degrees base 3, 90 degrees base four, etc), while the color is shifted up the spectrum after a fixed number of steps (red-orange-yellow-green-cyan-blue-purple-red). In Figure 2 we show a walk on the first billion base 4 digits of π. This may be viewed in more detail online at http://gigapan.org/gigapans/e76a680ea683a233677109fddd36304a. We note that the random walks in Figures 3 and 5 and look entirely different from the expected behavior of a genuine pseudorandom walk as in Figure 1, which is similar to the random walk in Figure 2.

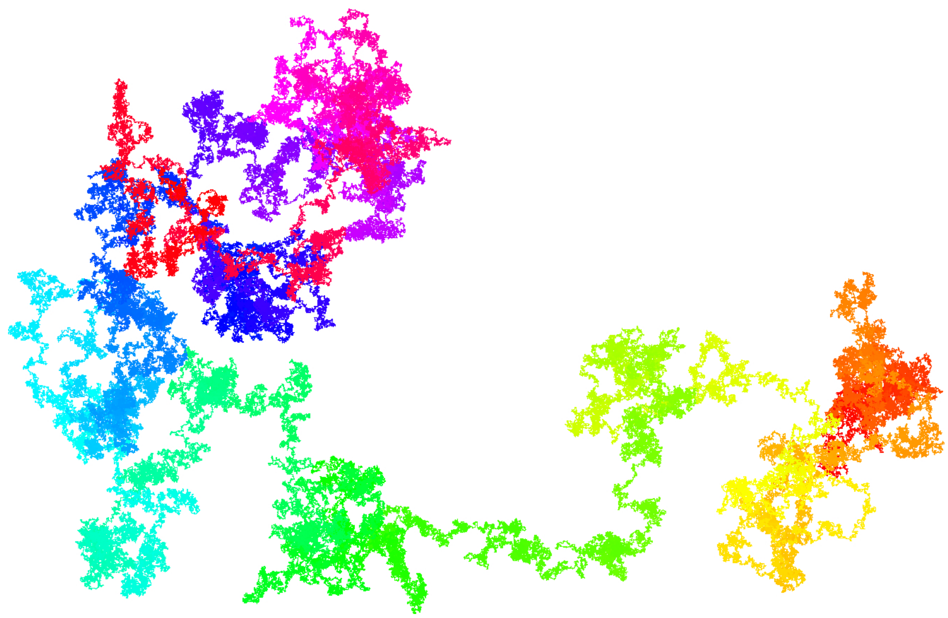

Figure 1: A uniform pseudo-random walk.

In the following, given some positive integer base b, we will say that a real number α is *b-normal* if every m-long string of base-b digits appears in the base-b expansion of α with precisely the expected limiting frequency $1/b^m$. It follows, from basic measure theory, that almost all real numbers are b-normal for any specific base b and even for all bases simultaneously. But proving normality for specific constants of interest in mathematics has proven remarkably difficult.

Borel was the first to conjecture that *all* irrational algebraic numbers are b-normal for *every* integer $b \geq 2$. Yet not a single instance of this conjecture has ever been proven. We do not even *know* for certain whether or not the limiting frequency of zeroes in the binary expansion of $\sqrt{2}$ is one-half, although numerous large statistical analyses have failed to show any significant deviation from statistical normals.

Recently two of the present authors, together with Richard Crandall and Carl

Pomerance, proved the following: If a real y has algebraic degree $D > 1$, then the number $\#(|y|, N)$ of 1-bits in the binary expansion of $|y|$ through bit position N satisfies

$$\#(|y|, N) \quad > \quad CN^{1/D} \tag{1}$$

for a positive number C (depending on y) and all sufficiently large N [1]. For example, there must be at least $\sqrt{N}$ 1-bits in the first N bits in the binary expansion of $\sqrt{2}$, in the limit. A related and more refined result has been obtained by Hajime Kaneko of Kyoto University in Japan. He obtained the bound in $C(\log N)^{3/2}/[(\log(6D))^{1/2}(\log\log N)^{1/2}]$ and extended his results to a very general class of bases and algebraic irrationals [10]. However, each of these results falls far short of establishing b-normality for any irrational algebraic in any base b, even in the single-digit sense.

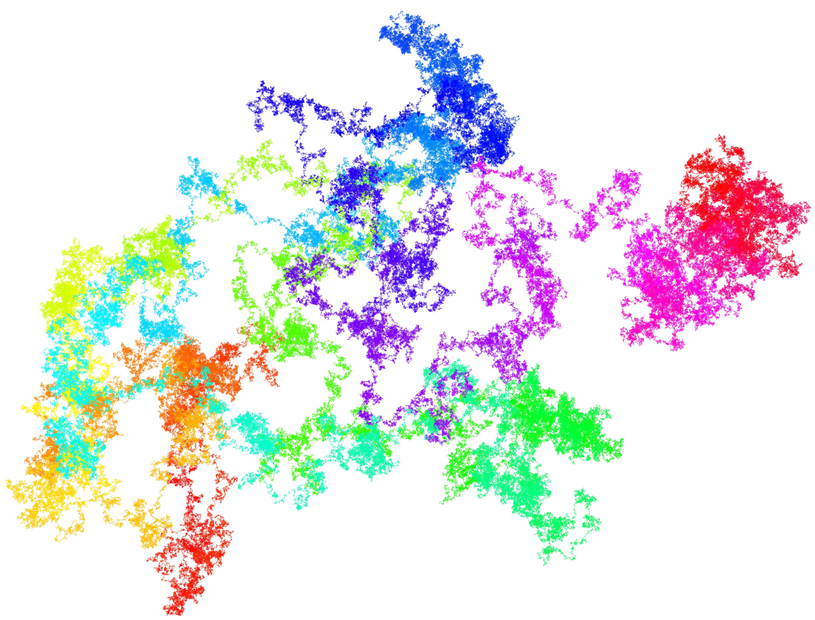

Figure 2: A random walk on the first two billion bits of π (normal?).

The same can be said for π and other basic constants, such as $e, \log 2$ and $\zeta(3)$. Clearly any result (one way or the other) for one of these constants would be a mathematical development of the first magnitude.

We do record the following known stability result [4, pp. 165–166]:

Theorem 1 *If α is normal in base b and r, s are positive rational numbers then $r\alpha + s$ is also normal in base b.*

3 The Champernowne number and relatives

The first mathematical constant proven to be 10-normal is the *Champernowne number*, which is defined as the concatenation of the decimal values of the positive integers, i.e., $C_{10} = 0.12345678910111213141516\ldots$, which can also be written as

$$C_{10} \;=\; \sum_{n=1}^{\infty} \sum_{k=10^{n-1}}^{10^n-1} \frac{k}{10^{kn-9\sum_{k=0}^{n-1} 10^k(n-k)}}. \tag{2}$$

Champernowne proved that C_{10} is 10-normal in 1933 [8]. This, was later extended to base-b normality (for base-b versions of the Champernowne constant).

In 1946, Copeland and Erdös established that the corresponding concatenation of primes $0.23571113171923\ldots$ and also the concatenation of composites $0.46891012141516\ldots$, among others, are also 10-normal [9]. In general they proved:

Theorem 2 ([9]) *If $a_1, a_2, \cdots$ is an increasing sequence of integers such that for every $\theta < 1$ the number of a_i's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal*

$$0.a_1 a_2 a_3 \cdots$$

is normal with respect to the base β in which these integers are expressed.

This clearly applies the Champernowne numbers (Figure 3) and to the primes of the form $ak + c$ with a and c relatively prime in any given base (Figure 4) and to the integers which are the sum of two squares (since every prime of the form $4k + 1$ is included).

In further illustration, using the primes in binary lead to normality in base two of the number

$$0.101110111110111101100011001110111111011111100101101001101011101111\ldots,$$

as shown as a random walk in Figure 5.

Some related results were established by Schmidt, including the following [15]. Write $p \sim q$ if there are positive integers r and s such that $p^r = q^s$. Then

Theorem 3 *If $p \sim q$, then any real number that is p-normal is also q-normal. However, if $p \not\sim q$, then there are uncountably many p-normal reals that are not q-normal.*

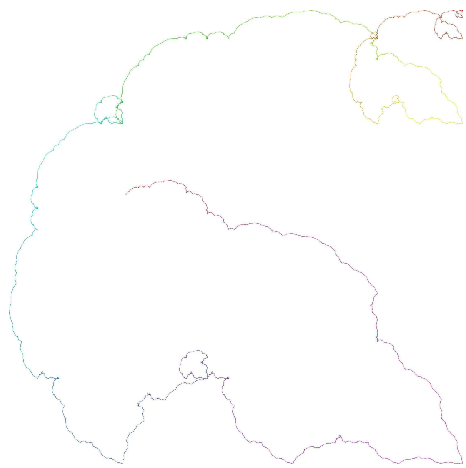

Figure 3: A 600,000 step walk on Champernowne's number base 4 (normal).

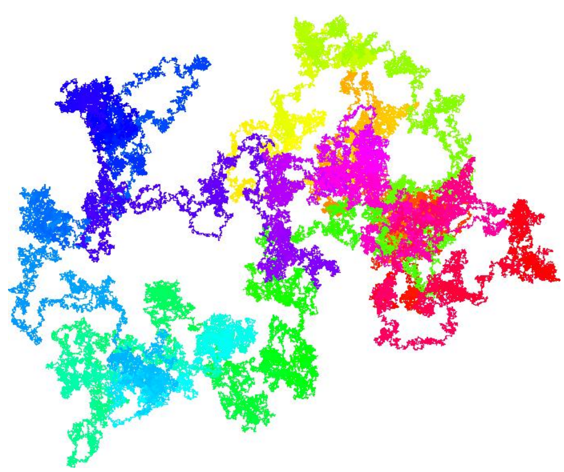

Figure 4: A million step walk on 23571113... base 2 (normal?).

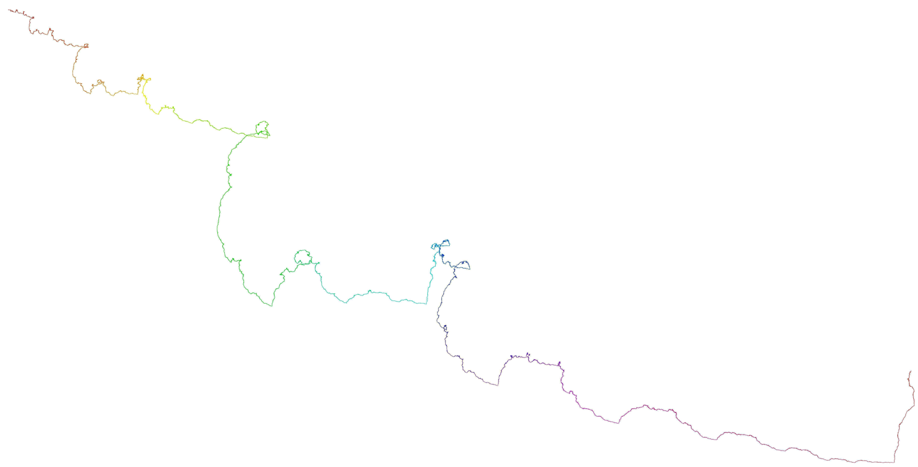

Figure 5: A random walk on the first 100,000 bits of the primes base two (normal).

Queffelec [14] described the above result in a recent survey which also presented the following theorem:

Theorem 4 (Korobov) *Numbers of the form* $\sum_k p^{-2^k} q^{-p^{2^k}}$*, where* $p > 1$ *and* $q > 1$ *are relatively prime, are* q*-normal.*

We are still completely in the dark as to the b-normality of "natural" constants of mathematics.

4 Normality for strings

Let x be a (finite) binary string. We denote by $N_i^m(x)$ the number of occurrences of the ith string of length m ($1 \le i \le 2^m$), ordered lexicographically, where $|x|_m = \lfloor |x|/m \rfloor$ is the number of (contiguous, non-overlapping) of length m strings in x. The *prefix* of length n of the infinite (binary) sequence $\mathbf{x} = x_1 x_2 \ldots x_m \ldots$ is denoted by $\mathbf{x} \upharpoonright n = x_1 x_2 \ldots x_n$.

Definition 1 ([6, 7]) *Let* $\varepsilon > 0$ *and* m *be a positive integer. We say:*

1. x *is* $(\varepsilon, m)-$*normal if, for every* $1 \le i \le 2^m$,

$$\left| \frac{N_i^m(x)}{|x|_m} - \frac{1}{2^m} \right| \le \varepsilon.$$

2. x is $m-$normal if, for every $1 \le i \le 2^m$,

$$\left| \frac{N_i^m(x)}{|x|_m} - \frac{1}{2^m} \right| \le \sqrt{\frac{\log_2 |x|}{|x|}}. \tag{3}$$

3. x is normal if it is $m-$normal for every $1 \le m \le \log_2(\log_2 |x|)$.

If for every positive integer n, the string $\mathbf{x} \upharpoonright n$ is normal, then $\mathbf{x}$ is normal, but the converse is not necessarily true (because $\mathbf{x}$ can be normal but with a different "speed").

5 Testing normality of prefixes of π

In 1996, one of the present authors (Bailey), together with Peter Borwein (brother of Jonathan Borwein) and Simon Plouffe, published what is now known as the BBP formula for π [2], [4, Ch. 3]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{4}$$

We had access to an extremely large dataset, thanks to recent record computations by Kondo and Yee, of π initially to five trillion hexadecimal (base 16) places in August 2010 and then to ten trillion in October 2011 [17]. We first converted these bits — which Kondo and Yee had confirmed by a computation with (4) — to a true binary string of bits using the Python module `binascii`.

All input lines contained an even number of characters so it was easy to convert pairs of hexadecimal digits to bytes.

```
import sys, binascii
for line in sys.stdin.readlines():
    sys.stdout.write(binascii.unhexlify(line.strip()))
```

For our normality test we needed to split a *big* binary string of length n into $\lfloor n/k \rfloor$ pieces (non-overlapping strings) of length $k = 1, 2, \ldots, \log\log n$. We use the term *string* to denote a binary string of length k. We then proceeded to calculate the minimum and maximum occurrences of such strings.

This calculation is done by running the following Algorithm 1 once for each different value of k.

Algorithm 1: Frequency range of strings of a given length.

Input: Binary string X, string length k

Output: Minimum and maximum counts over all possible 2^k strings of length k in string X

integer array $counts[0, \ldots, 2^k - 1] = [0, 0, \ldots, 0]$;

for $i = 0$ **to** $|X| - k$ ***step*** k **do**

  $w = integer(X[i, \ldots, i + k - 1])$;

  increment $counts[w]$;

return min($counts$), max($counts$);

It is essential to do an efficient streaming implementation of Algorithm 1 so that the actual bits of input X are only read into main memory as needed.

Finally to check that these minimum and maximum frequencies satisfy the expected range for the normality test we used the following Python code snippet to generate a table using our earlier formula (3):

```
import math, sys
n=int(sys.argv[1]) # n = |X|
r = int(math.floor(math.log(math.log(n,2),2))) # r = lg lg n
m1,m2=[0]*(r+1),[0]*(r+1)
sqrtV = math.sqrt(math.log(n,2)/n)
for k in range(1,r+1):
    floorNk = math.floor(n/k)
    m1[k] = int(math.floor(((1.0/2.0**i)-sqrtV)*floorNk))
    m2[k] = int(math.ceil((sqrtV+(1.0/2.0**k))*floorNk))
    print "expected range k=",k, "[",m1[k],"...",m2[k],"]"
```

We tested normality for the prefix of $N = 15,925,868,541,400$ bits of π—nearly 16 trillion bits—calculated with the *y-cruncher-multi-threaded pi* program [16] and we have found it to be within the normality range as described above. The frequency counts passed our `expectedCheck.py` test script as shown in Table 2.

6 Normality of π

We have tested the prefix of $N = 15,925,868,541,400$ bits of π—nearly 16 trillion bits—and we have found it to be normal as described above.

Table 2: Frequency summary for $N = 15,925,868,541,400$ bits of π.

m	min frequency found	max frequency found	expected range
1	**796293**3149184	**796293**5392216	7962907842460, ..., 7962960698940
2	**199073**2495242	**199073**5357049	1990720353555, ..., 1990746781795
3	**66357**6589836	**66357**9050172	663569046478, ..., 663586665305
4	**24884**1171873	**24884**2651924	248835088899, ..., 248848303020
5	**9953**5989611	**9953**7473460	99531392735, ..., 99541964032

Does this "information" tell us anything about the classical normality of π? In the next subsection, we will use a Poisson process model to provide an affirmative answer to this question.

6.1 A Poisson process model

We denote by

$$\mathbf{b} = b(1)\, b(2) \ldots b(n) \ldots$$

the (infinite) binary expansion of π (b is a computable function) and by

$$\mathbf{b} \restriction n = b(1)\, b(2) \ldots b(n)$$

the finite prefix of $\mathbf{b}$ of length n.

We base our model on the distribution on 1s' and 0's only, i.e., we work with $N_1^1(\mathbf{b} \restriction n)$, the number of occurrences of 1's in $\mathbf{b} \restriction n$, so $N_0^1(\mathbf{b} \restriction n) = n - N_1^1(\mathbf{b} \restriction n)$. A similar, slightly more elaborate model, can be developed for strings of any length.

The number $N_1^1(\mathbf{b} \restriction n)$ can be connected with π by means of a *counting (Poisson) process* [11]:

$$\begin{aligned} Y_n &= \#\{j \mid 1 \le j \le n,\ b(j) = 1\},\ n = 1, 2, \ldots \\ Y_0 &= 0, \end{aligned}$$

where $Y_n = N_1^1(\mathbf{b} \restriction n)$, $n = 1, 2, \ldots$

Theorem 5 *If π is normal, then $\{Y_n, n = 0, 1, 2...\}$ can be approximated by a homogenous Poisson process with intensity $\lambda = 0.5$.*

Proof. By construction, $\{Y_n, n = 0, 1, 2...\}$ is a Poisson process with an unspecified parameter λ. Hence Y_n is a random variable with parameter $n\lambda$ with the following properties: $E(Y_n) = V(Y_n) = n\lambda$, $\lim_{n\to\infty} Y_n = \infty$ almost sure.

We apply Chebysev's inequality, so for every $c > 0$,

$$P\left(\left|Y_n - E\left(Y_n\right)\right| < c\right) \geq 1 - \frac{V\left(Y_n\right)}{c^2},$$

we have

$$P\left(\left|Y_n - n\lambda\right| < c\right) \geq 1 - \frac{n\lambda}{c^2},$$

hence

$$P\left(\left|\frac{Y_n}{n} - \lambda\right| < \frac{c}{n}\right) \geq 1 - \frac{n\lambda}{c^2}.$$

In view of (3) we take

$$\frac{c}{n} = \varepsilon = \sqrt{\frac{\log_2 n}{n}},$$

so we obtain

$$P\left(\left|\frac{Y_n}{n} - \lambda\right| < \varepsilon\right) \geq 1 - \frac{n\lambda}{\left(n\varepsilon\right)^2} = 1 - \frac{\lambda}{\log_2 n}. \tag{5}$$

If π is normal, then

$$\left|\frac{N_1^1\left(x_{(n)}\right)}{n} - \frac{1}{2}\right| \leq \varepsilon = \sqrt{\frac{\log_2 n}{n}}$$

or

$$\left|\frac{Y_n}{n} - \frac{1}{2}\right| \leq \varepsilon = \sqrt{\frac{\log_2 n}{n}}. \tag{6}$$

If we identify the random event in relation (5) and the certain event in relation (6) we get $\lambda = 1/2$ and

$$P\left(\left|\frac{Y_n}{n} - \frac{1}{2}\right| < \sqrt{\frac{\log_2 n}{n}}\right) \geq 1 - \frac{1}{2\log_2 n}.$$

QED

A Poisson process with intensity λ has the following properties [12]:

- The Poisson process $\{Y_n, n = 0, 1, 2, ...\}$ has independent increments.
- For $n > r$, $Y_n - Y_r$ has a Poisson distribution with parameter $\lambda\left(n - r\right)$, and $Y_n - Y_r$ is independent of $\{Y_t, t \leq r\}$.

Let us denote the positions where 1s occur (*jump moments*) by

$$\tau_r = \inf \{n \mid Y_n = r\}, \; r = 1, 2, ...$$

Then

$$Y_n = 0, \; n < \tau_1,$$
$$Y_n = r, \; \tau_r \leq n < \tau_{r+1}.$$

With the convention $\tau_0 = 0$, we can introduce the *sojourn times,* or *inter-arrival times*

$$T_r = \tau_r - \tau_{r-1}, \; r = 1, 2, ...$$

Note that the sojourn times represent the distances between two successive 1s. Thus, for the string 10^s1 the sojourn time is $s + 1$.

- $\{T_r, r = 1, 2, ...\}$ is a sequence of independent, identical distributed random variables, with the Exponential distribution $Expo\,(\lambda)$ .Then

$$E\,(T_r) = \frac{1}{\lambda}, \; V\,(T_r) = \frac{1}{\lambda^2}.$$

Note that the jump moments $\tau_r = T_1 + ... + T_r$ have an Erlang distribution with parameters $(r; \lambda)$, hence

$$E\,(T_r) = \frac{r}{\lambda}, \; V\,(T_r) = \frac{r}{\lambda^2}.$$

Corollary 1 *If π is normal, then the sojourn times $\{T_r, r = 1, 2, ...\}$ form a sequence of independent, identical distributed random variables, with the Exponential distribution $Expo\,(1/2)$. Hence*

$$P\,(T_r > t_r, \; r = 1, ..., k) = \prod_{r=1}^{k} \left(\exp\left(-\frac{t_r}{2}\right)\right) = \exp\left(-\frac{1}{2}\sum_{r=1}^{k} t_r\right).$$

6.2 Testing the hypothesis "π is normal"

We test the hypothesis H: "π is normal" against the alternative H_A: "π is not normal". If H is true, then for every d there exists K_d such that the sojourn tine exceeds the value d if we wait long enough, up to the rank $(K_d + 1)$:

$$P\left(T_1 \leq d, ..., T_{K_d} \leq d, T_{K_d+1} > d \mid H \text{ true}\right) = \prod_{r=1}^{K_d} \left(1 - \exp\left(-\frac{d}{2}\right)\right) \cdot \exp\left(-\frac{d}{2}\right)$$
$$= \exp\left(-\frac{d}{2}\right)\left(1 - \exp\left(-\frac{d}{2}\right)\right)^{K_d} > 0.$$

We can base our decision of accepting/rejecting normality (hypothesis H) on the following implication: "π is a normal sequence" *implies* "for every d there exists K_d such that $P\left(T_1 \leq d, ..., T_{K_d} \leq d, T_{K_d+1} > d\right) > 0$"),

As one cannot explore the whole sequence π, we deal with an *evidence body* represented by a prefix of π, of length N. In this evidence body, we look for the largest value $d_{\max}$ for which a rank $K_{d\max}$ can be identified or, equivalently, we look for the first value $(d+1)$ which is not reached by the sojourn time T. Accordingly, the decision of accepting/rejecting the hypothesis H : "π is normal" is taken according to the following algorithm:

- If there is no such $d_{\max}$ in the evidence body, we conclude that the sequence π is normal.
- If $d_{\max}$ and the corresponding $K_{d_{\max}}$ exist, we can decide that the sequence π is not normal. The decision is based on the event

  $$\left\{T_1 \leq d_{\max}, ..., T_{K_{d_{\max}}} \leq d_{\max}, T_{K_{d_{\max}}+1} > d_{\max}\right\}$$

  whose probability is

$$P\left(T_1 \leq d_{\max}, ..., T_{K_{d_{\max}}} \leq d_{\max}, T_{K_{d_{\max}}+1} > d_{\max}\right)$$
$$= \exp\left(-\frac{d_{\max}}{2}\right)\left(1 - \exp\left(-\frac{d_{\max}}{2}\right)\right)^{K_{d_{\max}}}.$$

We interpret the above probability as the decision "π is normal" has credibility equal to

$$1 - \exp\left(-\frac{d_{\max}}{2}\right)\left(1 - \exp\left(-\frac{d_{\max}}{2}\right)\right)^{K_{d_{\max}}}.$$

Table 3: d and K_d values for 400 million bits of π.

d	1	2	3	4	5	6	7
K_d	9	1	14	3	46	56	41
d	8	9	10	11	12	13	14
K_d	78	1276	446	2090	18082	8633	4175
d	15	16	17	18	19	20	21
K_d	239183	5856	56453	218007	643030	363117	2787207
d	22	23	24	25	26	27	28
K_d	13733056	1003213	21127913	100317701	not found	85745944	not found
d	29						
K_d	not found						

6.3 Results

Suppose first that the evidence body is represented by a prefix of 400 million bits of π. The $d-$values and their corresponding ranks K_d are given in Table 4; $\max K_d$=100317701.

The value $d = 28$ has the property that for every K, the event

$$\{T_1 \leq 28, ..., T_K \leq 28, T_{K+1} > 28\}$$

has not been identified in the the evidence body, so, based on the algorithm in Section 8.2, the decision "π is not normal" has credibility

$$P\left(T_s \leq 27,\ s = 1, ..., 100317701, T_{100317702} > 27\right)$$
$$= \left(1 - \exp\left(-\frac{27}{2}\right)\right)^{100317701} \cdot \exp\left(-\frac{27}{2}\right) = 2.557\,6 \times 10^{-66}.$$

Suppose now that the evidence body has increased to the prefix of π of $N =$ 15925868541400 bits. The $d-$values and their corresponding ranks K_d are given in following Table 5; $\max K_d = 9274770297096$.

The value $d = 43$ has the property that for every K, the event

$$\{T_1 \leq 43, ..., T_K \leq 43, T_{K+1} > 43\}$$

has not been identified in the evidence body, so, based on the algorithm in Section 8.2, the decision "π is not normal" has credibility

$$P\left(T_s \leq 42,\ s = 1, ..., 9274770297096, T_{9274770297097} > 42\right)$$

Table 4: d and K_d values for 15925868541400 bits of π.

d	1	2	3	4	5
K_d	9	1	14	3	46
d	6	7	8	9	10
K_d	56	41	78	1276	446
d	11	12	13	14	15
K_d	2090	18082	8633	4175	239183
d	16	17	18	19	20
K_d	5856	56453	218007	643030	363117
d	21	22	23	24	25
K_d	2787207	13733056	1003213	21127913	100317701
d	26	27	28	29	30
K_d	273575848	85745944	234725219	611367301	1075713943
d	31	32	33	34	35
K_d	703644000	10621041176	27019219636	15063287853	10887127703
d	36	37	38	39	40
K_d	48115888750	19128531469	1218723032299	1334087352175	792460189481
d	41	42	43	44	45
K_d	9274770297096	4368224447710	not found	not found	not found

$$= \left(1 - \exp\left(-\frac{42}{2}\right)\right)^{9274770297096} \cdot \exp\left(-\frac{42}{2}\right) = 4.349\,7 \times 10^{-3064}.$$

This is perhaps 'incredible'?

7 Conclusion

A prime motivation in computing and analyzing digits of π is to explore the age-old question of whether and why these digits appear "random." Numerous computer-based statistical checks of the digits of π have failed to disclose any deviation from reasonable statistical norms. A new avenue for studying the normality of π was explored: we proved that the prefix of length 15, 925, 868, 541, 400 bits of π is normal when viewed as a binary string [6].

This result was used in a Poisson process model to show that the probability that π is not normal is extraordinarily small, reinforcing the empirical evidence we have presented evidence for the normality of π. In future work we intend to look methodically at other numerical constants.

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