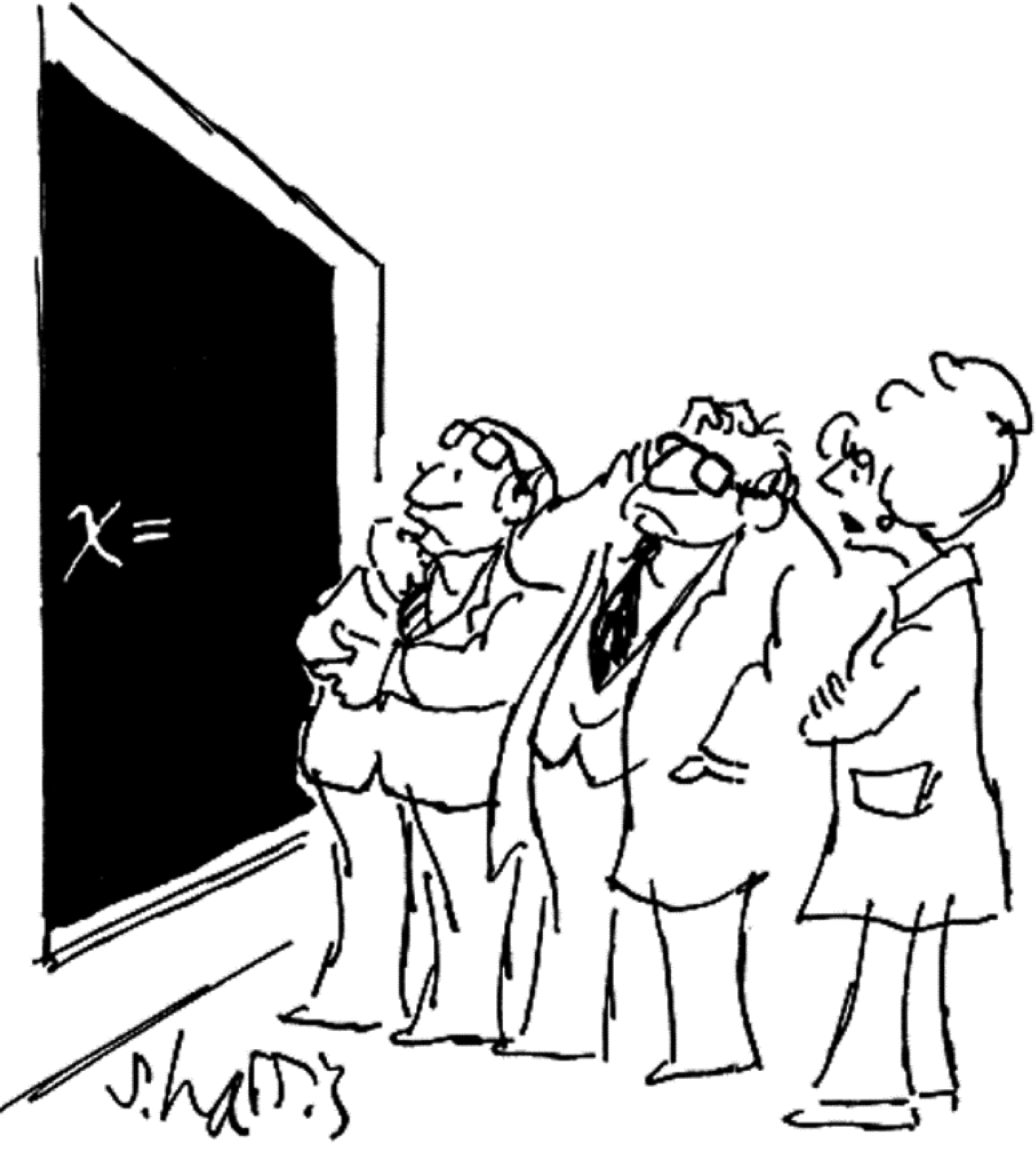
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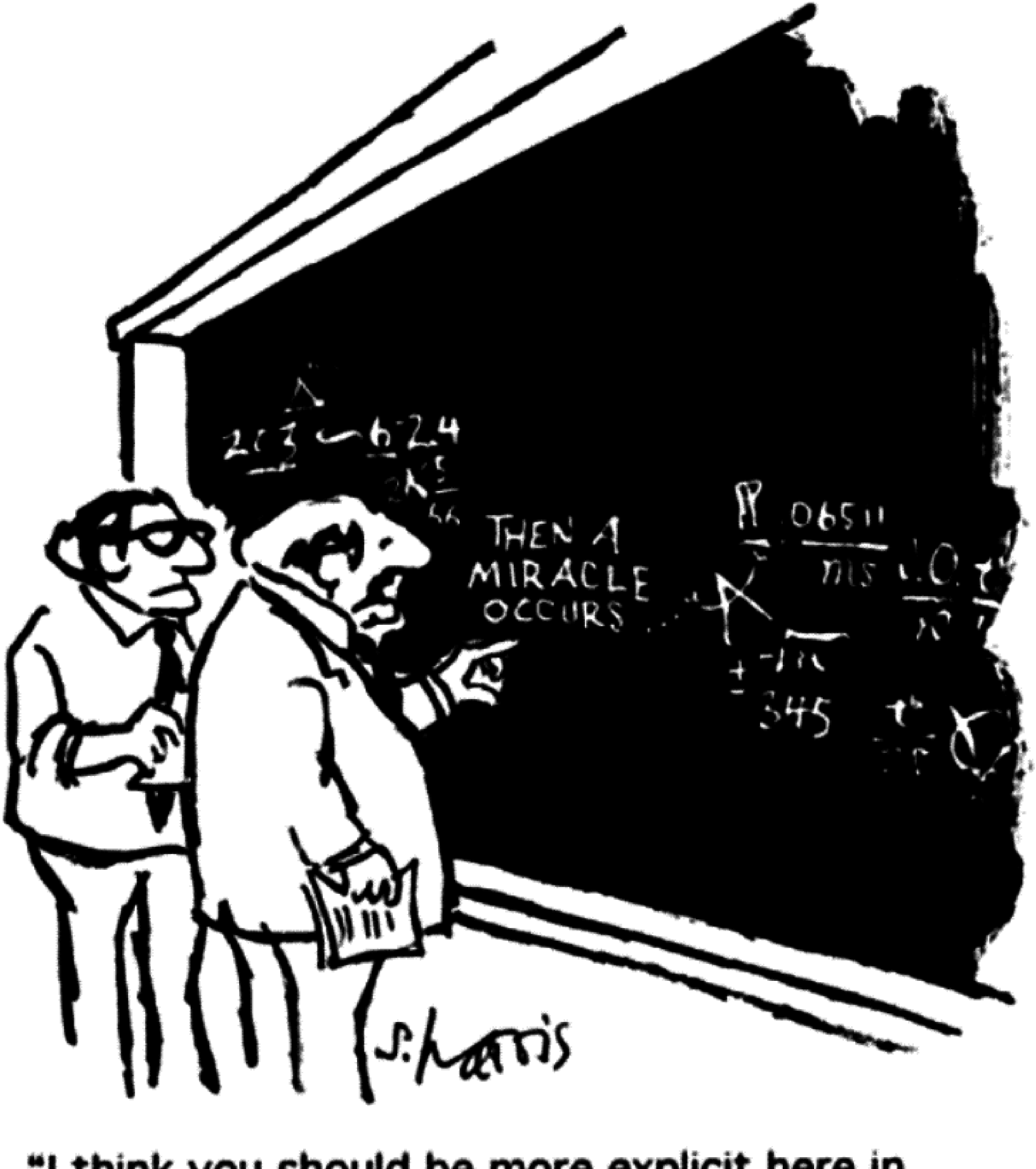


"I think you should be more explicit here in step two."

The Philosophical Implications of Experimental (Computational) Mathematics

Jonathan M. Borwein, FRSC

Research Chair in IT
Dalhousie University

DALHOUSIE UNIVERSITY
Inspiring Minds

Halifax, Nova Scotia, Canada

Philosophical Implications Lecture

November 2004

If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics. (Kurt Gödel, 1951)

URL: `www.expmath.info` 

AK Peters 2004 **Talk Revised**: 01–03–05

OUTLINE of PRESENTATION

A. PART 1. Philosophy of Experimental Math

> *Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged.* (Christof Koch, 2004)*

⇜ The abridged version ends here ⇝

B. PART 2. Finding or Proving Things

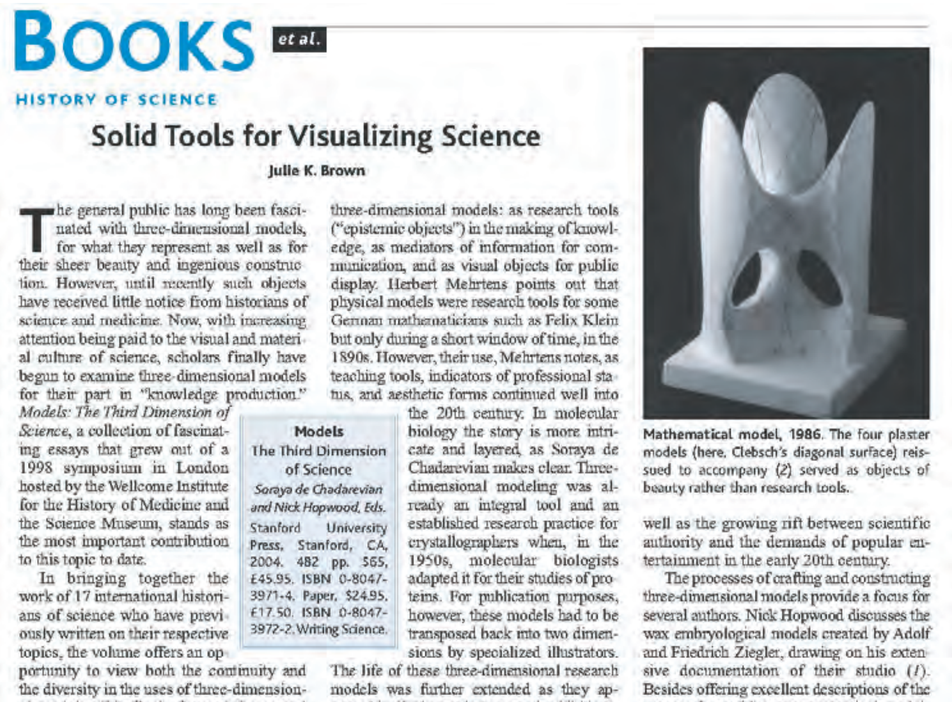
PART 3. Various Mathematical Models

PART 4. Concluding Remarks

*In "Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, OUP 2004.

DEFINITIONS

mathematics, n. *a group of related subjects, including algebra, geometry, trigonometry and calculus , concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.*



BOOKS *et al.*

HISTORY OF SCIENCE

Solid Tools for Visualizing Science

Julie K. Brown

The general public has long been fascinated with three-dimensional models, for what they represent as well as for their sheer beauty and ingenious construction. However, until recently such objects have received little notice from historians of science and medicine. Now, with increasing attention being paid to the visual and material culture of science, scholars finally have begun to examine three-dimensional models for their part in "knowledge production." *Models: The Third Dimension of Science*, a collection of fascinating essays that grew out of a 1998 symposium in London hosted by the Wellcome Institute for the History of Medicine and the Science Museum, stands as the most important contribution to this topic to date.

In bringing together the work of 17 international historians of science who have previously written on their respective topics, the volume offers an opportunity to view both the continuity and the diversity in the uses of three-dimensional models. [...] three-dimensional models: as research tools ("epistemic objects") in the making of knowledge, as mediators of information for communication, and as visual objects for public display. Herbert Mehrtens points out that physical models were research tools for some German mathematicians such as Felix Klein but only during a short window of time, in the 1890s. However, their use, Mehrtens notes, as teaching tools, indicators of professional status, and aesthetic forms continued well into the 20th century. In molecular biology the story is more intricate and layered, as Soraya de Chadarevian makes clear. Three-dimensional modeling was already an integral tool and an established research practice for crystallographers when, in the 1950s, molecular biologists adapted it for their studies of proteins. For publication purposes, however, these models had to be transposed back into two dimensions by specialized illustrators. The life of these three-dimensional research models was further extended as they ap[...]

Models
The Third Dimension of Science
Soraya de Chadarevian and Nick Hopwood, Eds.
Stanford University Press, Stanford, CA, 2004. 482 pp. \$65, £45.95. ISBN 0-8047-3971-4. Paper, \$24.95, £17.50. ISBN 0-8047-3972-2. Writing Science.

Mathematical model, 1986. The four plaster models (here, Clebsch's diagonal surface) reissued to accompany (2) served as objects of beauty rather than research tools.

well as the growing rift between scientific authority and the demands of popular entertainment in the early 20th century.

The processes of crafting and constructing three-dimensional models provide a focus for several authors. Nick Hopwood discusses the wax embryological models created by Adolf and Friedrich Ziegler, drawing on his extensive documentation of their studio (1). Besides offering excellent descriptions of the [...]

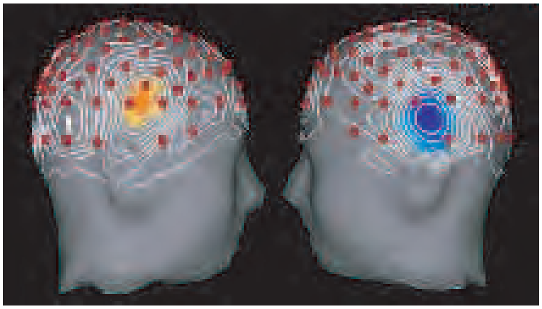
Science November 19, 2004

induction, n. *any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.*

PART I. EXPERIMENTAL PHILOSOPHY

Intention. To discuss Experimental M*a*thodology*

Insight taking place



- Its philosophy
- its history
- its current practice
- its proximate future

Enigma

And using concrete entertaining (I hope) examples to explore

- implications for mathematics
- and for math philosophy

Thereby, to persuade you the traditional accounting of mathematical learning and research is largely an ahistorical caricature

*Bailey, Moll and I invite you to the 2nd Experimental Mathematics Workshop at Tulane, April 1–3 2005, and a January 2006 MAA short course.

I shall talk broadly about experimental and heuristic mathematics, giving accessible, primarily visual and symbolic, examples. The typographic to digital culture shift is vexing in math, viz:

- There is still no truly satisfactory way of displaying mathematics on the web

- We respect authority* but value authorship deeply

- And we care more about the reliability of our literature than does any other science

The traditional central role of proof in mathematics is arguably under siege.

- Via examples, I intend to ask:

*Judith Grabiner, "Newton, Maclaurin, and the Authority of Mathematics," MAA, December 2004

MY QUESTIONS

★ What constitutes secure mathematical knowledge?

★ When is computation convincing? Are humans less fallible?

- What tools are available? What methodologies?

- What of the 'law of the small numbers'?

- Who cares for certainty? What is the role of proof?

★ How is mathematics actually done? How should it be?

DEWEY on HABITS

Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. ⋯ Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species." * (John Dewey)

* *The Influence of Darwin on Philosophy*, 1910. Dewey knew 'Comrade Van' in Mexico.

and MY ANSWERS

$\models$ "Why I am a computer assisted fallibilist"

★ Rigour (proof) follows Reason (discovery)

★ Excessive focus on rigour drove us away from our wellsprings

- Many ideas are false. Not all truths are provable. Not all provable truths are worth proving ...

★ Near certainly is often as good as it gets—intellectual context (community) matters

- Complex human proofs are fraught with error (FLT, simple groups, $\cdots$)

★ Modern computational tools dramatically change the nature of available evidence

- Many of my more sophisticated examples originate in the boundary between mathematical physics and number theory and involve the ζ-function, $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$, and its relatives.

They often rely on the sophisticated use of *Integer Relations Algorithms* — recently ranked among the 'top ten' algorithms of the century.

- Integer Relation methods were first discovered by our colleague **Helaman Ferguson** the mathematical sculptor.

- Other winners: Monte Carlo,Fortran Compiler, QuickSort, Simplex Method, FFT, QR, ...

See `www.cecm.sfu.ca/projects/IntegerRelations/`

FOUR FORMS of EXPERIMENTS

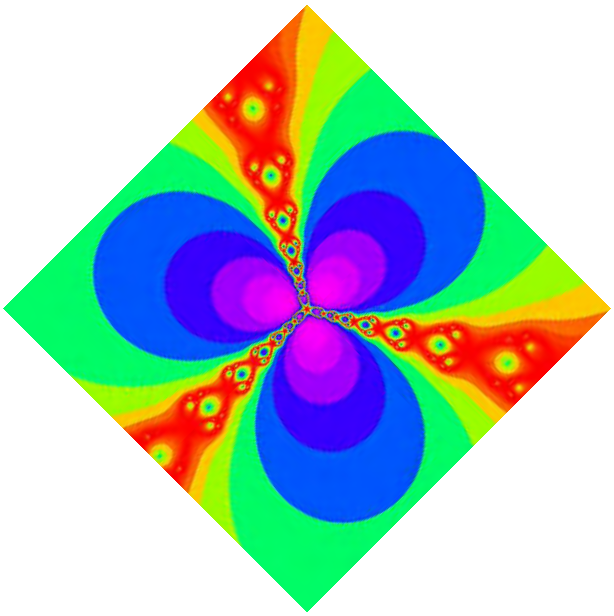
• 1. **Kantian** example: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."

• 2. The **Baconian** experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."

• 3. **Aristotelian** demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."

• 4. The most important is **Galilean**: "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

- The only form which will make Experimental Mathematics a serious enterprise.



A Julia set

From Peter Medawar (1915–87) *Advice to a Young Scientist* (1979)

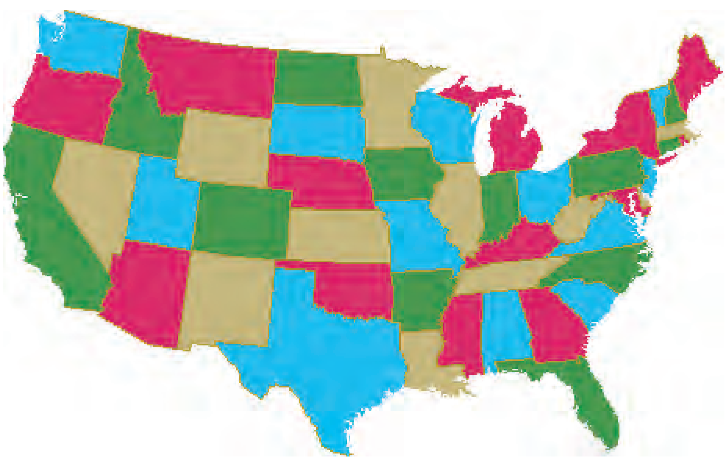
A PARAPHRASE of HERSH

In any event mathematics is and will remain a uniquely human undertaking. Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our computational setting:

1. *Mathematics is human*. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.

2. *Mathematical knowledge is fallible*. As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "**fallibilism**" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations*.

3. There are different versions of proof or rigor. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977 (1997), is just one example of an emerging nontraditional standard of rigor.

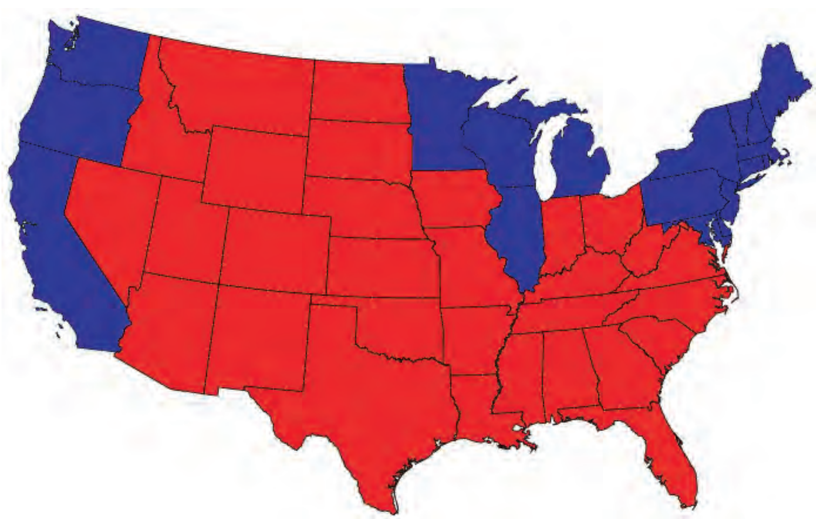


A 4-coloring

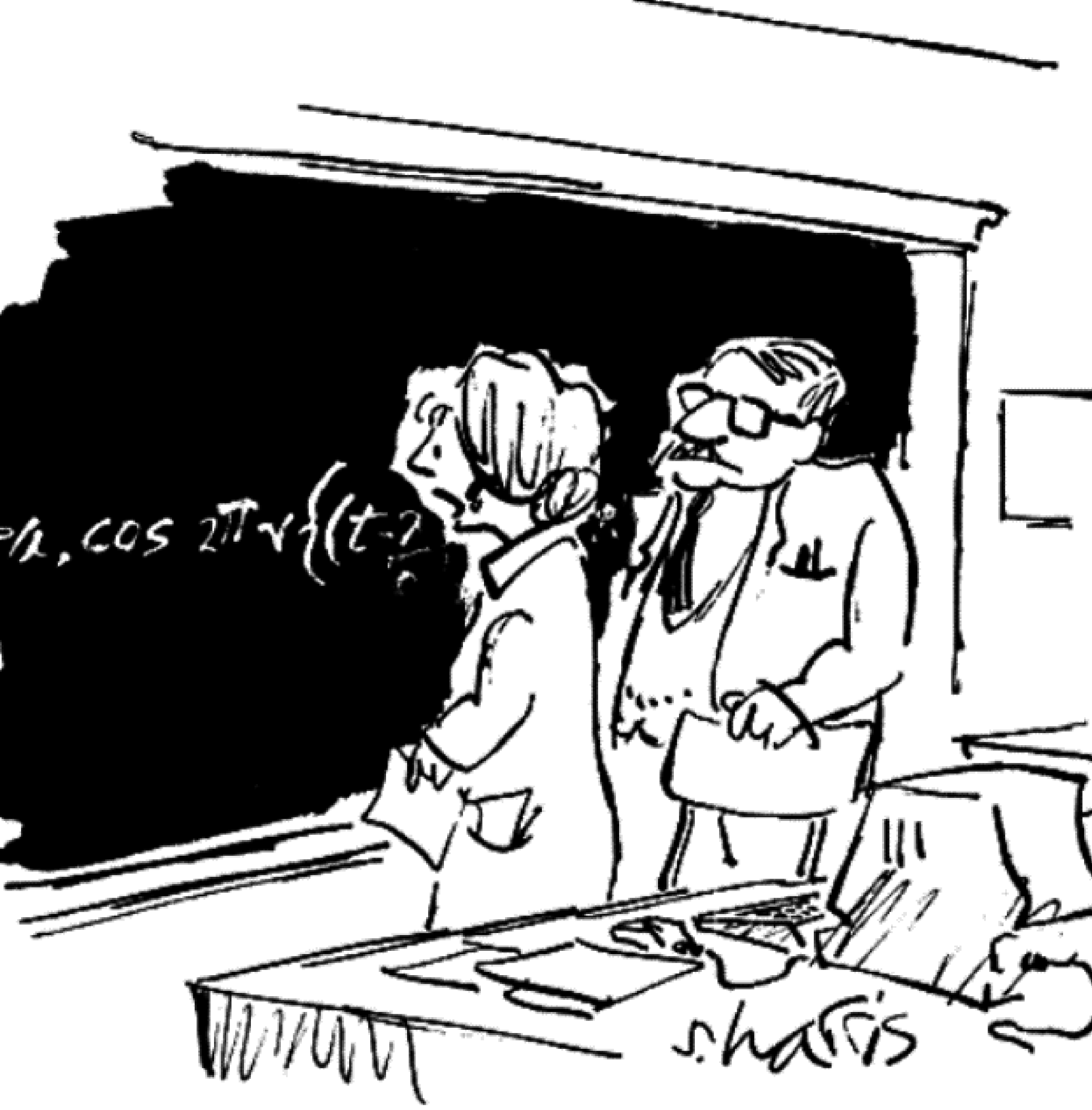
4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. Aristotelian logic isn't necessarily always the best way of deciding.

5. *Mathematical objects are a special variety of a social-cultural-historical object*. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

- "Fresh Breezes in the Philosophy of Mathematics", *MAA Monthly*, Aug 1995, 589–594.



A 2-coloring?

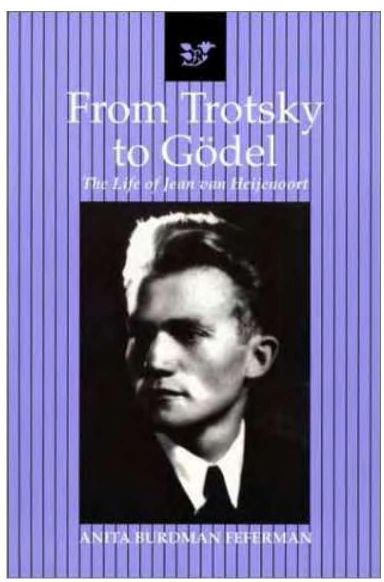


"IT'S AN EXCELLENT PROOF, BUT IT LACKS WARMTH AND FEELING."

THREE Humanist VIGNETTES: I

> *By 1948, the Marxist-Leninist ideas about the proletariat and its political capacity seemed more and more to me to disagree with reality ... I pondered my doubts, and for several years the study of mathematics was all that allowed me to preserve my inner equilibrium. Bolshevik ideology was, for me, in ruins. I had to build another life.*

Jean Van Heijenoort (1913-1986) *With Trotsky in Exile*, in Anita Feferman's *From Trotsky to Gödel*



II. It's Obvious . . .

Aspray: Since you both [Kleene and Rosser] had close associations with Church, I was wondering if you could tell me something about him. What was his wider mathematical training and interests? What were his research habits? I understood he kept rather unusual working hours. How was he as a lecturer? As a thesis director?

Rosser: In his lectures he was painstakingly careful. There was a story that went the rounds. *If Church said it's obvious, then everybody saw it a half hour ago. If Weyl says it's obvious, von Neumann can prove it. If Lefschetz says it's obvious, it's false.* *

*One of several versions in *The Princeton Mathematics Community in the 1930s*. This one in Transcript Number 23 (**PMC23**)

III. The Evil of Bourbaki

"There is a story told of the mathematician Claude Chevalley (1909–84), who, as a true Bourbaki, was extremely opposed to the use of images in geometric reasoning.

He is said to have been giving a very abstract and algebraic lecture when he got stuck. After a moment of pondering, he turned to the blackboard, and, trying to hide what he was doing, drew a little diagram, looked at it for a moment, then quickly erased it, and turned back to the audience and proceeded with the lecture. ...

... The computer offers those less expert, and less stubborn than Chevalley, access to the kinds of images that could only be imagined in the heads of the most gifted mathematicians, ... " (Nathalie Sinclair[a])

[a]Chapter in *Making the Connection: Research and Practice in Undergraduate Mathematics*, MAA Notes, 2004 in Press.

SYDNEY BRENNER

And it is one of the ironies of this entire field that were you to write a history of ideas in the whole of DNA, simply from the documented information as it exists in the literature - that is, a kind of Hegelian history of ideas - you would certainly say that Watson and Crick depended on Von Neumann, because von Neumann essentially tells you how it's done. But of course no one knew anything about the other. It's a great paradox to me that this connection was not seen. Of course, all this leads to a real distrust about what historians of science say, especially those of the history of ideas. (Sidney Brenner)

The 2002 Nobelist talking about von Neumann's essay on The General and Logical Theory of Automata on pages 35–36 of My life in Science as told to Lewis Wolpert.

MY MOTIVATION and GOALS

INSIGHT – demands speed $\equiv$ **micro-parallelism**

- For rapid verification.

- For validation; proofs *and* refutations; "monster barring".

★ What is "easy" changes: HPC & HPN blur, merging disciplines and collaborators — democratizing mathematics but challenging authenticity.

- **Parallelism** $\equiv$ more space, speed & stuff.

- **Exact** $\equiv$ hybrid $\equiv$ symbolic '$+$' numeric (*Maple meets NAG*).

- In analysis, algebra, geometry & topology.

. . . Moreover

- Towards an Experimental **Mathodology** — philosophy and practice.

▶ **Intuition is acquired** — mesh computation and mathematics.

- **Visualization** — **3** is a lot of dimensions.

▶ “Monster-barring” (Lakatos) and “Caging”:

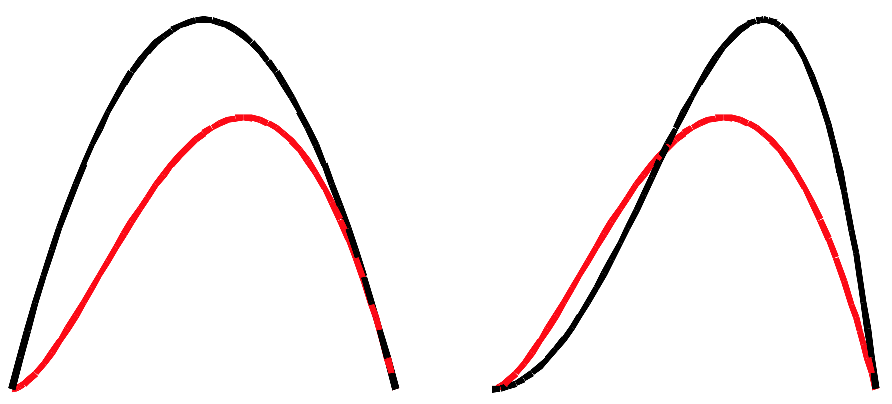
  – randomized checks: equations, linear algebra, primality.

  – graphic checks: equalities, inequalities, areas.

... Graphic Checks

- Comparing $y-y^2$ and y^2-y^4 to $-y^2\ln(y)$ for $0 < y < 1$ pictorially is a much more rapid way to divine which is larger than traditional analytic methods.

- It is clear that in the later case they cross, it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate a proof.



Graphical comparison of
$y - y^2$ and $y^2 - y^4$ to $-y^2\ln(y)$ (red)

OUR EXPERIMENTAL MATHODOLOGY

1. Gaining insight and intuition
2. Discovering new patterns and relationships
3. Graphing to expose math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it *merits* formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

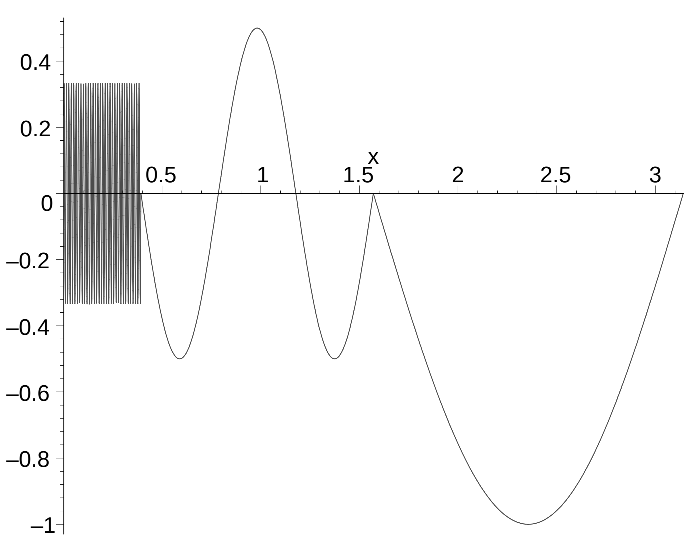
A BRIEF HISTORY OF RIGOUR

- Greeks: trisection, circle squaring, cube doubling and $\sqrt{2}$

- Newton and Leibniz: fluxions/infinitesimals

- Cauchy and Fourier: limits and continuity

- Frege and Russell, Gödel and Turing



For continuous functions **Fourier series need not converge** in 1810, 1860 or 1910?

THE PHILOSOPHIES OF RIGOUR

- Everyman: Platonism—stuff exists (1936)

- Hilbert: Formalism—math is invented; formal symbolic games without meaning

- Brouwer: Intuitionism-—many variants; ('embodied cognition')

- Bishop: Constructivism—tell me how big; (not 'social constructivism')

℧ Last two deny *excluded middle*: $A \vee \tilde{A}$ and resonate with computer science—as does some of formalism.

Ξ *Absolutism versus Fallibilism.*

SOME SELF PROMOTION

- Today *Experimental Mathematics* is being discussed widely

MATH

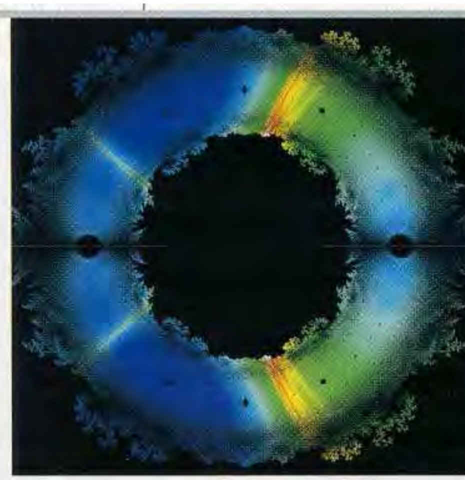
A Digital Slice of Pi

THE NEW WAY TO DO PURE MATH: EXPERIMENTALLY **BY W. WAYT GIBBS**

"**One of the greatest ironies** of the information technology revolution is that while the computer was conceived and born in the field of pure mathematics, through the genius of giants such as John von Neumann and Alan Turing, until recently this marvelous technology had only a minor impact within the field that gave it birth." So begins *Experimentation in Mathematics*, a book by Jonathan M. Borwein and David H. Bailey due out in September that documents how all that has begun to change. Computers, once looked on by mathematical researchers with disdain as mere calculators, have gained enough power to enable an entirely new way to make fundamental discoveries: by running experiments and observing what happens.

The first clear evidence of this shift emerged in 1996. Bailey, who is chief technologist at the National Energy Research Scientific Computing Center in Berkeley, Calif., and several colleagues developed a computer program that could uncover integer relations among long chains of real numbers. It was a problem that had long vexed mathematicians. Euclid discovered the first integer relation scheme—a way to work out the greatest common divisor of any two integers—around 300 B.C. But it wasn't until 1977 that Helaman Ferguson and Rodney W. Forcade at last found a method to detect relations among an arbitrarily large set of numbers. Building on that work, in 1995 Bailey's group turned its computers loose on some of the fundamental constants of math, such as log 2 and pi.

To the researchers' great surprise, after months of calculations the machines came up with novel formulas for these and other nat-



COMPUTER RENDERINGS of mathematical constructs can reveal hidden structure. The bands of color that appear in this plot of all solutions to a certain class of polynomials (specifically, those of the form $\pm 1 \pm x \pm x^2 \pm x^3 \pm \ldots \pm x^n = 0$, up to $n = 18$) have yet to be explained by conventional analysis.

From Scientific American, May 2003

MATH LAB

Computer experiments are transforming mathematics

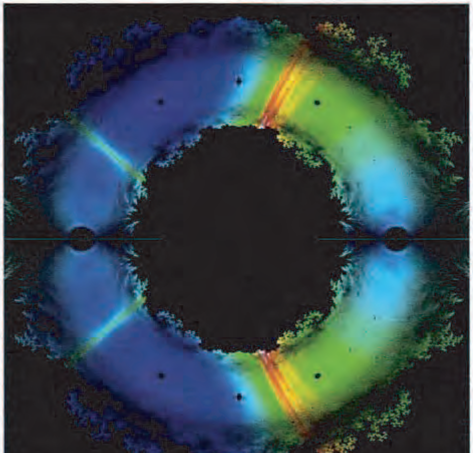
BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results



UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

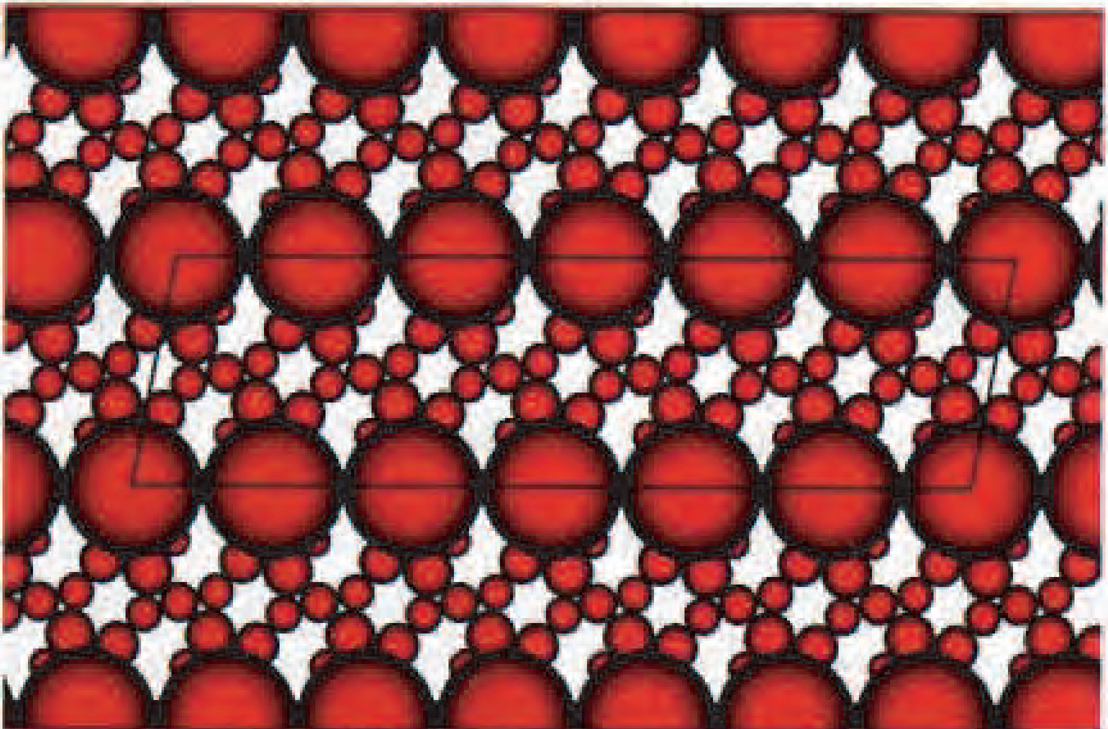
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

From Science News April 2004



STRAIGHT CIRCLES — When mathematicians Colin Adams and Eric Schoenfeld created this image while playing with the computer program Snappea last year, they were stunned to see perfectly straight chains of spheres. The observation led them to an unexpected discovery about knots.

Bailey and Jonathan Borwein advance the controversial thesis that mathematics should move toward a more empirical approach. In it, formal proof would not be the only acceptable way to establish mathematical knowledge.

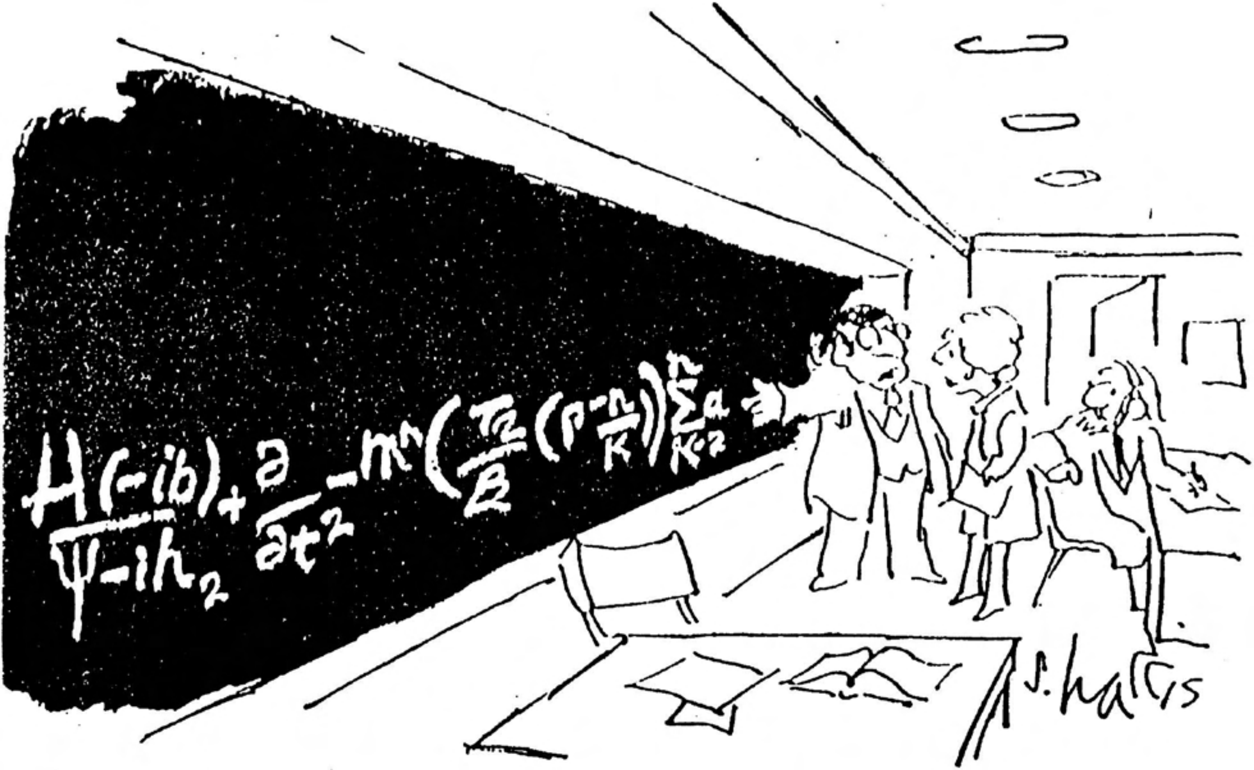
Mathematicians, Bailey and Borwein argue, should be free to work more like other scientists do, developing hypotheses through experimentation and then testing them in further experiments. Formal proof is still the ideal, they say, but it is not the only path to mathematical truth.

"When I started school, I thought mathematics was about proofs, but now I think it's about having secure mathematical knowledge," Borwein says. "We claim that's not the same thing."

> "One thing that's happening is you can discover many more things than you can explain."
>
> — JONATHAN BORWEIN
> DALHOUSIE UNIVERSITY

Bailey and Borwein point out that mathematical proofs can run to hundreds of pages and require such specialized knowledge that only a few people are capable of reading and judging them.

"We feel that in many cases, computations constitute very strong evidence, evidence that is at least as compelling as some of the more complex formal proofs in the literature," Bailey and Borwein say in *Mathematics by Experiment*.



"But this *is* the simplified version for the general public."

PART II. FINDING vs PROVING THINGS

Donald Knuth* asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty} \left\{ \frac{k^k}{k!\,e^k} - \frac{1}{\sqrt{2\,\pi\,k}} \right\} = -0.084069508727655\ldots.$$

2000 CE. It is easy to compute 20 or 200 digits of this sum The 'smart lookup' facility in the *Inverse Symbolic Calculator*† rapidly returns

$$0.084069508727655 \approx \frac{2}{3} + \frac{\zeta\,(1/2)}{\sqrt{2\,\pi}}.$$

We thus have a prediction which Maple 9.5 on a laptop confirms to 100 places in under 6 seconds and to 500 in 40 seconds.

Arguably we are done. □

*Posed as *MAA Problem* 10832, November 2002.

†At www.cecm.sfu.ca/projects/ISC/ISCmain.html

In the same vein . . .

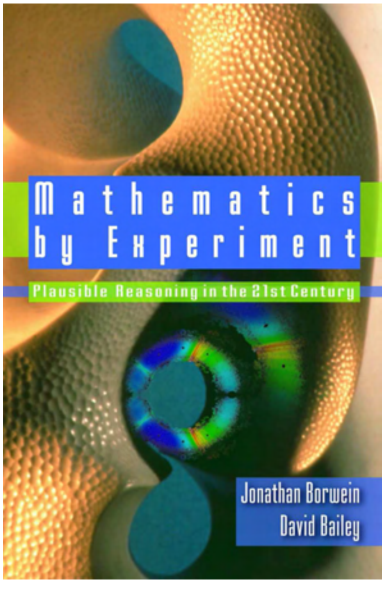
Consider the following two *Euler sum identities* both discovered heuristically.

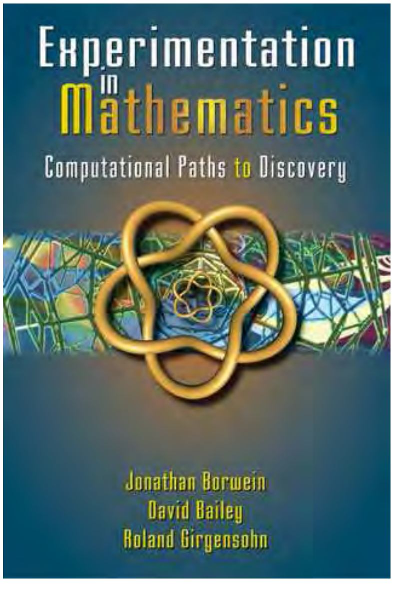
- Both merit quite firm belief—more so than many proofs.

  Why?

- Only the first warrants significant effort for its proof.

  Why and **Why Not**?





I. A MULTIPLE ZETA VALUE

Euler sums or *MZVs* are a wonderful generalization of the classical ζ function.

For natural numbers

$$\zeta(i_1, i_2, \ldots, i_k) := \sum_{n_1 > n_2 > n_k > 0} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$

◇ Thus $\zeta(a) = \sum_{n \geq 1} n^{-a}$ is as before and

$$\zeta(a, b) = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2^b} + \cdot + \frac{1}{(n-1)^b}}{n^a}$$

✓ k is the sum's *depth* and $i_1 + i_2 + \cdots + i_k$ is its *weight*.

- This clearly extends to alternating and character sums.

- MZV's satisfy many striking identities, of which the simplest are

$$\zeta(2,1) = \zeta(3) \qquad 4\zeta(3,1) = \zeta(4).$$

- MZV's have recently found interesting interpretations in high energy physics, knot theory, combinatorics ...

✓ Euler found and partially proved theorems on **reducibility** of depth 2 to depth 1 ζ's

- $\zeta(6,2)$ is the lowest weight **'irreducible'**.

✓ High precision *fast ζ-convolution* (see *EZ-Face/Java*) allows use of integer relation methods and leads to important dimensional (reducibility) conjectures and amazing identities.

A Striking CONJECTURE open for all $n > 2$ is:

$$\boxed{8^n \zeta(\{-2,1\}_n) \stackrel{?}{=} \zeta(\{2,1\}_n)}$$

There is abundant evidence amassed since it was found in 1996.

© For example, very recently Petr Lisonek checked the first 85 cases to 1000 places in about 41 HP hours with only the *expected error*. And N=163 in ten hours.

- This is the *only* identification of its type of an Euler sum with a distinct MZV.

- Can even just the case $n = 2$ be proven *symbolically* as is the case for $n = 1$?

II. A CHARACTER EULER SUM

Let

$$[2b,-3](s,t) := \sum_{n>m>0} \frac{(-1)^{n-1}}{n^s}\frac{\chi_3(m)}{m^t},$$

where χ_3 is the character modulo 3. Then for $N = 0, 1, 2, \ldots$ $[2b,-3](2N+1,1)$

$$\begin{aligned}
&= \frac{L_{-3}(2N+2)}{4^{1+N}} - \frac{1+4^{-N}}{2} L_{-3}(2N+1)\log(3) \\
&+ \sum_{k=1}^{N} \frac{1-3^{-2N+2k}}{2} L_{-3}(2N-2k+2)\,\alpha(2k) \\
&- \sum_{k=1}^{N} \frac{1-9^{-k}}{1-4^{-k}} \frac{1+4^{-N+k}}{2} L_{-3}(2N-2k+1)\,\alpha(2k+1) \\
&- 2L_{-3}(1)\,\alpha(2N+1)\,.
\end{aligned}$$

✓ Here α is the *alternating zeta function* and L_{-3} is the *primitive L-series modulo 3*.

✓ One first evaluates such sums as integrals

COINCIDENCE OR FRAUD

- Coincidences do occur

The approximations

$$\pi \approx \frac{3}{\sqrt{163}} \log(640320)$$

and

$$\pi \approx \sqrt{2}\,\frac{9801}{4412}$$

occur for deep number theoretic reasons—the first good to 15 places, the second to eight

By contrast

$$e^{\pi} - \pi = \mathbf{19.99909997}9189475768\ldots$$

most probably for no good reason.

✓ This seemed more bizarre on an eight digit calculator

Likewise, as spotted by Pierre Lanchon recently

$$e = \overline{\mathbf{10.10110111111000010}}101000101100\ldots$$

while

$$\pi = 11.0010\overline{\mathbf{0100001111110110101}}01000\ldots$$

have 19 bits agreeing in base two—with one read right to left

- More extended coincidences are almost always contrived ...

- And strong heuristics exist for believing results like the three preceding examples

® But recall the *Skewes number* and the *Merten Conjecture* counter-examples

DICTIONARIES ARE LIKE TIMEPIECES

► Samuel Johnson observed of watches that "the best do not run true, and the worst are better than none." The same is true of tables and databases. Michael Berry "would give up Shakespeare in favor of Prudnikov, Brychkov and Marichev."

- That excellent compendium contains

$$(1) \qquad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^2\left(k^2 - kl + l^2\right)} = \frac{\pi^{\propto}\sqrt{3}}{30},$$

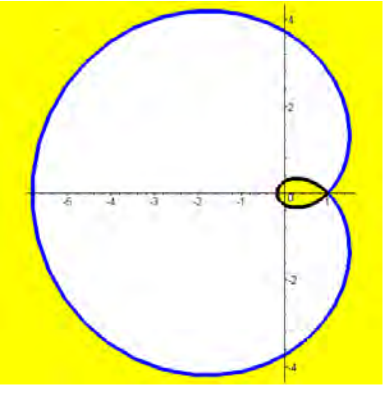
where the "$\propto$" is *probably* "4" [volume 1, entry 9, page 750].

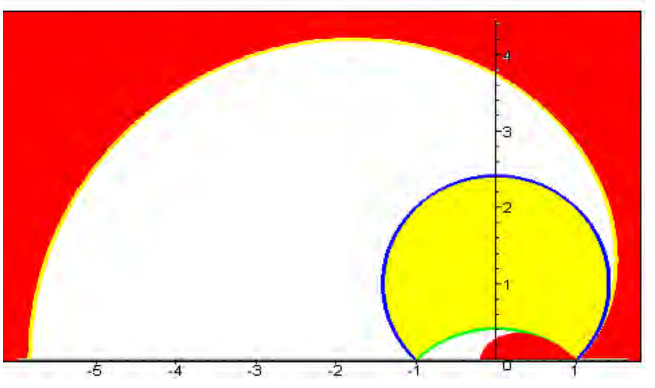
★ Integer relation methods suggest that no reasonable value of $\propto$ works

Forensic Mathematics (CSI-Math). What is intended in (1)?

POLYA and HEURISTICS

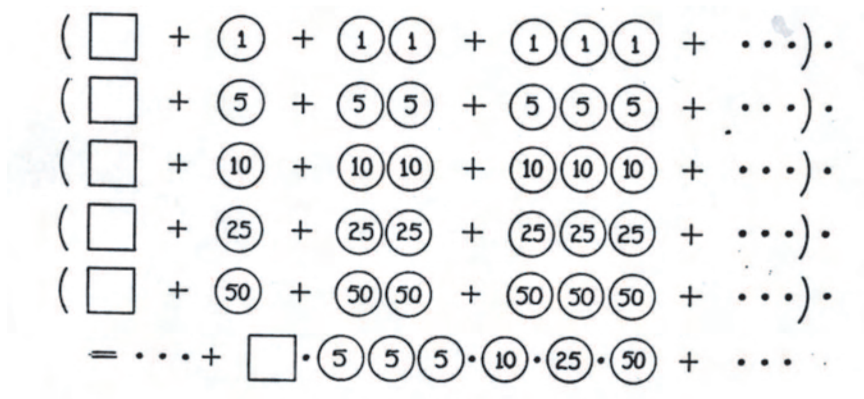
"[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication." (George Polya)*





*In *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving, 1968.*

Polya on Picture-writing



Polya's illustration of the change solution

Polya, in his 1956 *American Mathematical Monthly* article provided three provoking examples of converting pictorial representations of problems into generating function solutions. We discuss the first one.

1. *In how many ways can you make change for a dollar?*

This leads to the (US currency) generating function $\sum_{k\geq 0} P_k x^k =$

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

which one can easily expand using a *Mathematica* command,

```
Series[
1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)*(1-x^50))
,{x,0,100}]
```

to obtain $P_{100} = 292$ (**243** for Canadian currency, which lacks a 50 cent piece but has a dollar coin).

- Polya's diagram is shown in the Figure.*

*Illustration courtesy the Mathematical Association of America

- To see why, we use geometric series and consider the so called *ordinary generating function*

$$\frac{1}{1-x^{10}} = 1 + x^{10} + x^{20} + x^{30} + \cdots$$

for dimes and

$$\frac{1}{1-x^{25}} = 1 + x^{25} + x^{50} + x^{75} + \cdots$$

for quarters etc.

- We multiply these two together and compare coefficients

$$\begin{aligned}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}} &= 1 + x^{10} + x^{20} + x^{25} \\ &+ x^{30} + x^{35} + x^{40} + x^{45} \\ &+ 2\,x^{50} + x^{55} + 2\,x^{60} + \cdots\end{aligned}$$

We argue that the *coefficient* of x^{60} on the right is precisely the number of ways of making 60 cents out of identical dimes and quarters.

- This is easy to check with a handful of change or a calculator, The general question with more denominations is handled similarly.

- I leave it open whether it is easier to decode the generating function from the picture or vice versa. In any event, symbolic and graphic experiment provide abundant and mutual reinforcement and assistance in concept formation.

> "*In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting.*" (George Polya)

While by 'beginner' George Polya intended young school students, I suggest this is equally true of anyone engaging for the first time with an unfamiliar topic in mathematics.

SIMON and RUSSELL on INDUCTION

This skyhook-skyscraper construction of science from the roof down to the yet unconstructed foundations was possible because the behaviour of the system at each level depended only on a very approximate, simplified, abstracted characterization at the level beneath.[13]

This is lucky, else the safety of bridges and airplanes might depend on the correctness of the "Eightfold Way" of looking at elementary particles.

- ◇ Herbert A. Simon, *The Sciences of the Artificial*, MIT Press, 1996, page 16. (An early experimental computational scientist.)

[13]... *More than fifty years ago Bertrand Russell made the same point about the architecture of mathematics. See the "Preface" to Principia Mathematica "... the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question allows us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reason rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises." Contemporary preferences for deductive formalisms frequently blind us to this important fact, which is no less true today than it was in 1910.*

FROM ENIAC: Integrator and Calculator

SIZE/WEIGHT: ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons

SPEED/MEMORY: A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000 — 1,000 times faster than any earlier machine. The first stored-memory computer, ENIAC could store 200 digits.

ARCHITECTURE: Data flowed from one accumulator to the next. After each accumulator finished a calculation, it communicated its results to the next in line

The accumulators were connected to each other manually

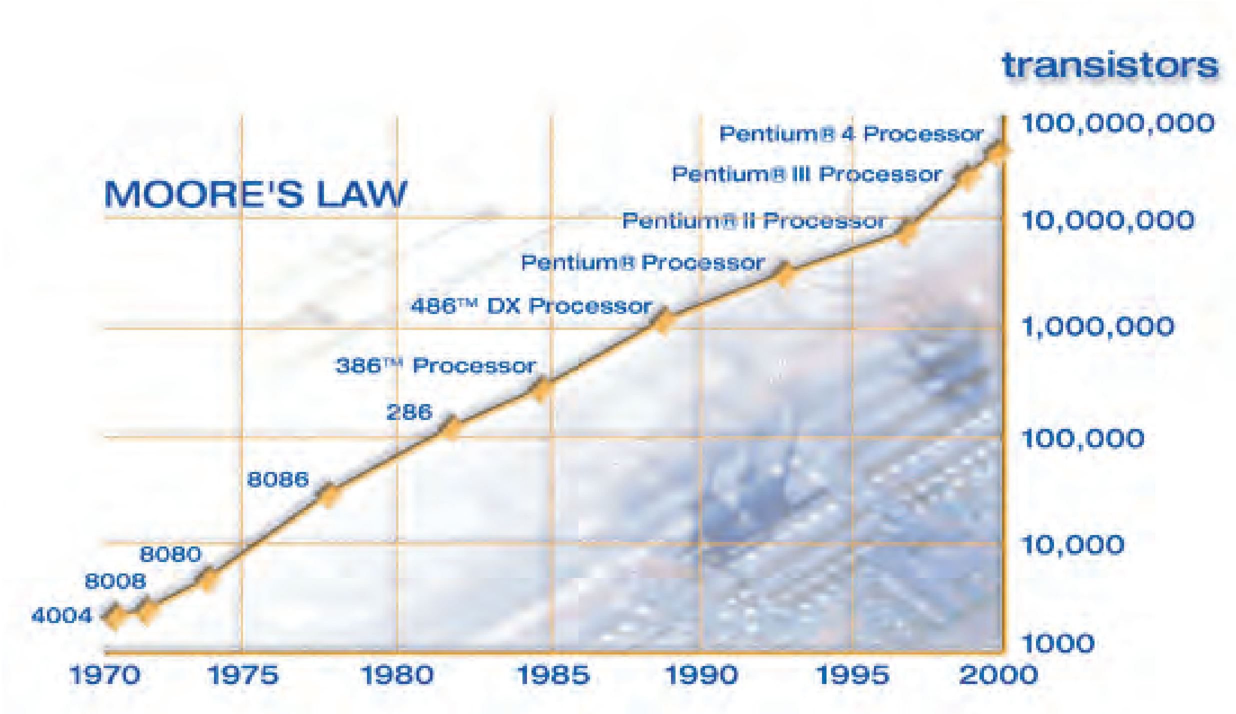
- The 1949 computation of π to 2,037 places suggested by von Neumann, took 70 hours

- It would have taken roughly 100,000 ENIACs to store the Smithsonian's picture!

⊗ Now after 40 years of Moore's law ...

> "Moore's Law" is now taken to be the assertion that semiconductor technology approximately doubles in capacity and performance roughly every 18 to 24 months

... To MOORE'S LAW



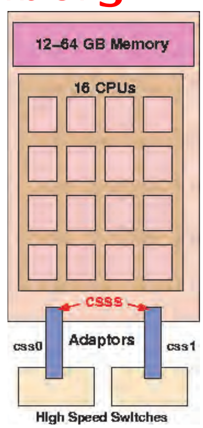
The complexity for minimum component costs has increased at a rate of roughly a factor of two per year. ... *Over the longer term, the rate of increase is a bit more uncertain, although there is no reason to believe it will not remain nearly constant for at least 10 years.* (Gordon Moore,* Intel co-founder, 1965)

*'Expect at least another decade.' (Moore et al)

- An astounding record of sustained exponential progress without peer in history of technology

- Math tools are now being implemented on parallel platforms, providing *much* greater power to the research mathematician

NERSC's 6656cpu Seaborg



727-fold speed-up of **quadrature** on the 1K G5's at **Virginia Tech** reduces **3hrs** to **15secs**

- Amassing huge amounts of processing power will not solve many mathematical problems. There are few math 'Grand-challenge problems' —more value in very rapid 'Aha's.

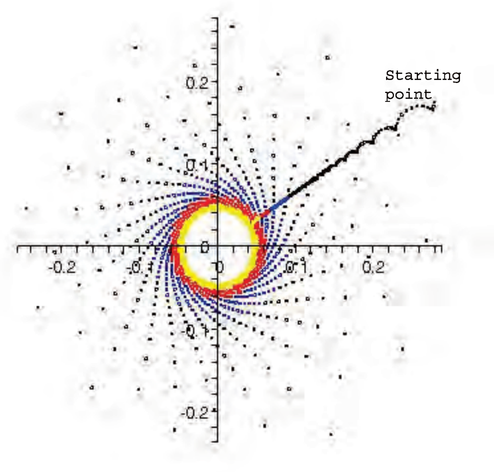
VISUAL DYNAMICS

• In recent continued fraction work, we needed to study the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

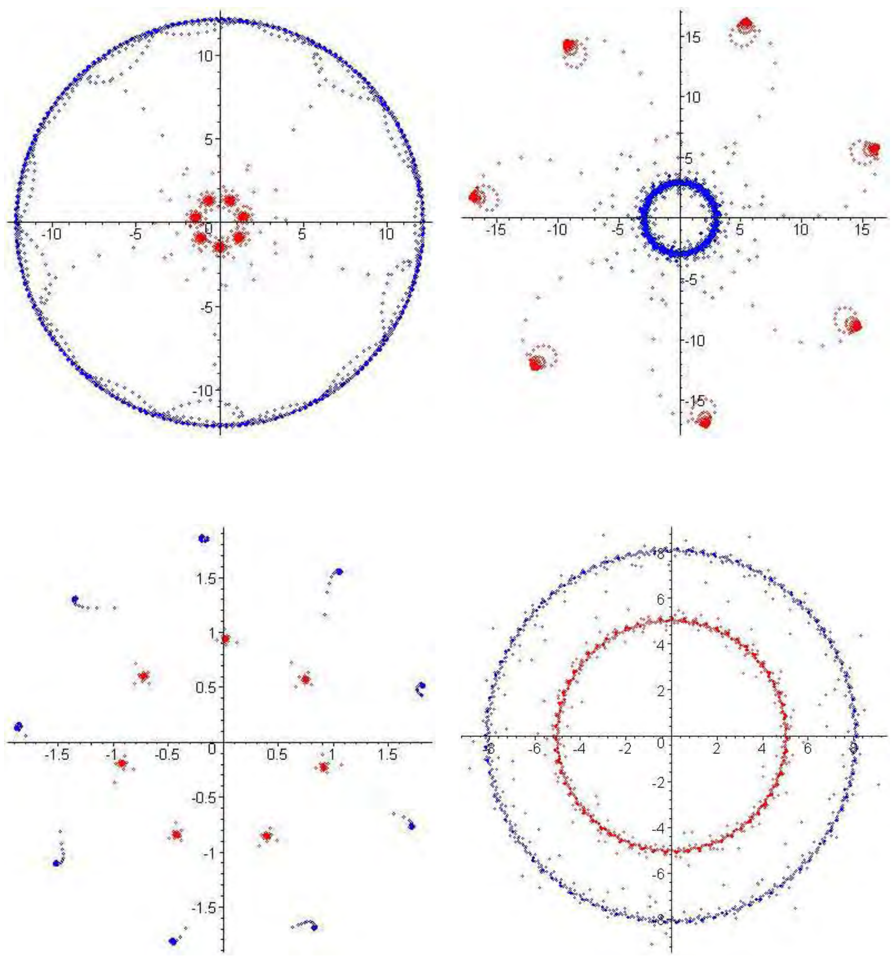
where $\omega_n = a^2, b^2$ for n even, odd respectively.
✓ Think of this as a **black box**.

▷ Numerically all one *sees* is $t_n \to 0$ slowly.
▷ Pictorially we *learn* significantly more*:



*... "Then felt I like a watcher of the skies, when a new planet swims into his ken." (*Chapman's Homer*)

- Scaling by $\sqrt{n}$, and coloring odd and even iterates, fine structure appears. We now predict and validate:



The attractors for various $|a| = |b| = 1$

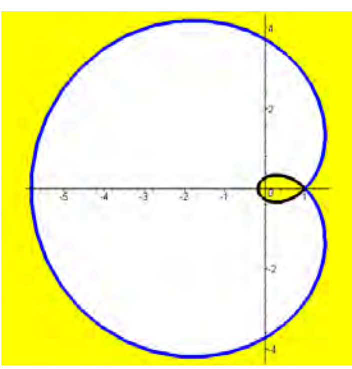
RAMANUJAN'S FRACTION

Chapter 18 of *Ramanujan's Second Notebook* studies the beautiful:

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}} \tag{1.1}$$

for **real, positive** $a, b, \eta > 0$. Remarkably, $\mathcal{R}$ satisfies an *AGM relation*

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2} \tag{1.2}$$



A scatter plot experiment discovered the domain of convergence for $a/b \in \mathbf{C}$. This is now fully explained with a *lot* of dynamics work.

HADAMARD and GAUSS

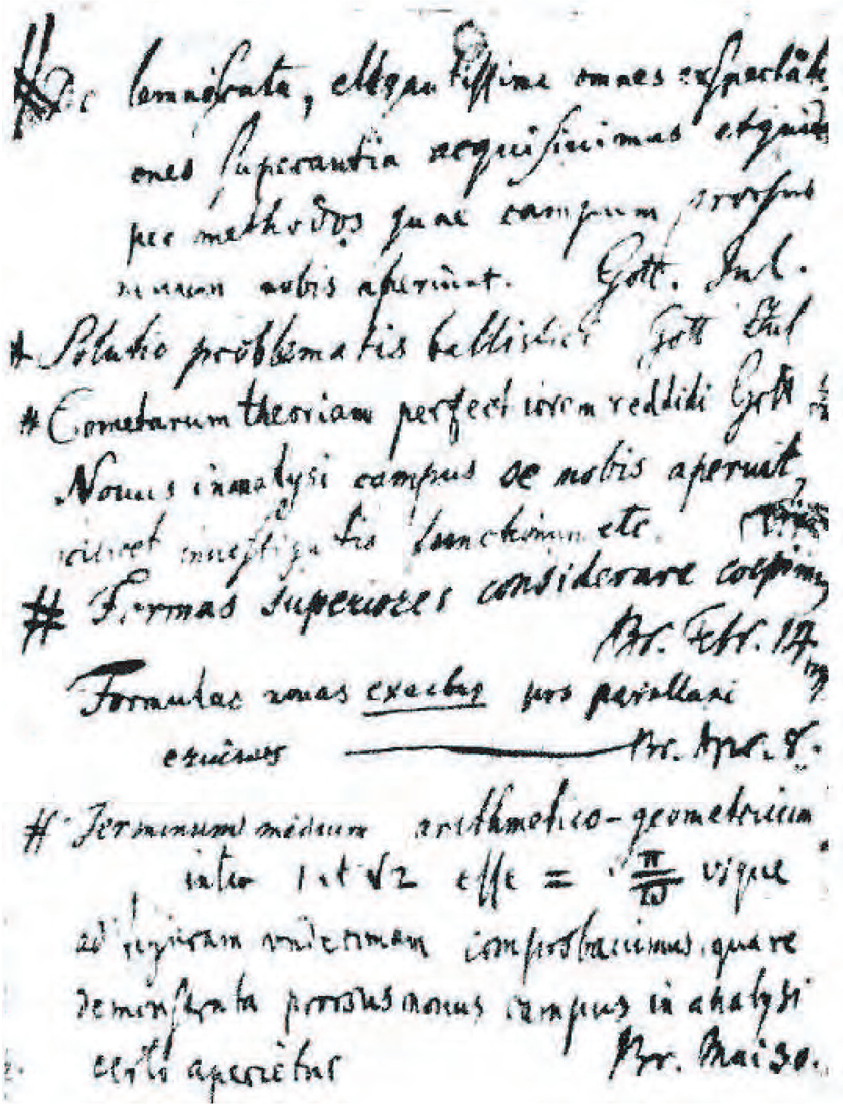
The object of mathematical rigor is to sanction and legitimize the conquests of intuition, *and there was never any other object for it.*

◇ J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.

Pauca sed Matura

Carl Friedrich Gauss, who drew (carefully) and computed a great deal, once noted, *I* **have** *the result, but I do not yet know how to get it.**

*Likewise the quote!



Novus in analysi campus se nobis aperuit

An excited young Gauss writes: "A new field of analysis has appeared to us, evidently in the study of functions etc." (October 1798)

HALES and KEPLER

- Kepler's conjecture: **the densest way to stack spheres is in a pyramid** is the oldest problem in discrete geometry.

- The most interesting recent example of computer assisted proof. Published in *Annals of Math* with an "only 99% checked" disclaimer.

- This has triggered very varied reactions. (In Math, Computers Don't Lie. Or Do They? *NYT* 6/4/04)

- Famous earlier examples: The Four Color Theorem and The non existence of a projective plane of order 10.

- The three raise and answer quite distinct questions—both real and specious.

Does the proof stack up?

Think peer review takes too long? One mathematician has waited four years to have his paper refereed, only to hear that the exhausted reviewers can't be certain whether his proof is correct. George Szpiro investigates.

P. TURNLEY/S. SHERBELL/R. BICKEL/CORBIS; H. SITTON/GETTY

Grocers the world over know the most efficient way to stack spheres — but a mathematical proof for the method has brought reviewers to their knees.

Just under five years ago, Thomas Hales made a startling claim. In an e-mail he sent to dozens of mathematicians, Hales declared that he had used a series of computers to prove an idea that has evaded certain confirmation for 400 years. The subject of his message was Kepler's conjecture, proposed by the German astronomer Johannes Kepler, which states that the densest arrangement of spheres is one in which they are stacked in a pyramid — much the same way as grocers arrange oranges.

Soon after Hales made his announcement, reports of the breakthrough appeared on the front pages of newspapers around the world. But today, Hales's proof remains in limbo. It has been submitted to the prestigious *Annals of Mathematics*, but is yet to appear in print. Those charged with checking it say that they believe the proof is correct, but are so exhausted with the verification process that they cannot definitively rule out any errors. So when Hales's manuscript finally does appear in the *Annals*, probably during the next year, it will carry an unusual editorial note — a statement that parts of the paper have proved impossible to check.

At the heart of this bizarre tale is the use of computers in mathematics, an issue that has split the field. Sometimes described as a 'brute force' approach, computer-aided proofs often involve calculating thousands of possible outcomes to a problem in order to produce the final solution. Many mathematicians dislike this method, arguing that it is inelegant. Others criticize it for not offering any insight into the problem under consideration. In 1977, for example, a computer-aided proof was published for the four-colour theorem, which states that no more than four colours are needed to fill in a map so that any two adjacent regions have different colours[1,2]. No errors have been found in the proof, but some mathematicians continue to seek a solution using conventional methods.

Pile-driver

Hales, who started his proof at the University of Michigan in Ann Arbor before moving to the University of Pittsburgh, Pennsylvania, began by reducing the infinite number of possible stacking arrangements to 5,000 contenders. He then used computers to calculate the density of each arrangement. Doing so was more difficult than it sounds. The proof involved checking a series of mathematical inequalities using specially written computer code. In all, more than 100,000 inequalities were verified over a ten-year period.

Robert MacPherson, a mathematician at the Institute for Advanced Study in Princeton, New Jersey, and an editor of the *Annals*, was intrigued when he heard about the proof. He wanted to ask Hales and his graduate student Sam Ferguson, who had assisted with the proof, to submit their finding for publication, but he was also uneasy about the computer-based nature of the work.

The *Annals* had, however, already accepted a shorter computer-aided proof — the paper, on a problem in topology, was published this March[3]. After sounding out his colleagues on the journal's editorial board, MacPherson asked Hales to submit his paper. Unusually, MacPherson assigned a dozen mathematicians to referee the proof — most journals tend to employ between one and three. The effort was led by Gábor Fejes Tóth of the Alfréd Rényi Institute of Mathematics in Budapest, Hungary, whose father, the mathematician László Fejes Tóth, had predicted in 1965 that computers would one day make a proof of Kepler's conjecture possible.

It was not enough for the referees to rerun Hales's code — they had to check whether the programs did the job that they were supposed to do. Inspecting all of the code and its inputs and outputs, which together take up three gigabytes of memory space, would have been impossible. So the referees limited themselves to consistency checks, a reconstruction of the thought processes behind each step of the proof, and then a

S. TERRY/SPL

study of all of the assumptions and logic used to design the code. A series of seminars, which ran for full academic years, was organized to aid the effort.

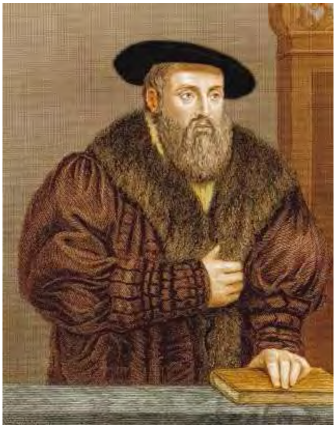
But success remained elusive. Last July, Fejes Tóth reported that he and the other referees were 99% certain that the proof is sound. They found no errors or omissions, but felt that without checking every line of the code, they could not be absolutely certain that the proof is correct.

For a mathematical proof, this was not enough. After all, most mathematicians believe in the conjecture already — the proof is supposed to turn that belief into certainty. The history of Kepler's conjecture also gives reason for caution. In 1993, Wu-Yi Hsiang, then at the University of California, Berkeley, published a 100-page proof of the conjecture in the *International Journal of Mathematics*[4]. But shortly after publication, errors were found in parts of the proof. Although Hsiang stands by his paper, most mathematicians do not believe it is valid.

After the referees' reports had been considered, Hales says that he received the following letter from MacPherson: "The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy … One can speculate whether their process would have converged to a definitive answer had they had a more clear manuscript from the beginning, but this does not matter now."

UNIV. PITTSBURGH CIDDE

Pyramid power: Thomas Hales believes that computers will succeed where humans have failed in verifying his proof.



Star player: Johannes Kepler's conjecture has kept mathematicians guessing for 400 years.

The last sentence lets some irritation shine through. The proof that Hales delivered was by no means a polished piece. The 250-page manuscript consisted of five separate papers, each a sort of lab report that Hales and Ferguson filled out whenever the computer finished part of the proof. This unusual format made for difficult reading. To make matters worse, the notation and definitions also varied slightly between the papers.

Rough but ready

MacPherson had asked the authors to edit their manuscript. But Hales and Ferguson did not want to spend another year reworking their paper. "Tom could spend the rest of his career simplifying the proof," Ferguson said when they completed their paper. "That doesn't seem like an appropriate use of his time." Hales turned to other challenges, using traditional methods to solve the 2,000-year-old honeycomb conjecture, which states that of all conceivable tiles of equal area that can be used to cover a floor without leaving any gaps, hexagonal tiles have the shortest perimeter[5]. Ferguson left academia to take a job with the US Department of Defense.

Faced with exhausted referees, the editorial board of the *Annals* decided to publish the paper — but with a cautionary note. The paper will appear with an introduction by the editors stating that proofs of this type, which involve the use of computers to check a large number of mathematical statements, may be impossible to review in full. The matter might have ended there, but for Hales, having a note attached to his proof was not satisfactory.

This January, he launched the Flyspeck project, also known as the Formal Proof of Kepler. Rather than rely on human referees, Hales intends to use computers to verify every step of his proof. The effort will require the collaboration of a core group of about ten volunteers, who will need to be qualified mathematicians and willing to donate the computer time on their machines. The team will write programs to deconstruct each step of the proof, line by line, into a set of axioms that are known to be correct. If every part of the code can be broken down into these axioms, the proof will finally be verified.

Those involved see the project as doing more than just validating Hales's proof. Sean McLaughlin, a graduate student at New York University, who studied under Hales and has used computer methods to solve other mathematical problems, has already volunteered. "It seems that checking computer-assisted proofs is almost impossible for humans," he says. "With luck, we will be able to show that problems of this size can be subjected to rigorous verification without the need for a referee process."

But not everyone shares McLaughlin's enthusiasm. Pierre Deligne, an algebraic geometer at the Institute for Advanced Study, is one of the many mathematicians who do not approve of computer-aided proofs. "I believe in a proof if I understand it," he says. For those who side with Deligne, using computers to remove human reviewers from the refereeing process is another step in the wrong direction.

Despite his reservations about the proof, MacPherson does not believe that mathematicians should cut themselves off from computers. Others go further. Freek Wiedijk, of the Catholic University of Nijmegen in the Netherlands, is a pioneer of the use of computers to verify proofs. He thinks that the process could become standard practice in mathematics. "People will look back at the turn of the twentieth century and say 'that is when it happened'," Wiedijk says.

Whether or not computer-checking takes off, it is likely to be several years before Flyspeck produces a result. Hales and McLaughlin are the only confirmed participants, although others have expressed an interest. Hales estimates that the whole process, from crafting the code to running it, is likely to take 20 person-years of work. Only then will Kepler's conjecture become Kepler's theorem, and we will know for sure whether we have been stacking oranges correctly all these years. ■

George Szpiro writes for the Swiss newspapers *NZZ* and *NZZ am Sonntag* from Jerusalem, Israel. His book *Kepler's Conjecture* (Wiley, New York) was published in February.

1. Appel, K. & Haken, W. *Illinois J. Math.* **21,** 429–490 (1977).
2. Appel, K., Haken, W. & Koch, J. *Illinois J. Math.* **21,** 491–567 (1977).
3. Gabai, D., Meyerhoff, G. R. & Thurston, N. *Ann. Math.* **157,** 335–431 (2003).
4. Hsiang, W.-Y. *Int. J. Math.* **4,** 739–831 (1993).
5. Hales, T. C. *Discrete Comput. Geom.* **25,** 1–22 (2001).

Flyspeck
➧ **www.math.pitt.edu/~thales/flyspeck/index.html**

"It says it's sick of doing things like inventories and payrolls, and it wants to make some break-throughs in astrophysics."

PART III. MATHEMATICAL MODELS

Felix Klein's heritage

Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. . . .


Hiroshi Sugimoto for The New York Times

It's hard to imagine that these plaster forms, so starkly beautiful, were originally used to teach advanced students trigonometry. Called stereometric models, they were manufactured in turn-of-the-century Germany to help scholars grasp complex mathematical formulas. Last year, the Japanese photographer Hiroshi Sugimoto shot each object, the tallest of which is less than a foot high, from below at close range so that they appear monumental. His series of photographs, "Mathematical Forms," reimagine these scientific models as things of wonder. They embody Sugimoto's belief that art is possible even without artistic intention.

Mathematical Form 0009

Conic surface of revolution
with constant negative curvature.

$$x = a \sinh v \cos u$$
$$y = a \sinh v \sin u$$
$$z = \int_0^v \sqrt{1 - a^2 \cosh^2 t}\, dt$$
$$(0 < a < 1,\ 0 \leq u < 2\pi)$$

Hiroshi Sugimoto for The New York Times

Mathematical Form 0001

Helicoid: minimal surface.

$$x = a \sinh v \cos u$$
$$y = a \sinh v \sin u$$
$$z = au$$
$$(0 \leq u < 2\pi, -\infty < v < \infty)$$

Hiroshi Sugimoto for The New York Times

Mathematical Form 0002

Hypersphere: constant curvature surface, revolution of hyperbolic type.

$$x = a \cosh v \cos u$$
$$y = a \cosh v \sin u$$
$$z = \int_0^v \sqrt{1\text{-}a^2 \sinh^2 t}\, dt$$
$$(a > 0, 0 \leq u < 2\pi)$$

Hiroshi Sugimoto for The New York Times

Mathematical Form 0003

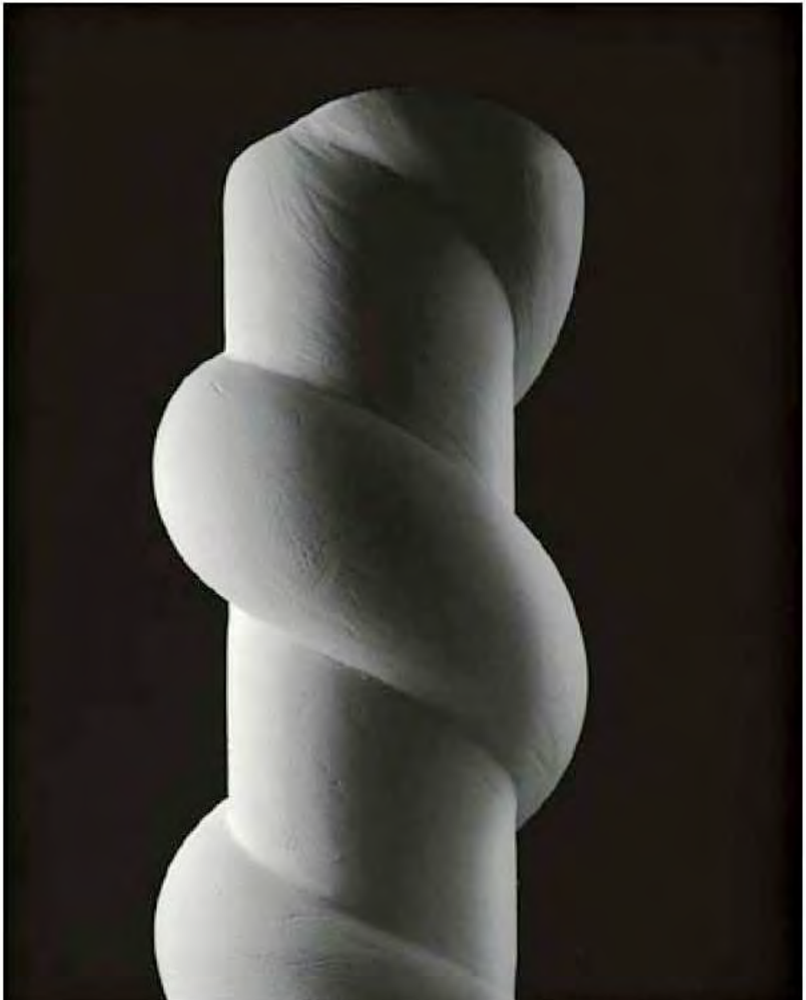
Dini's surface.

$$x = \frac{\cos u}{\cosh v}$$

$$y = \frac{\sin u}{\cosh v}$$

$$z = v - \tan v + au$$

$$(0 \leq u < 2\pi, -\infty < v < \infty)$$



Hiroshi Sugimoto for The New York Times

Mathematical Form 0005

Generalized helicoid of positive constant curavture.

$$x = a\cos v\cos u$$
$$y = a\cos v\sin u$$
$$z = \int_0^v \sqrt{1-a^2\sin^2 t}\,dt$$
$$(a > 0, 0 \leq u < 2\pi)$$

Hiroshi Sugimoto for The New York Times

Mathematical Form 0006

Kuen's surface:
constant negative curvature.

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = \log\tan\frac{v}{2} + a\cos v \ (0 < v < \pi)$$
$$\varphi = u - \arctan u$$
$$a = \frac{2}{1 + u^2\sin^2 v}$$
$$r = a\sqrt{1+u^2}\sin v$$

Hiroshi Sugimoto for The New York Times

Mathematical Form 0012

Clebsch diagonal surface: cubic, with 27 lines.

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0$$

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

$$(x_0 : x_1 : x_2 : x_3 : x_4) \in \mathrm{R}P^4$$

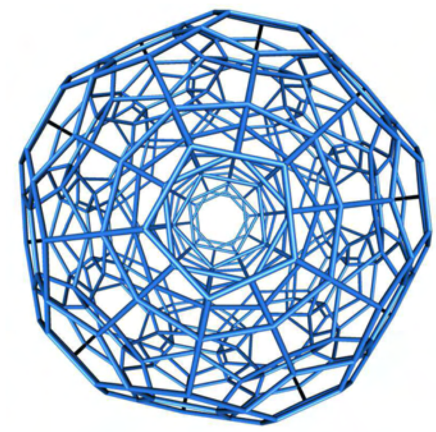
Mathematical forms from the collection of the University of Tokyo

I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane. Augustus de Morgan (1806–71).

- First President of the LMS, he was equally influential as an educator and a researcher

- There is evidence young children see more naturally in three than two dimensions

Donald Coxeter's (1907–2003) **octahedral kaleidoscope** built in Liverpool (circa 1925)



4D Coxeter polytope with 120 do- decahedral faces

- In a **1997** paper, Coxeter showed his friend M.C. Escher, knowing no math, had achieved "mathematical perfection" in etching *Circle Limit III*. "Escher did it by instinct," Coxeter wrote, "I did it by trigonometry."

David Mumford recently noted that Donald Coxeter placed great value on working out details of complicated explicit examples:

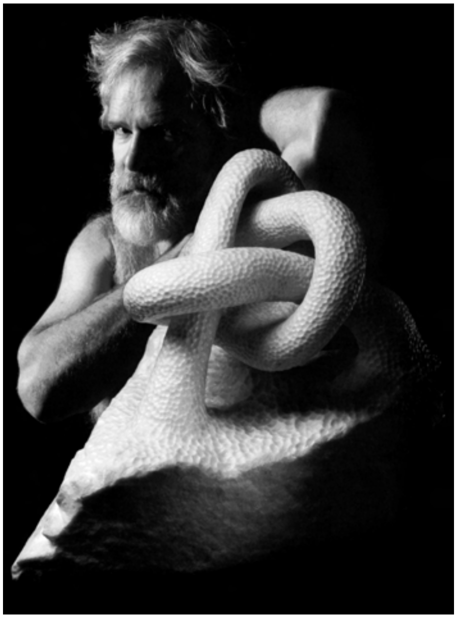
In my book, Coxeter has been one of the most important 20th century mathematicians —not because he started a new perspective, but because he deepened and extended so beautifully an older esthetic. The classical goal of geometry is the exploration and enumeration of geometric configurations of all kinds, their symmetries and the constructions relating them to each other.

The goal is not especially to prove theorems but to discover these perfect objects and, in doing this,theorems are only a tool that imperfect humans need to reassure themselves that they have seen them correctly. (David Mumford, 2003)

20th C. MATHEMATICAL MODELS

Ferguson's **"Eight-Fold Way"** sculpture

The Fergusons won the 2002 Communications Award, of the Joint Policy Board of Mathematics. The citation runs:

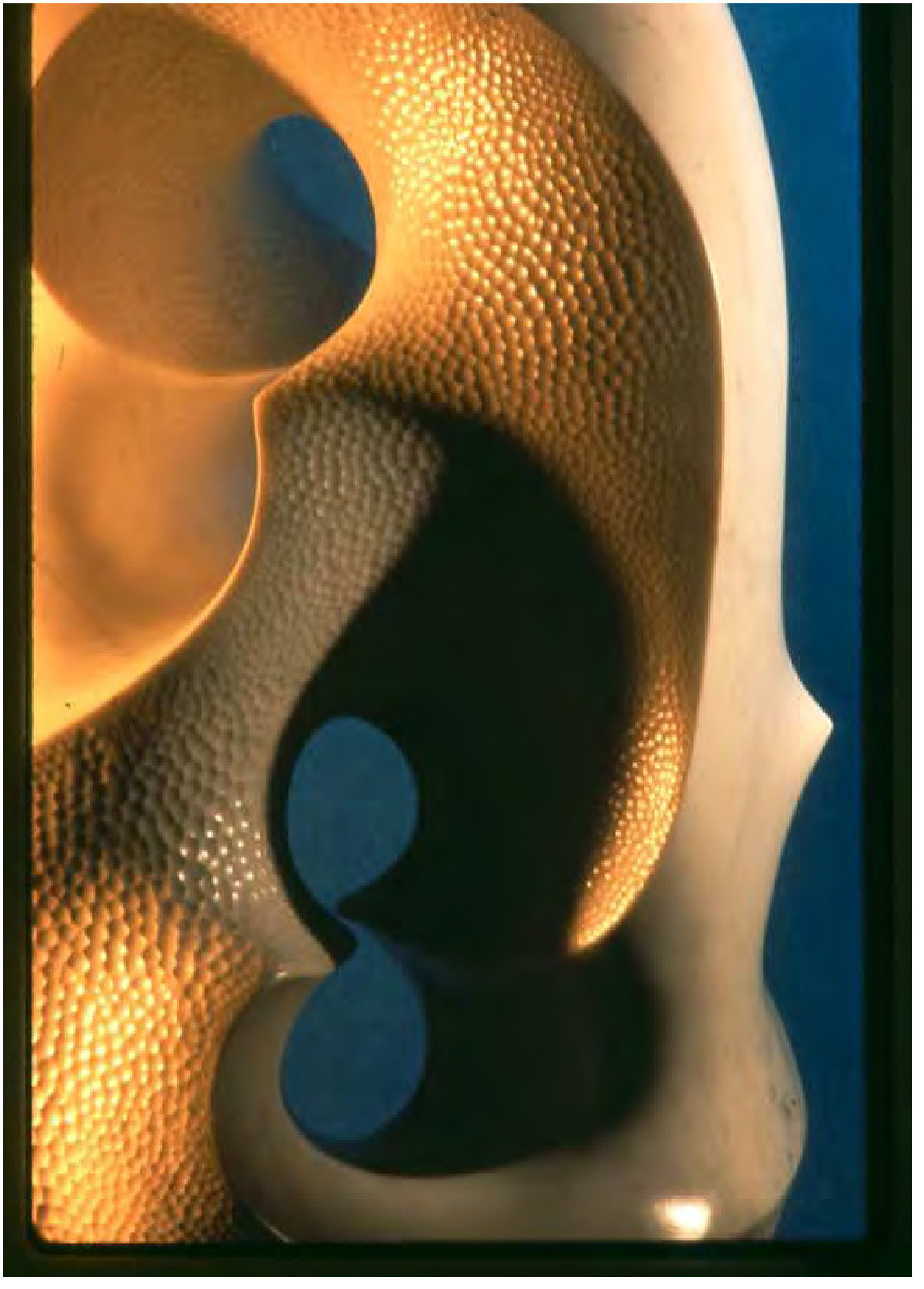


They *have dazzled the mathematical community and a far wider public with exquisite sculptures embodying mathematical ideas, along with artful and accessible essays and lectures elucidating the mathematical concepts.*

It has been known for some time that the *hyperbolic volume* V of the **figure-eight knot complement** is

$$V = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\sum_{k=n}^{2n-1}\frac{1}{k}$$

$$= 2.029883212819307250042405108549\ldots$$

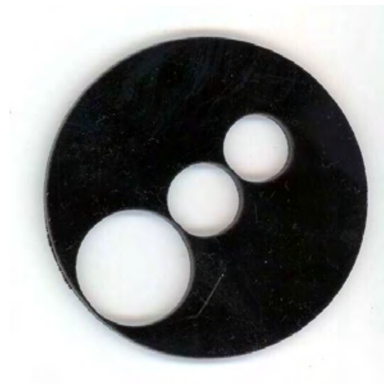


Ferguson's **"Figure-Eight Knot Complement"** sculpture

In 1998, British physicist David Broadhurst conjectured $V/\sqrt{3}$ is a *rational linear combination* of

$$C_j = \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n(6n+j)^2}. \tag{2}$$

Ferguson's
subtractive image
of the
BBP Pi formula



Indeed, as Broadhurst found, using PSLQ (*Ferguson's Integer Relation Algorithm*):

$$V = \frac{\sqrt{3}}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n} \times \left\{ \frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2} \right\}.$$

- Entering the following code in the *Mathematician's Toolkit*, at **www.expmath.info**:

```
v = 2 * sqrt[3] * sum[1/(n*binomial[2*n,n])
  * sum[1/k,{k, n,2*n-1}], {n, 1, infinity}]

  pslq[v/sqrt[3],
  table[sum[(-1)^n/(27^n*(6*n+j)^2),
  {n, 0, infinity}], {j, 1, 6}]]
```

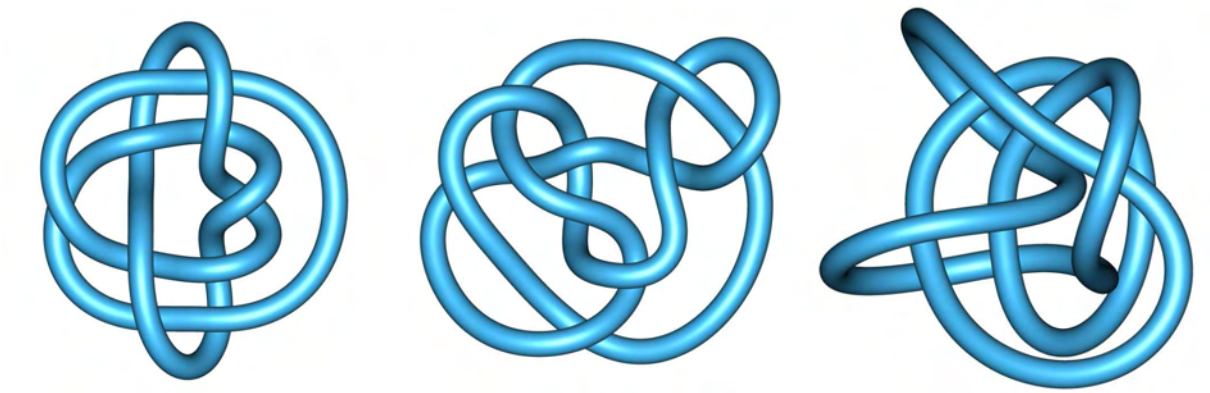
  recovers the solution vector

  (9, -18, 18, 24, 6, -2, 0)

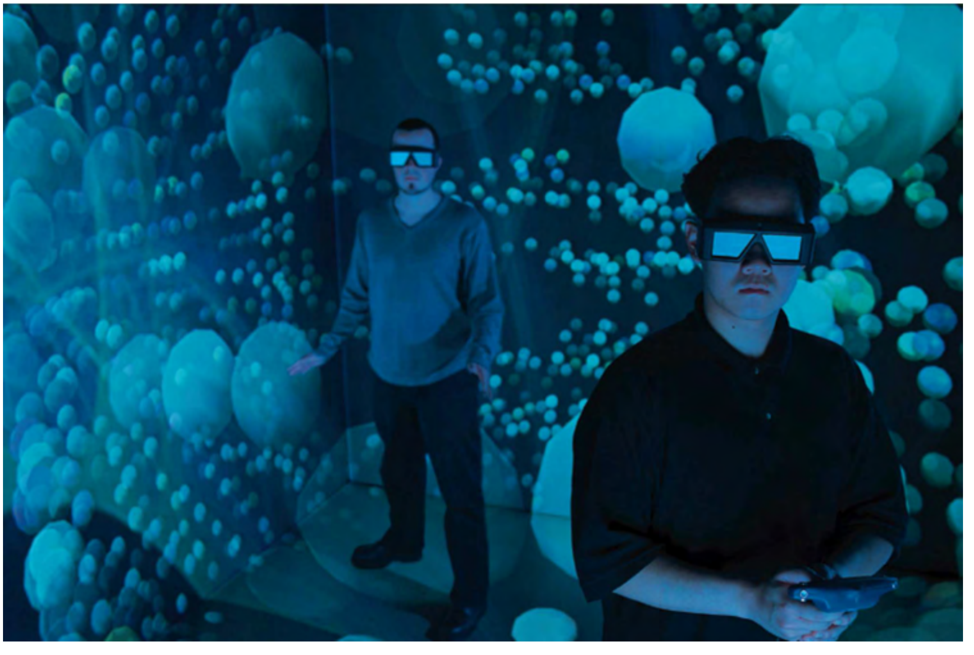
- The *first proof* that this formula holds is given in our recent book

- The formula is inscribed on each cast of the sculpture—marrying both sides of Helaman!

21st C. MATHEMATICAL MODELS



Knots 10_{161} (L) and 10_{162} (C) agree (R)*



In a NewMedia Cave or Plato's?

*KnotPlot: from Little (1899) to Perko (1974) and on

MORE of OUR 'METHODOLOGY'

1. (*High Precision*) computation of object(s)

2. *Pattern Recognition* of *Real Numbers*

   $$\text{identify}(\sqrt{2.}+\sqrt{3.}) = \sqrt{2}+\sqrt{3}$$

   (Inverse Calculator and 'identify')* or *Sequences* (Salvy & Zimmermann's 'gfun', Sloane and Plouffe's *Encyclopedia*).

3. Much use of 'Integer Relation Methods':†

   ✓ "Exclusion bounds" are especially useful
   ✓ Great test bed for "Experimental Math"

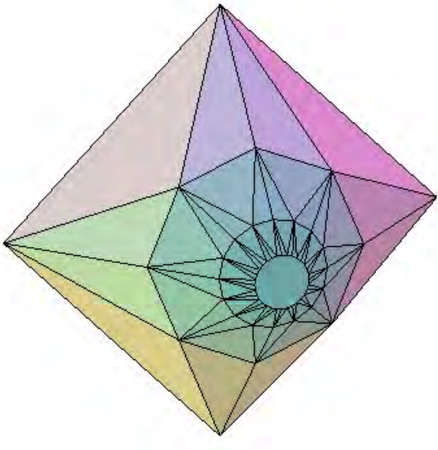
4. Some automated theorem proving (Wilf-Zeilberger etc)

*ISC space limits: from 10Mb in 1985 to 10Gb today.

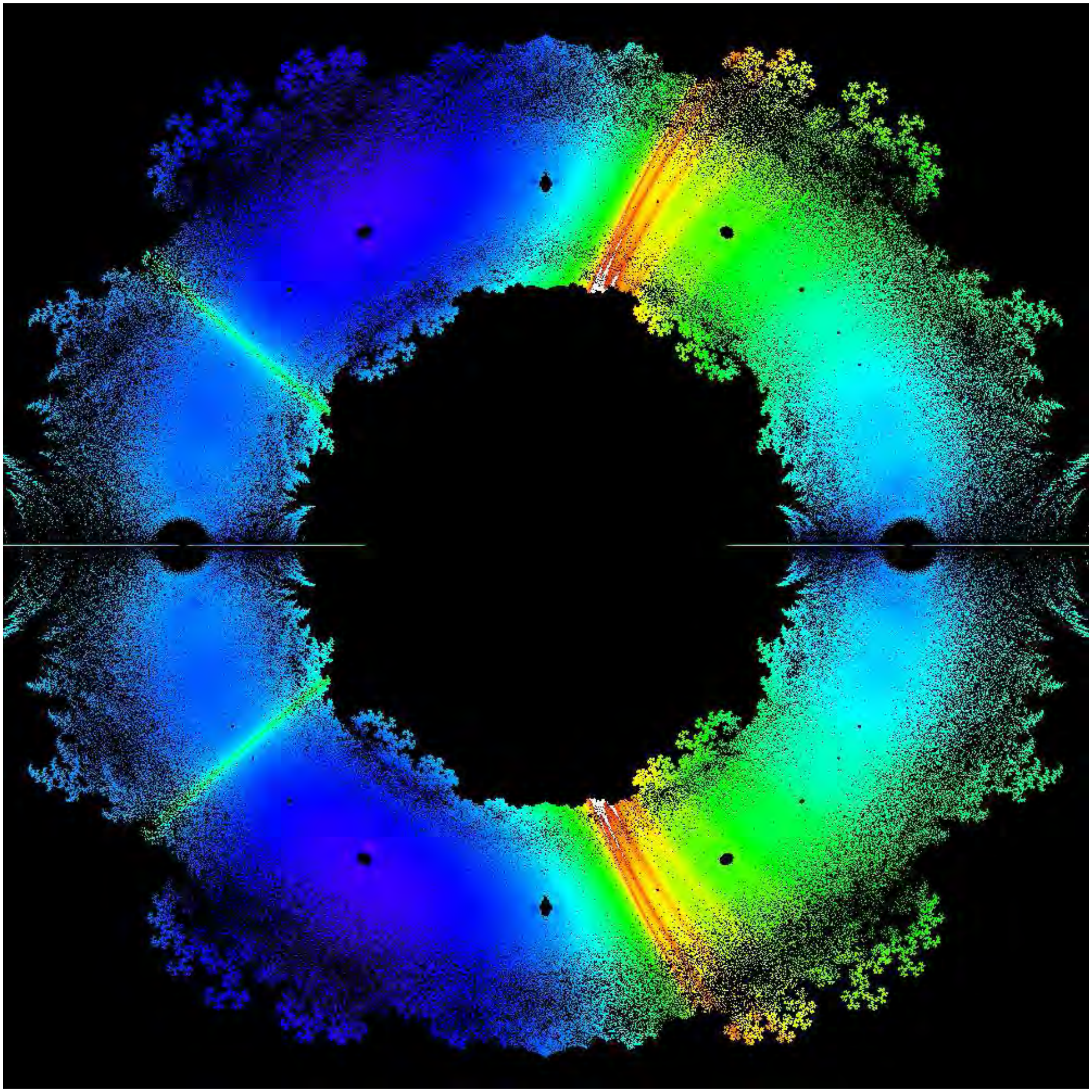
†*PSLQ, LLL, FFT*. Top Ten "Algorithm's for the Ages," Random Samples, Science, Feb. 4, 2000.

JOHN MILNOR

If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with.



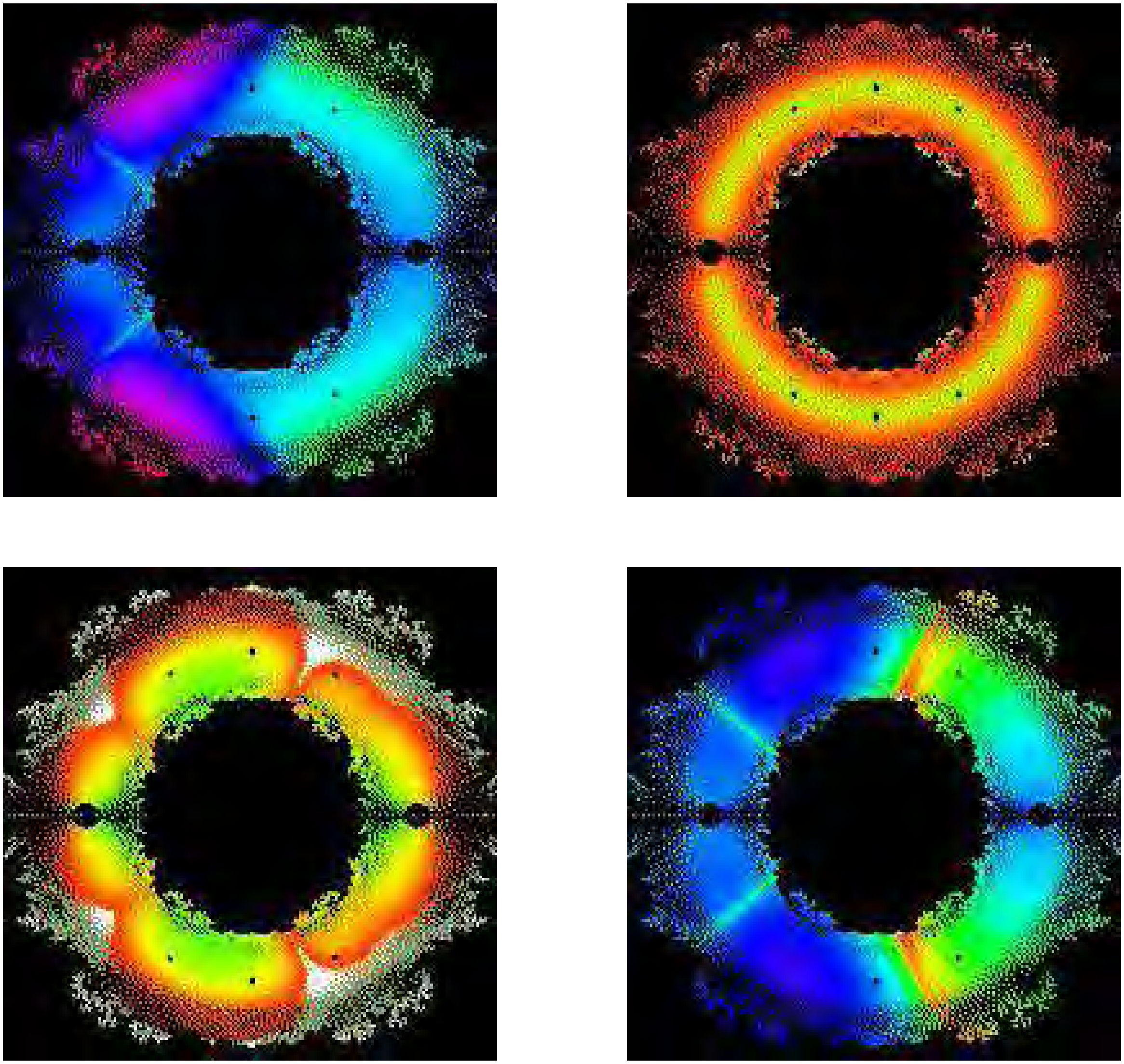
ZEROES of $0-1$ POLYNOMIALS



Data mining in polynomials

- The striations are unexplained!

WHAT YOU DRAW is WHAT YOU SEE



The price of metaphor is eternal vigilance (Arturo Rosenblueth & Norbert Wiener)

SEEING PATTERNS in PARTITIONS

The number of *additive partitions* of n, $p(n)$, is *generated* by

$$(3) \qquad 1 + \sum_{n \geq 1} p(n) q^n = \frac{1}{\prod_{n \geq 1}(1 - q^n)}.$$

Thus, $p(5) = 7$ since

$$\begin{aligned} 5 &= 4+1 = 3+2 = 3+1+1 = 2+2+1 \\ &= 2+1+1+1 = 1+1+1+1+1. \end{aligned}$$

Developing (3) is an introduction to enumeration via *generating functions* as discussed in Polya's change example.

Additive partitions are harder to handle than multiplicative factorizations, but they may be introduced in the elementary school curriculum with questions like: *How many 'trains' of a given length can be built with Cuisenaire rods*?

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable deep congruences like

$$p(\mathbf{5n+4}) \equiv 0 \mod \mathbf{5}, \qquad p(\mathbf{7n+5}) \equiv 0 \mod \mathbf{7}$$

$$p(\mathbf{11n+6}) \equiv 0 \mod \mathbf{11},$$

from relatively limited data like $P(q) =$

$$\begin{aligned} 1 &+ q + 2\,q^2 + 3\,q^3 + \underline{5}\,q^{\mathbf{4}} + \overline{7}\,q^{\mathbf{5}} + 11\,q^6 + 15\,q^7 \\ &+ 22\,q^8 + \underline{30}\,q^{\mathbf{9}} + 42\,q^{10} + 56\,q^{11} + \overline{77}\,q^{\mathbf{12}} \\ &+ 101\,q^{13} + \underline{135}\,q^{\mathbf{14}} + 176\,q^{15} + 231\,q^{16} \\ &+ 297\,q^{17} + 385\,q^{18} + \overline{\underline{490}}\,q^{\mathbf{19}} \\ &+ 627\,q^{20}b + 792\,q^{21} + 1002\,q^{22} \\ &+ \cdots + p(200)q^{200} + \cdots \end{aligned} \tag{4}$$

- Cases $5n+4$ and $7n+5$ are flagged in (4).

- Of course, it is easier to (heuristically) confirm than find these fine examples of **Mathematics: the science of patterns***

*Keith Devlin's 1997 book.

IS HARD or EASY BETTER?

A modern computationally driven question is *How hard is $p(n)$ to compute?*

- In **1900**, it took the father of combinatorics, Major Percy MacMahon (1854–1929), months to compute $p(200)$ using recursions developed from (3).

- By **2000**, *Maple* would produce $p(200)$ in seconds if one simply demands the 200'th term of the Taylor series. A few years earlier it required being careful to compute the series for $\prod_{n\geq 1}(1-q^n)$ *first* and *then* the series for the *reciprocal* of that series!

- This baroque event is occasioned by *Euler's pentagonal number theorem*

$$\boxed{\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)n/2}.}$$

- The reason is that, if one takes the series for (3), the software has to deal with **200** terms on the bottom. But the series for $\prod_{n\geq 1}(1-q^n)$, has only to handle the **23** non-zero terms in series in the pentagonal number theorem.

- If introspection fails, we can find the *pentagonal numbers* occurring above in *Sloane* and Plouffe's on-line '**Encyclopedia of Integer Sequences**': `www.research.att.com/personal/njas/sequences/eisonline.html`.

- This ex post facto algorithmic analysis can be used to facilitate independent student discovery of the pentagonal number theorem, and like results.

- The difficulty of estimating the size of $p(n)$ analytically—so as to avoid enormous or unattainable computational effort—led to some marvellous mathematical advances*.

- The corresponding ease of computation may now act as a retardant to insight.

- New mathematics is discovered only when prevailing tools run totally out of steam.

- This raises a caveat against mindless computing:

  > *Will a student or researcher discover structure when it is easy to compute without needing to think about it? Today, she may thoughtlessly compute* $p(500)$ *which a generation ago took much, much pain and insight*.

*By researchers including Hardy and Ramanujan, and Rademacher

The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description of the world.

It is a style that has shaped the physical but not the biological sciences, and its success in Newtonian mechanics, general relativity and quantum mechanics is among the miracles of mankind. But the era in thought that the calculus made possible is coming to an end. *Everyone feels this is so and everyone is right.*

$\cdots$ and $\cdots$

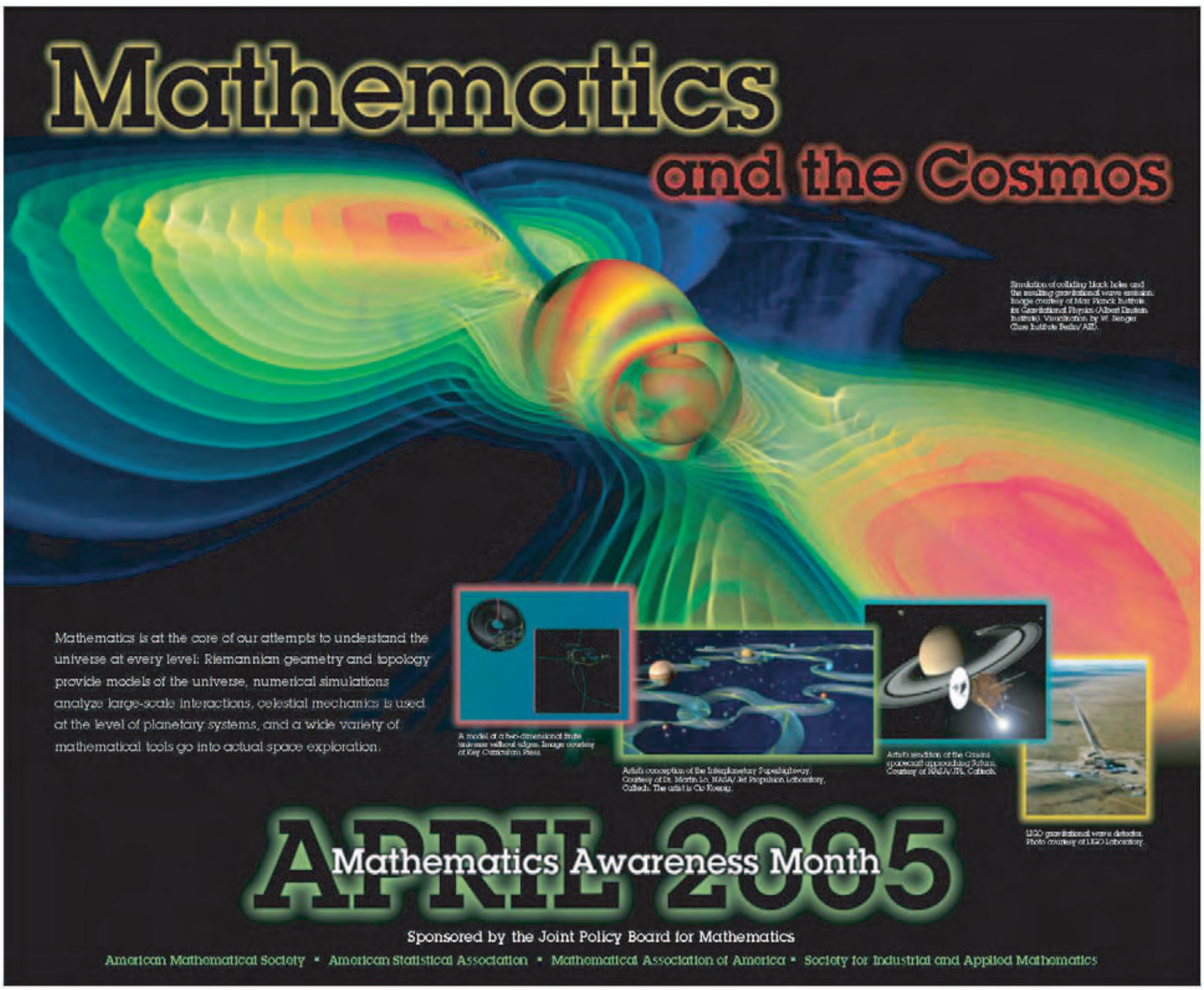
The computer has in turn changed the very nature of mathematical experience, *suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen.* (David Berlinski, 1997)*

- As all sciences rely more on 'dry experiments', via computer simulation, the boundary between physics (e.g.,*string theory*) and mathematics (e.g., *by experiment*) is again delightfully blurred.

- An early exciting example is provided by gravitational boosting:

*In "Ground Zero", a Review of *The Pleasures of Counting*, by T. W. Koerner.

- Interactive graphics will become an integral part of mathematics: gravitational boosting, gravity waves, Lagrange points ...

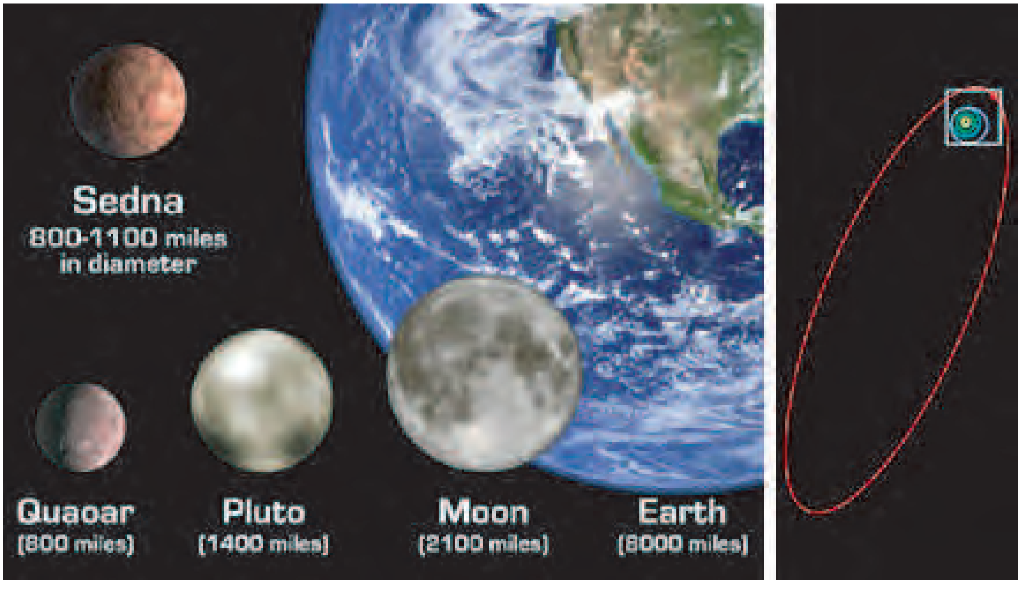


Gravitational Boosting

"The Voyager Neptune Planetary Guide" (JPL Publication 89–24) has an excellent description of Michael Minovitch' computational and unexpected discovery of *gravitational boosting* (also known as slingshot magic) at the Jet Propulsion Laboratory in 1961.

The article starts by quoting Arthur C. Clarke

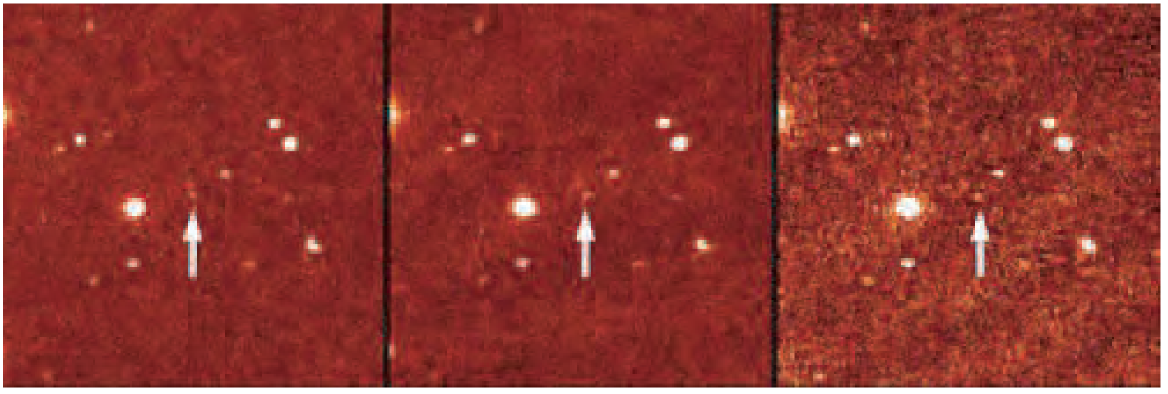
"Any sufficiently advanced technology is indistinguishable from magic."



Sedna And Friends in 2004

Until he showed *Hohmann transfer ellipses* were not least energy paths to the outer planets:

"most planetary mission designers considered the gravity field of a target planet to be somewhat of a nuisance, to be cancelled out, usually by onboard Rocket thrust."



- Without a boost from the orbits of Saturn, Jupiter and Uranus, the Earth-to-Neptune Voyager mission (achieved in 1989 in around a decade) would have taken over 30 years!

- We would still be waiting; longer to see Sedna confirmed (8 billion miles away—3 times further than Pluto).

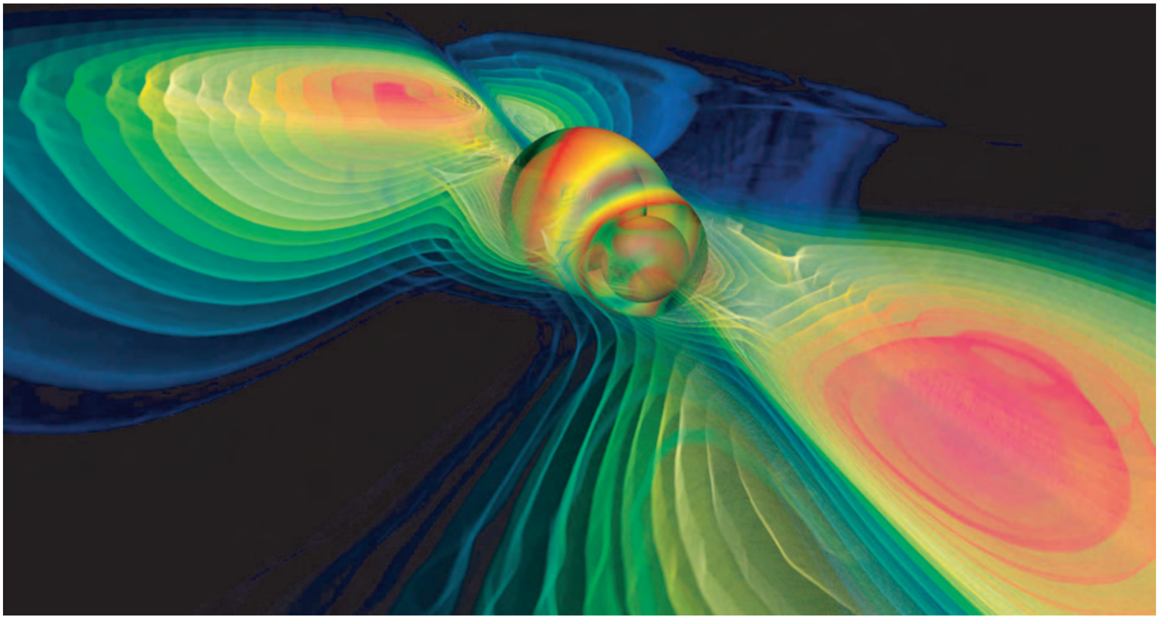
LIGO: Math and the Cosmos

Einstein's theory of general relativity describes how massive bodies curve space and time; it realizes gravity as movement of masses along shortest paths in curved space-time.

- A subtle mathematical inference is that relatively accelerating bodies will produce ripples on the curved space-time surface, propagating at the speed of light: **gravitational waves**.

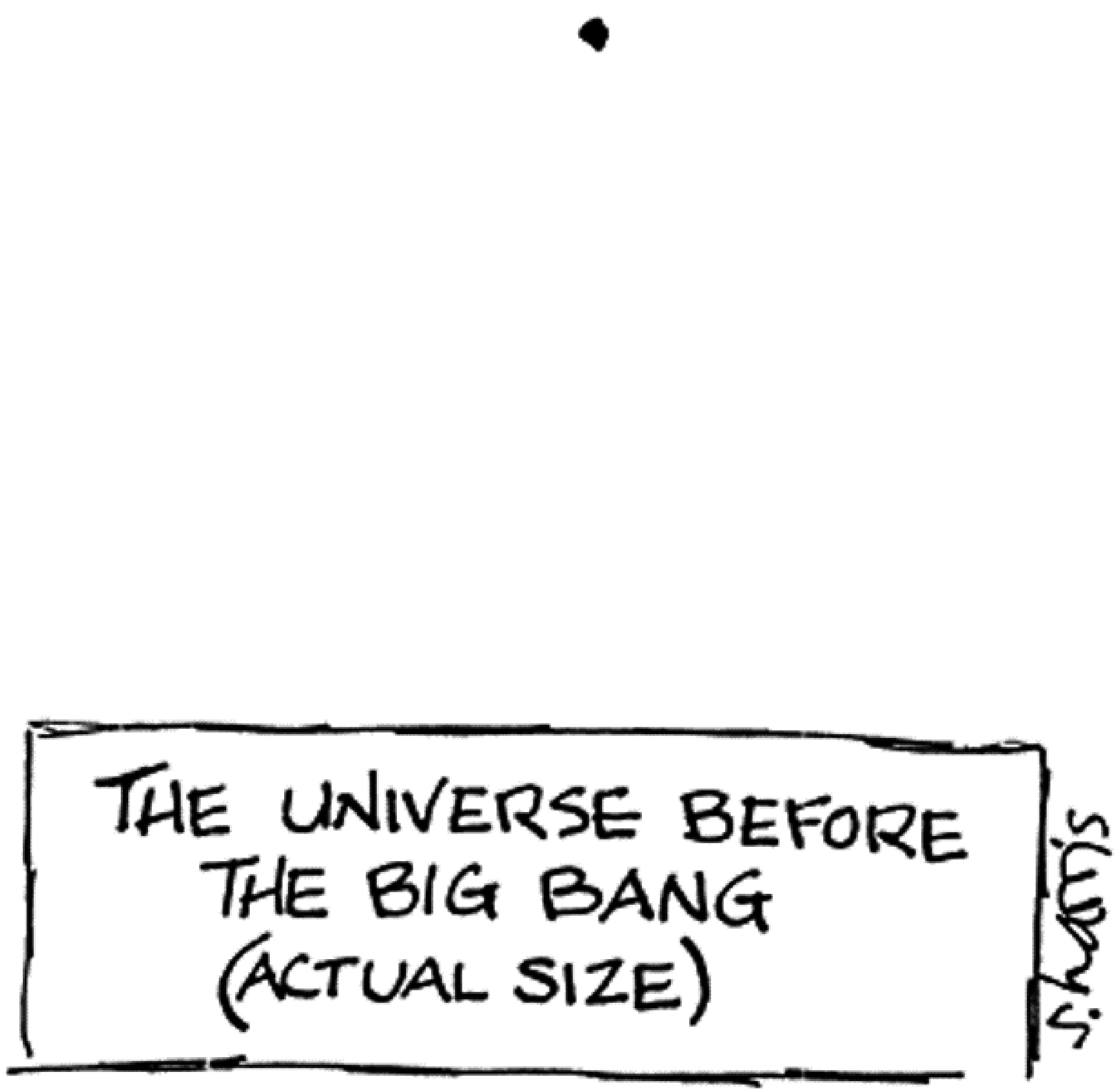
These extraordinarily weak cosmic signals hold the key to a new era of astronomy *if only* we can build detectors and untangle the mathematics to interpret them. The signal to noise ratio is tiny!

LIGO, the Laser Interferometer Gravitational-Wave Observatory, is such a developing US gravitational wave detector.



One of the first 3D simulations of the gravitational waves arising when two black holes collide

- Only recently has the computational power existed to realise such simulations, on computers such as at WestGrid.


THE UNIVERSE BEFORE
THE BIG BANG
(ACTUAL SIZE)
S.Harris

PART 4. CONCLUSIONS

> *The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. ⋯ in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced.* (Thomas Kuhn)

- In *Who Got Einstein's Office?* (Beurling)

> *And Max Planck, surveying his own career in his Scientific Autobiography, sadly remarked that "a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."* (Einstein)

HILBERT

Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution.

...

Besides it is an error to believe that rigor in the proof is the enemy of simplicity. (David Hilbert, 1900)

- In his '23' "Mathematische Probleme" lecture to the Paris International Congress, 1900*

*See Ben Yandell's fine account in *The Honors Class*, AK Peters, 2002.

CHAITIN

I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. I believe that Euclid's statement that an axiom is a self-evident truth is a big mistake. The Schrödinger equation certainly isn't a self-evident truth! And the Riemann Hypothesis isn't self-evident either, but it's very useful. A physicist would say that there is ample experimental evidence for the Riemann Hypothesis and would go ahead and take it as a working assumption.*

*There is no evidence that Euclid ever made such a statement. However, the statement does have an undeniable emotional appeal.

In this case, we have ample experimental evidence for the truth of our identity and we may want to take it as something more than just a working assumption. We may want to introduce it formally into our mathematical system. (Greg Chaitin, 1994)*

A tangible **Riemann surface for Lambert-**W

*A like article is in the 2004 *Mathematical Intelligencer*.

CARATHÉODORY and CHRÉTIEN

> *I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science.* (Constantin Carathéodory, 1936)

- Addressing the MAA (retro-digital data-mining?)

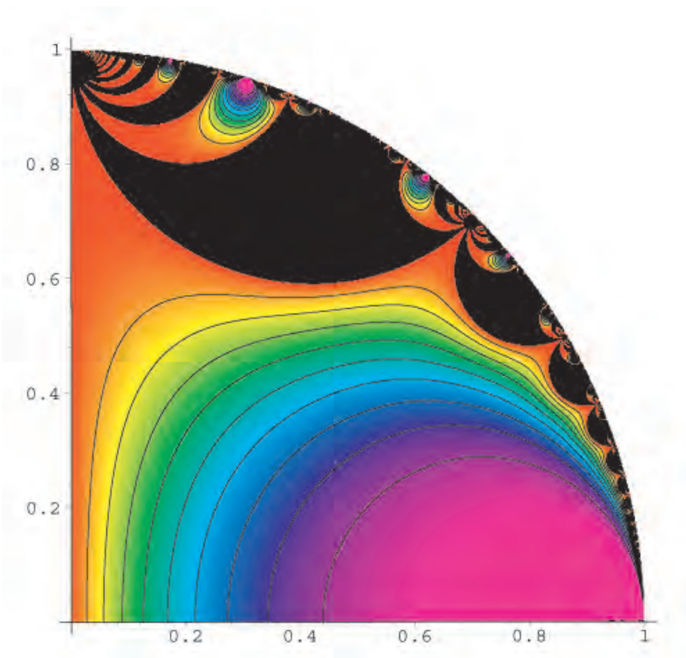
> *A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven.* (Jean Chrétien)

The then Prime Minister, explaining in 2002 how Canada would determine if Iraq had WMDs, sounds a lot like Bertrand Russell!

GAUSS

- Boris Stoicheff's often enthralling biography of Herzberg* records Gauss writing:

  It is not knowledge, but the act of learning, not possession but the act of getting there which generates the greatest satisfaction.



Fractal similarity
in
Gauss' discovery of modular functions

*Gerhard Herzberg (1903-99) fled Germany for Saskatchewan in 1935 and won the 1971 Chemistry Nobel

FINAL COMMENTS

★ The traditional deductive accounting of Mathematics is a largely ahistorical caricature*

★ Mathematics is primarily about secure knowledge not proof, and the aesthetic is central

- Proofs are often out of reach — understanding, even certainty, is not

- Packages can make concepts accessible (Linear relations, Galois theory, Groebner bases)

- While progress is made "*one funeral at a time*" (Niels Bohr), "*you can't go home again*" (Thomas Wolfe).

*Quotations are at `jborwein/quotations.html`

"The computer is claiming its intelligence is real, and ours is artificial."

HOW NOT TO EXPERIMENT

Pooh Math

'Guess and Check'

while

Aiming Too High

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3. D.H. Bailey and J.M. Borwein, "Experimental Mathematics: Recent Developments and Future Outlook," pp, 51-66 in Vol. I of *Mathematics Unlimited* — 2001 *and Beyond,* B. Engquist & W. Schmid (Eds.), Springer-Verlag, 2000. [CECM 99:143]

*All references are at `www.cecm.sfu.ca/preprints`.

4. J. Dongarra, F. Sullivan, "The top 10 algorithms," *Computing in Science & Engineering*, **2** (2000), 22–23. (See `personal/jborwein/algorithms.html`.)

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6. J.M. Borwein and D.H. Bailey), **Mathematics by Experiment: Plausible Reasoning in the 21st Century**, and **Experimentation in Mathematics: Computational Paths to Discovery**, (with R. Girgensohn,) AK Peters Ltd, 2003-04.

7. J.M. Borwein and T.S Stanway, " "Knowledge and Community in Mathematics," *The Mathematical Intelligencer*, in Press, 2004.

- The web site is at **www.expmath.info**