Future Challenges for Variational Analysis

ANZMC2008

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Nonlinear Optimization

ABSTRACT Modern non-smooth analysis is now roughly thirty-five years old. I shall briefly assess where the subject stands today both as theory and regarding applications.

I will also discuss open problems and current challenges for the subject.

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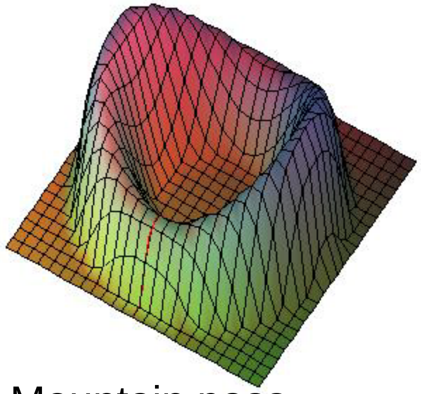
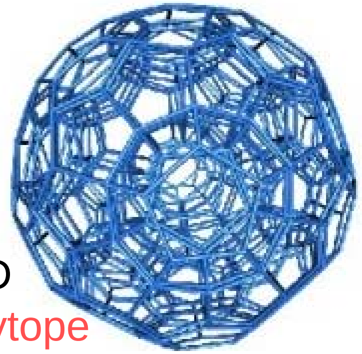
OUTLINE

- **First Order Theory**
- **High Order Theory**
  - second (and higher)
- **Applications**
  - inside Mathematics
  - outside Mathematics
- I'll mention **successes** and **failures**

- Each item has open questions even in the convex case (CA)
  - some technical and specialized
  - some broader and general
- To work fruitfully in VA it helps to understand CA and smooth analysis (SA)
  - they are the motivating foundations
  - and often provide the key technical tools



Mountain pass



A facet of Coxeter's favourite 4D convex polytope

LIPSCHITZ PRECURSORS

Pshenichnyi (1968)

- **descriptive**
- **Large class of good "**quasi-differentiable**" functions**

$$f'(x;h) := \limsup_{t \to 0^+} \frac{f(x+th) - f(x)}{t}$$

- required to be convex in h

Clarke (1972)

- **prescriptive**
- **New directional derivative**

$$f^\circ(x;h) := \limsup_{t \to 0^+, y \to x} \frac{f(y+th) - f(y)}{t}$$

- built to be convex in h

Both capture smooth and convex functions and are closed under + and $\vee$

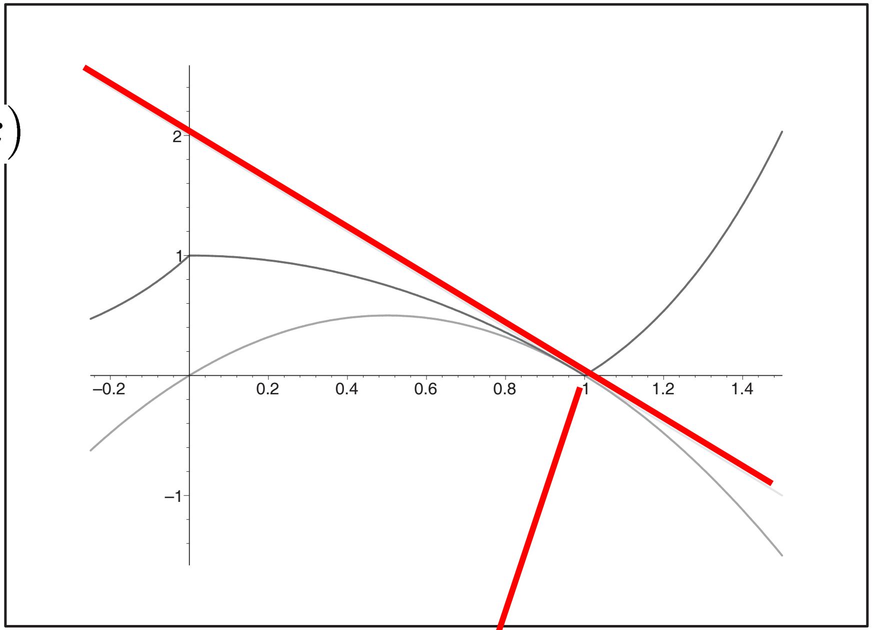
FIRST ORDER THEORY

- **Subgradient**
  - one-sided Fréchet $\partial_F f(x)$
    (Gâteaux or Hadamard)
- Viscosity subgradient $\partial_F^v f(x)$
  - derivative of smooth (local) minorants
  - in nice space $\partial_F f(x) = \partial_F^v f(x)$
- (Fuzzy) Sum Rule

$$\partial_F(f+g)(x) \subseteq \partial_F f(x_1) + \partial_F g(x_2) + \varepsilon B_{X^*}$$

- Normal cones $N_{\mathrm{epi}\,f} := \partial \iota_{\mathrm{epi}\,f}$
- Variational principles (VP)

A lsc function f,
a viscosity subgradient in red,
and a smooth minorant g



$$f(y) - g(y) \geq f(x) - g(x) \text{ for } y \text{ near } x$$

STATE OF THE THEORY

ACHIEVEMENTS

- Limiting subgradients

$$\partial f(x) := \lim_{y \to_f x} \partial_F f(x)$$

- Coderivatives of multis

$$D^*\Omega(x,y)(y^*) = \{x^*: (x^*, -y^*) \in N_{\mathrm{gph}(\Omega)}(x,y)\}$$

- VP+Viscosity+Sum Rule
  - yield **fine** 1st order theory for real-valued functions
  - esp. in Lipschitz case or in finite dimensions
  - seq normal compactness needed more generally; also for
- Metric regularity: locally

$$Kd(\Omega(x), y) \geq d(x, \Omega^{-1}(y))$$

  - and its extensions
  - implicit functions (Dontchev-Rock)
  - alternating projections (Bauschke-Combettes)

I now ignore epsilons and the like

LIMITATIONS

- Inapplicable outside of Asplund space (reflexive, …)
  - $\partial_H f$ extensions fiddly, limited
- Theoretically very beautiful
  - hard to compute even for 'nice' functions

Generically non-expansive functions have $\partial f(x) \equiv B_{X^*}$

  - **SNC** restriction is fundamental not technical

- better results rely on restricting classes of functions (and spaces)
  - E.g., prox-normal, lower C_2,
  - essentially smooth (B-Moors)

B-Sciffer (08) separable explicit construction for $\partial f(x) \equiv B_{X^*}$

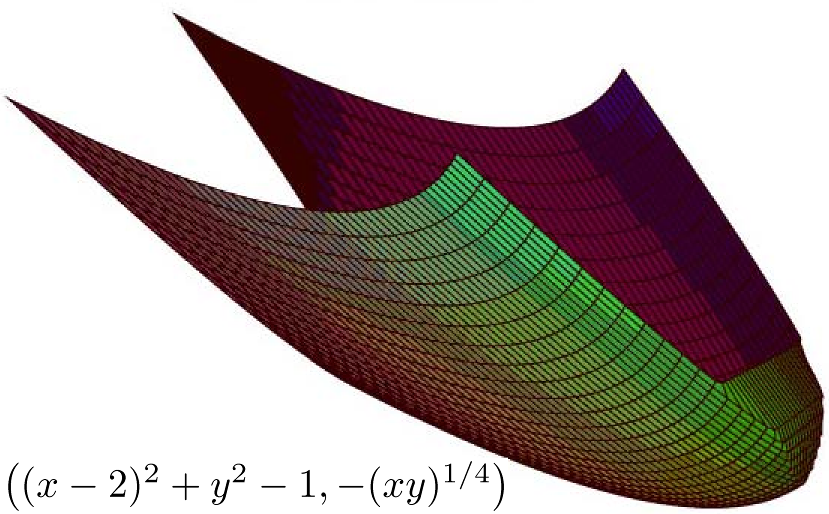
OPEN QUESTIONS

CONVEX

- Find a reflexive Legendre function generalization for convex functions without points of continuity such (neg) Shannon entropy

$$\mathcal{B}(x) := \int_0^1 x(t)\ \log x(t)\,\mathrm{d}t$$

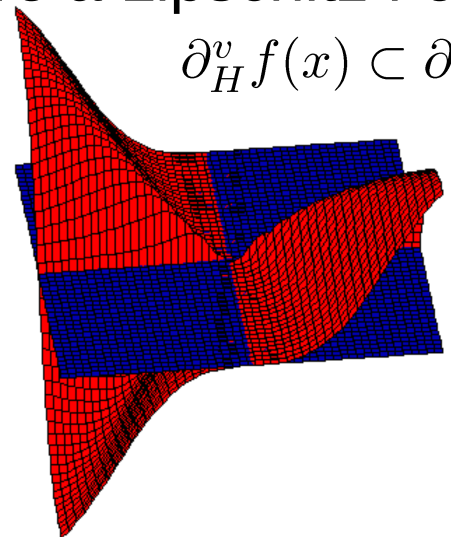
AN ESSENTIALLY STRICTLY CONVEX FUNCTION WITH NONCONVEX SUBGRADIENT DOMAIN AND WHICH IS NOT STRICTLY CONVEX



$\max\left((x-2)^2+y^2-1, -(xy)^{1/4}\right)$

NONCONVEX

Is there a Lipschitz f on ℓ^2 with $\partial_H^v f(x) \subset \partial_H f(x)$?



$\frac{xy^3}{x^2+y^4}$ has $0 \in \partial_H f(0)$ but $0 \notin \partial_H^v f(0)$

- Is the product of a separable and a weak Asplund space still weak Asplund?
  - true for GDS, but WASP is the largest class on which refined NA calculus could be expected to work

SECOND ORDER THEORY

Recall for **convex functions** the *difference quotient* of f by

$$\Delta_t f(x) : h \mapsto \frac{f(x+th) - f(x)}{t};$$

and the *second-order difference quotient* of f by

$$\Delta_t^2 f(x) : h \mapsto \frac{f(x+th) - f(x) - t\langle \nabla f(x), h\rangle}{\frac{1}{2}t^2}.$$

Analogously let

$$\Delta_t[\partial f](x) : h \mapsto \frac{\partial f(x+th) - \nabla f(x)}{t}.$$

For any $t > 0$,

$$\Delta_t f(x)$$

is closed, proper, convex and nonnegative.

Quite beautifully

$$\partial \left[\frac{1}{2}\Delta_t^2 f(x)\right] = \Delta_t[\partial f](x).$$

Theorem Alexandrov (1939) *In Euclidean space a real-valued continuous convex function admits a second order Taylor expansion at almost all points (w.r.t. Lebesgue measure).*

My favourite proof is a specialization of Mignot's 1976 extension to monotone operators [RW98, BV09]

The result relies on many happy coincidences in Euclidean space

The convex case is quite subtle and so the paucity of definitive non-convex results is no surprise

STATE OF THE THEORY

ACHIEVEMENTS

- Some lovely patterns and fine theorems in Euclidean space
  - but no definitive corpus of results
  - nor even canonical definitions outside of the convex case
  - interesting work by Jeyakumar 2006, Duta and others.
  - Many have noted

$$\partial^2 f(x) := \partial \nabla_G f(x)$$

is a fine object when function is Lipschitz smooth in separable Banach space (Rademacher's Thm. applies)

- Fundamental results by Ioffe-Penot (TAMS 1997) on limiting 2-subjets (and coderivatives)
- Fine refined calculus of 'efficient subset' of sub-hessians (Eberhard-Wenczel, SVA 2007)

LIMITATIONS

- Little 'deep' work in infinite dim.
  - i.e., if obvious extensions fail
  - even in Hilbert space
- Outside separable Hilbert space general results are not to be expected [BV09]
  - research should focus on structured classes: integral functionals (Moussaoui-Seeger TAMS 1999)
  - composite convex functions

2-subjet: all order two expansions of C^2 minorants agreeing at x. Now take limits

$$\overline{\partial}^2 f(x) := \limsup_{y \to_f x} \underline{\partial}^2 f(y)$$

yields sum rule in para-convex case,etc.

- extensions by Eberhard-Ralph, others

OPEN QUESTIONS

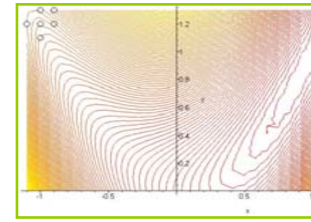
CONVEX

- Does every continuous convex function on separable Hilbert space admit a second order Gateaux expansion at at least one point (or on a dense set of points)?
  - fails in non-separable Hilbert space or in $\ell_p(\mathbf{N}), p \neq 2$
  - fails in the Fréchet sense even in $\ell_2(\mathbf{N})$

NONCONVEX

- Are there sizeable classes of functions for which subjets or other useful second order expansions can be built in separable Hilbert space?
  - I have no precise idea what "useful" means
  - even in convex case this is a tough request; then use
  - Lasry-Lions regularization (Penot, Eberhard, …)

APPLICATIONS



Pattern search

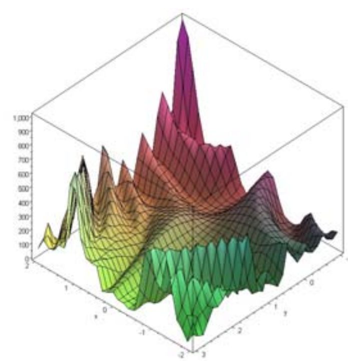
SUCCESSES

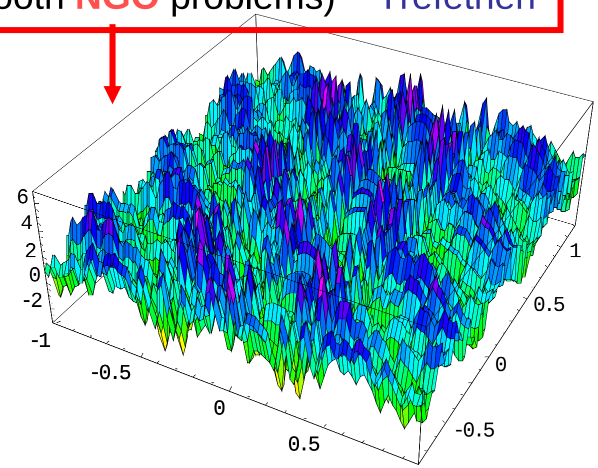
- Tools now part of pure non-linear and functional analysis
- Convergence theory for “pattern search” derivative-free optimization algorithms [Ma08]
- Eigenvalue and singular value optimization theory [BL05]
  - 2nd order (Lewis and Sendov)
- Differential Inclusions and Optimal Control
  - approx Maximum Principle
  - Hamilton-Jacobi equations
- Non-convex mathematical economics and MPECS
- Exact penalty and universal barrier methods [BV09]
  - oodles more counting convex analysis [BV04]

FAILURES

- Limited numeric successes
  - even in convex case excluding spectral & SDP code (somewhat)
  - bundle methods (Lemaréchal et al)
  - Vanderbei’s LOCO package
  - composite convex & smoothing
  - need for “structured nonsmooth” optimization (a la Boyd)

Pinter (Smooth **NGO** problems) Trefethen





$$\min \exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4?$$

$$(x^*, y^*) \approx (-0.024627, 0.211789)$$
with minimum ≈ -3.30687

THREE MORE OPEN QUESTIONS

1. The Chebyshev problem (Klee 1961)

   If every point in H has a unique nearest point in C is C convex? (Yes: Motzkin-Bunt in Euclidean space [BL05])

2. Existence of nearest points (proximal boundary?)

   Do some (many) points in H have a nearest point in C in every renorm of H ? (Yes: if the set is bounded or the norm is Kadec-Klee [BZ05])

3. Universal barrier functions in infinite dimensions

   Is there an analogue for H of the universal barrier function so important in Euclidean space? (I doubt it [BV09])

Convex Functions

The Chebyshev problem (Klee 1961) A set is Chebyshev if every point in H has a unique nearest point in C

Theorem If C is weakly closed and Chebyshev then C is convex. So in Euclidean space Chebyshev iff convex.

Four Euclidean variational proofs ([BL05], Opt Letters 07, [BV09])

1. **Brouwer's theorem** (Cheb. implies **sun** implies convex)
2. **Ekeland's theorem** (Cheb. implies **approx. convex** implies convex)
3. **Fenchel duality** (Cheb. iff d_C^2 is Frechet) use f^* smooth implies f convex for

$$\left(\frac{\iota_C + \|\cdot\|^2}{2}\right)^* = \frac{\|\cdot\|^2 + d_C^2}{2}$$

4. Inverse geometry also shows if there is a counter-example it can be a **Klee cavern** (Asplund) the closure of the complement of a convex body. **WEIRD**

- **Counterexamples** exist in incomplete inner product spaces. #2 seems most likely to work in Hilbert space.
- **Euclidean case** is due to Motzkin-Bunt

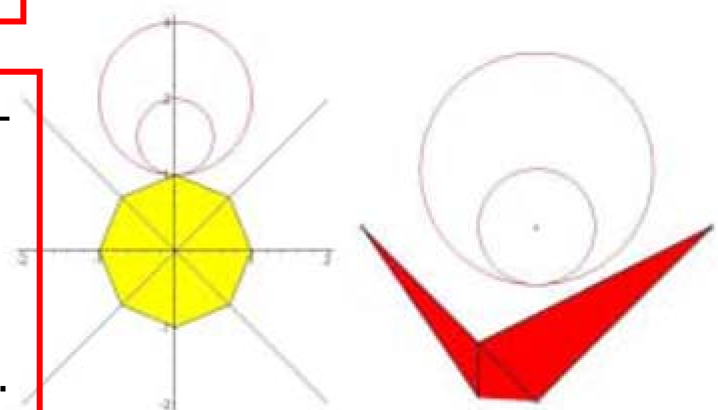


FIGURE 1. Suns and approximate convexity.

Existence of nearest points

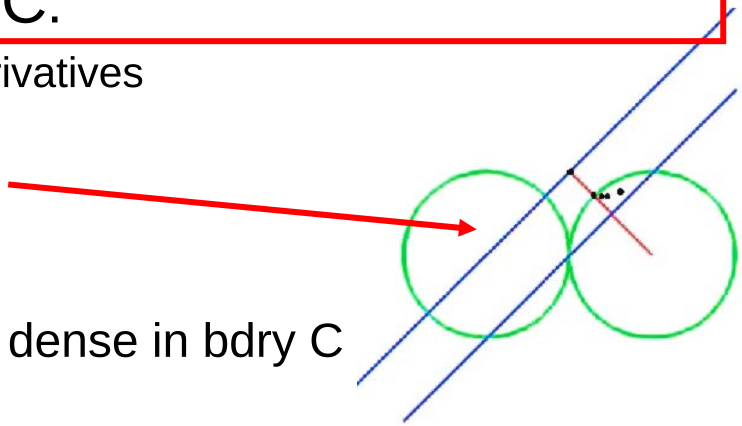
Do some (many) points in H have a nearest point in C in every renorm of H ?

Theorem (Lau-Konjagin, 76-86) A norm on a reflexive space is Kadec-Klee iff for every norm-closed C in X best approximations exist generically (densely) in X\ C.

Nicest proof is via dense existence of Frechet subderivatives

$$\phi \in \partial_F d_C(x)$$

The KK property forces approximate minimizers to line up.



- There are non KK norms with proximal points dense in bdry C
- If C is closed and bounded then there are some points with nearest points (RNP)
- So a counterexample has to be a weird unbounded set in a rotten renorm (BFitzpatrick 89, [BZ09])

A norm is **Kadec-Klee** norm if weak and norm topologies agree on the unit sphere.

Hence all LUR norms are Kadec-Klee.

Universal barrier functions in infinite dimensions

– Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

Theorem (Nesterov-Nemirovskii) For any open convex set A in n-space, the function

$$F(x) := \lambda_N \left((A - x)^o \right)$$

is an essentially smooth, log-convex barrier function for A.

– This relies heavily on the existence of **Haar** measure (Lebesgue).
– Amazingly for A the semidefinite matrix cone we recover – log det, etc

In Hilbert space the only really nice examples I know are similar to:

$$\phi(T) := \mathrm{trace}(T) - \log\det(I + T)$$

is a strictly convex Frechet differentiable barrier function for the Hilbert-Schmidt operators with I+T > 0.

We are able to build barriers in great generality but not "universally" [BV09]

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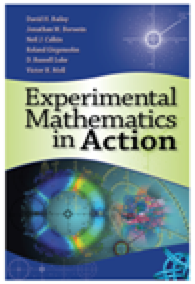

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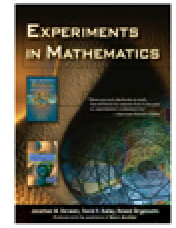

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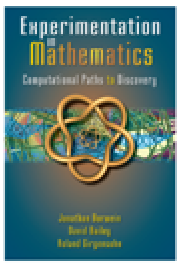

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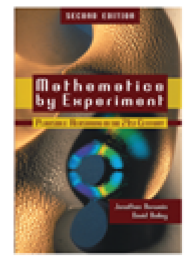

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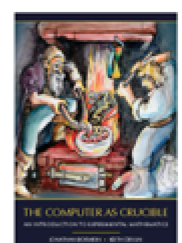

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