Seeing Things by Walking on Real Numbers

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(Joint work with Francisco Aragón, David Bailey and Peter Borwein)

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http://carma.newcastle.edu.au/meetings/evims/

57th AustMS Meeting
Number Theory Session
Sydney, October 2013

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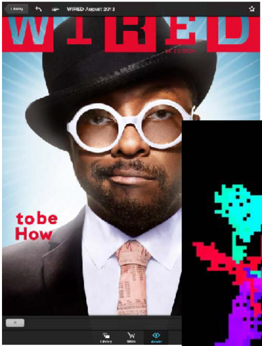
Contents:

One message is "Try drawing numbers"

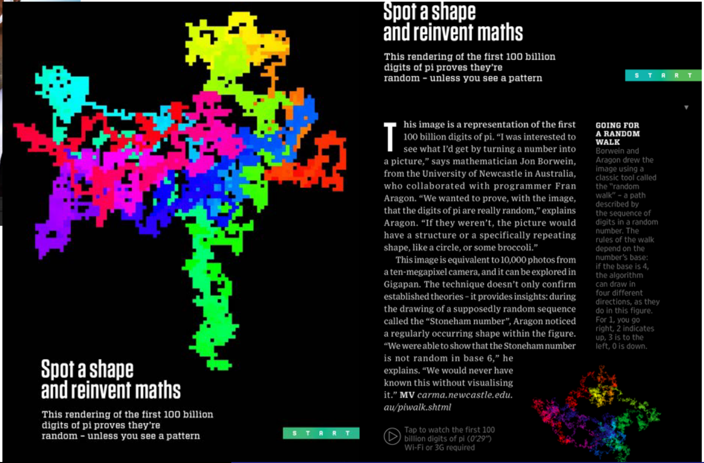
Contents

My collaborators

Outreach: images and animations led to high-level research which went viral



Wired UK August 2013



Spot a shape and reinvent maths

This rendering of the first 100 billion digits of pi proves they're random – unless you see a pattern

START

This image is a representation of the first 100 billion digits of pi. "I was interested to see what I'd get by turning a number into a picture," says mathematician Jon Borwein, from the University of Newcastle in Australia, who collaborated with programmer Fran Aragon. "We wanted to prove, with the image, that the digits of pi are really random," explains Aragon. "If they weren't, the picture would have a structure or a specifically repeating shape, like a circle, or some broccoli."

This image is equivalent to 10,000 photos from a ten-megapixel camera, and it can be explored in Gigapan. The technique doesn't only confirm established theories – it provides insights: during the drawing of a supposedly random sequence called the "Stoneham number", Aragon noticed a regularly occurring shape within the figure. "We were able to show that the Stoneham number is not random in base 6," he explains. "We would never have known this without visualising it." **MV** *carma.newcastle.edu.au/piwalk.shtml*

GOING FOR A RANDOM WALK

Borwein and Aragon drew the image using a classic tool called the "random walk" – a path described by the sequence of digits in a random number. The rules of the walk depend on the number's base: if the base is 4, the algorithm can draw in four different directions, as they do in this figure. For 1, you go right, 2 indicates up, 3 is to the left, 0 is down.

Tap to watch the first 100 billion digits of pi (0'29") Wi-Fi or 3G required

Contents

Some early conclusions: So I am sure they get made

Key ideas: randomness, normality of numbers, planar walks, and fractals

How not to experiment

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- using computer algebra, numerical computation and graphics: SNaG
- computations, tables and pictures are experimental data
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- and what you know you can usually use
- you do not need to know much before you start research (as we shall see)

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DHB and JMB, **Exploratory Experimentation in Mathematics** (2011), www.ams.org/notices/201110/rtx111001410p.pdf

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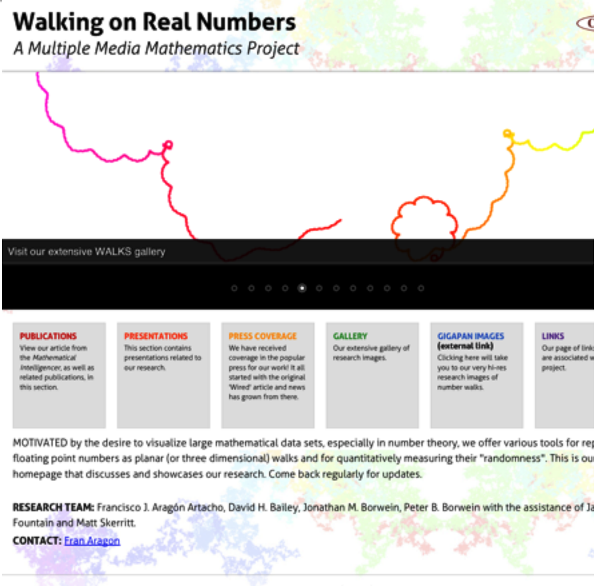

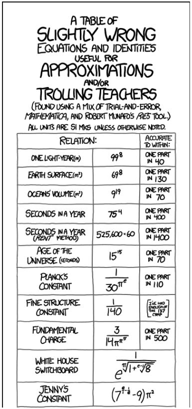

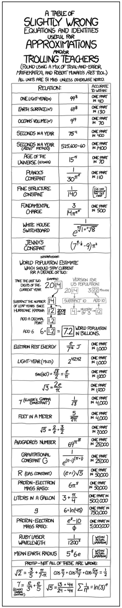

Almost all I mention is at `http://carma.newcastle.edu.au/walks/`

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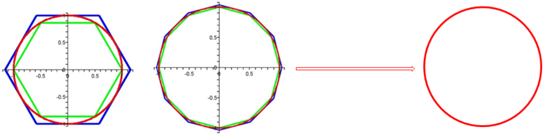

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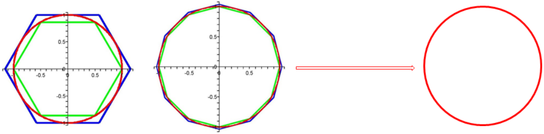

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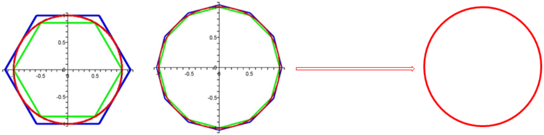
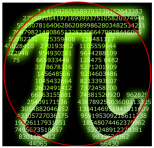


$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$ to obtain the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$



Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be "random":

$$\pi = 3.141592653589793238462643383279502884197169399375\dots$$

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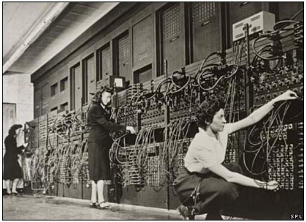
Are they really?

- **1949 ENIAC** (*Electronic Numerical Integrator and Calculator*) computed of π to **2,037** decimals (in **70** hours)—proposed by polymath John von Neumann (**1903-1957**) to shed light on distribution of π (and of e).



Two continued fractions

Change representations often

Gauss map. Remove the integer, invert the fraction and repeat: for 3.1415926 and 2.7182818 to get the fractions below.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}$$

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$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \dots}}}}}}}}}$$

Leonhard Euler (**1707-1783**) named e and π.

"Lisez Euler, lisez Euler, c'est notre maître à tous." Simon Laplace (**1749-1827**)

Are the digits of π random?

Digit	Ocurrences
0	99,993,942
1	99,997,334
2	100,002,410
3	99,986,911
4	**100,011**,958
5	**99,998**,885
6	100,010,387
7	99,996,061
8	100,001,839
9	100,000,273
Total	**1,000,000,000**

Table : Counts of first billion digits of π. Second half is 'right' for law of large numbers.

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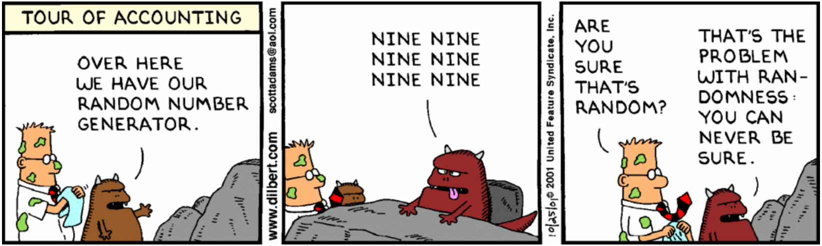
- There are equally many zeroes and ones in the binary expansion of π
- Or pretty much anything else...

What is “random”? A hard question



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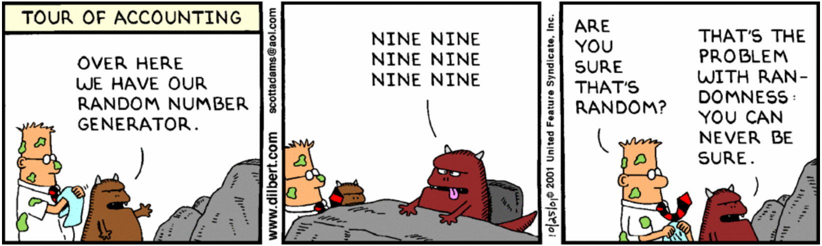

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- Without structure (noise)?
- Algorithmically random (π is not)?
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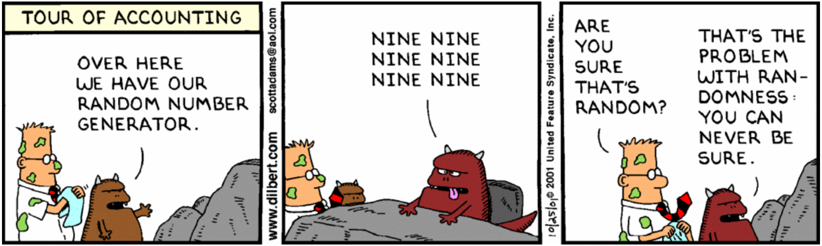

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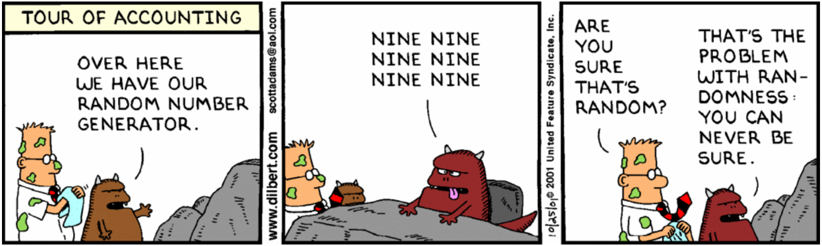

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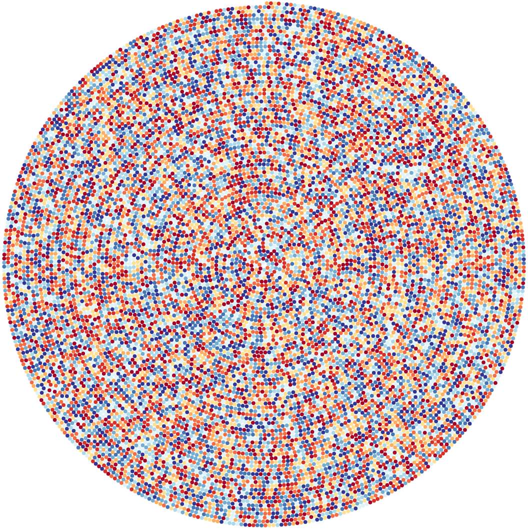
Best Theorem [BBCP, 04] (Feeble but hard) Asymptotically all degree d algebraics have at least $n^{1/d}$ ones in binary (should be $n/2$)

Randomness in Pi?

http://mkweb.bcgsc.ca/pi/art/



Normality

A property random numbers must possess

Definition

A real constant α is b-normal if, given the positive integer $b \geq 2$ (the base), every m-long string of base-b digits appears in the base-b expansion of α with precisely the expected limiting frequency $1/b^m$.

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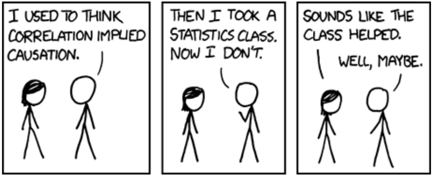

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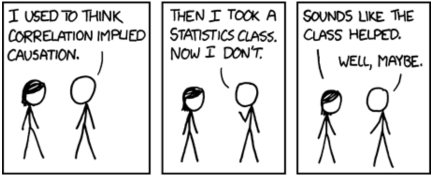

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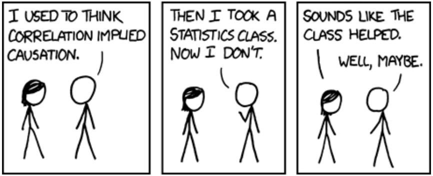
- Indeed, almost all real numbers are b-normal simultaneously for all positive integer bases ("absolute normality").
- Unfortunately, it has been very difficult to prove normality for any number in a given base b, much less all bases simultaneously.



Normal numbers *concatenation* numbers

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- The first constant proven 10-normal (and already proven transcendental by Mahler) was:

$$C_{10} := 0.123456789101112131415161718\ldots$$

  - **1933** by David Champernowne (1912-2000) as a student
  - Champernowne constant (**2012** proven not strongly normal)

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- Normality proofs are not known for $\pi, e, \log 2, \sqrt{2}$ etc.

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9	100,000,273	09	9,999,276	009	999,059
		10	9,997,289	010	998,884
		11	9,997,964	011	1,001,188
		⋮	⋮	⋮	⋮
		99	10,003,709	099	999,201
				⋮	⋮
				999	1,000,905
TOTAL	1,000,000,000	**TOTAL**	1,000,000,000	**TOTAL**	1,000,000,000

Table : Counts for the first billion digits of π.

Is π 16-normal

That is, in Hex?

0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	**62500**216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	**1,000,000,000,000**

$\hookleftarrow$ Counts of first trillion hex digits

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8	**62500**216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	**1,000,000,000,000**

$\hookleftarrow$ Counts of first trillion hex digits

- **2011** Ten trillion hex digits computed by Yee and Kondo – and seem very normal

Is π 16-normal

That is, in Hex?

0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
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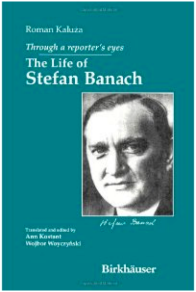
- They are $353CB3F7F0C9ACCFA9AA215F2$

See www.karrels.org/pi/index.html

OCTOPI

Stefan Banach (1892-1945) Another Nazi casuality

A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. [1]



[1] Only the best get stamps. Quoted in `www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html`.

Contents

What is a (base four) random walk ?

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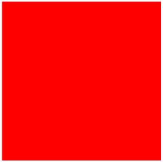

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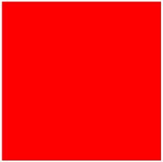

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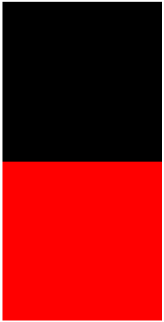

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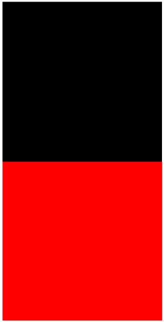

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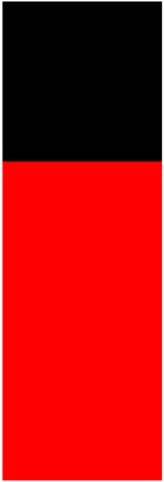

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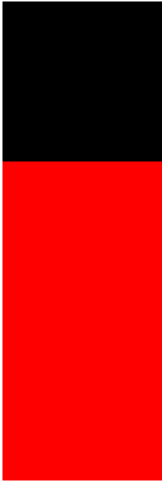

$$2 = \leftarrow$$

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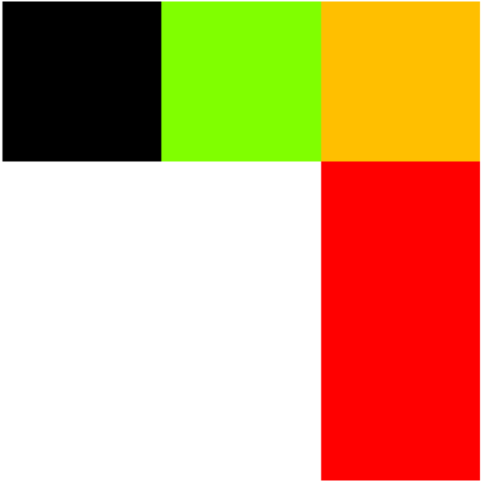

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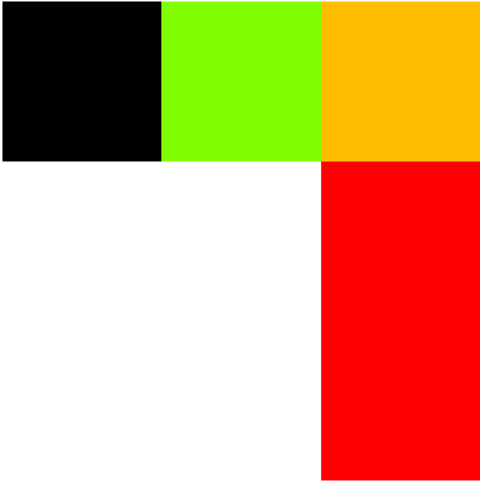

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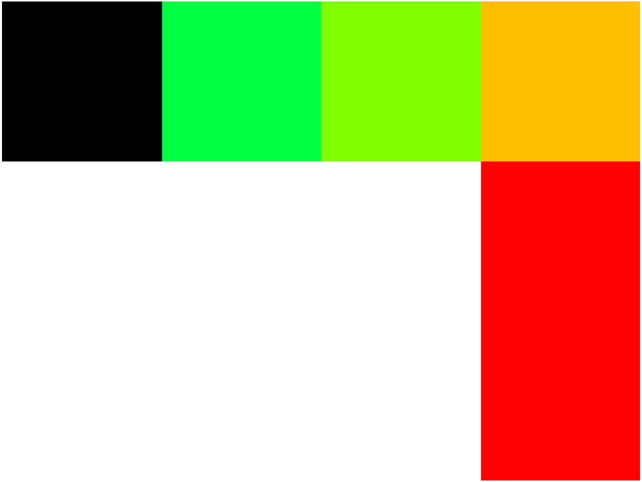

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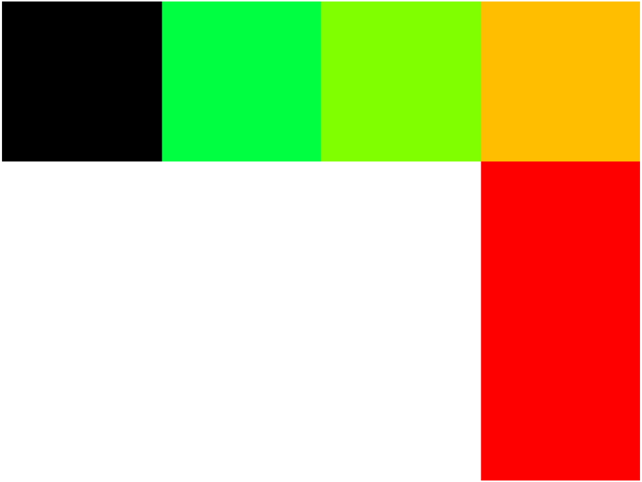

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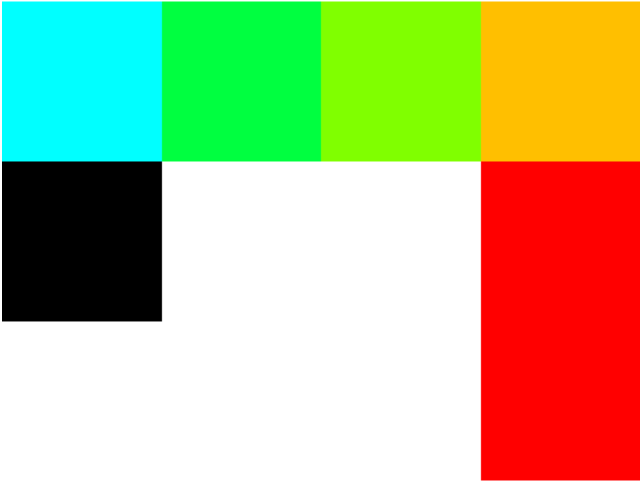

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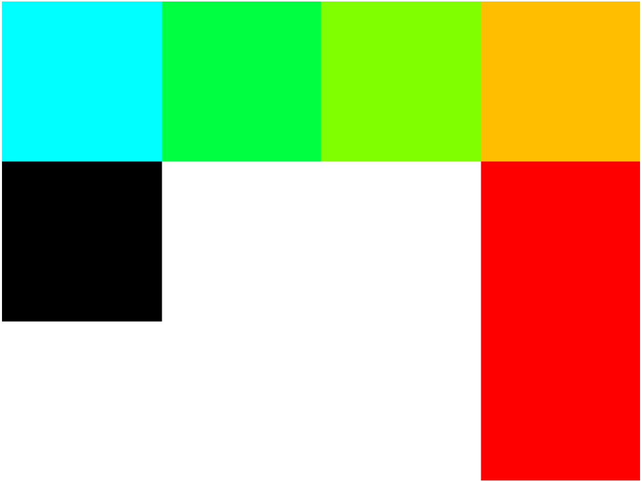

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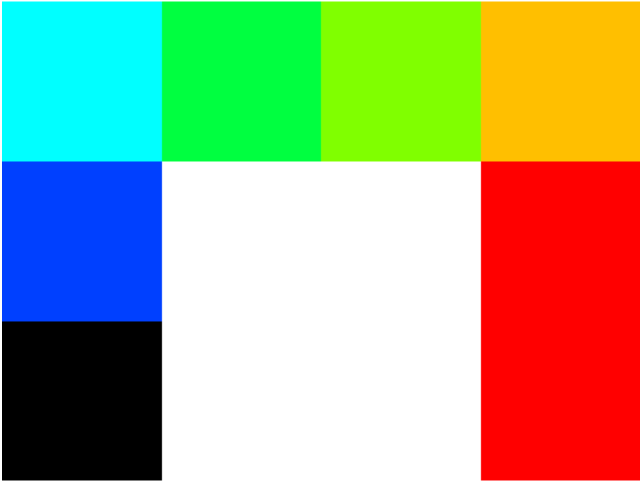

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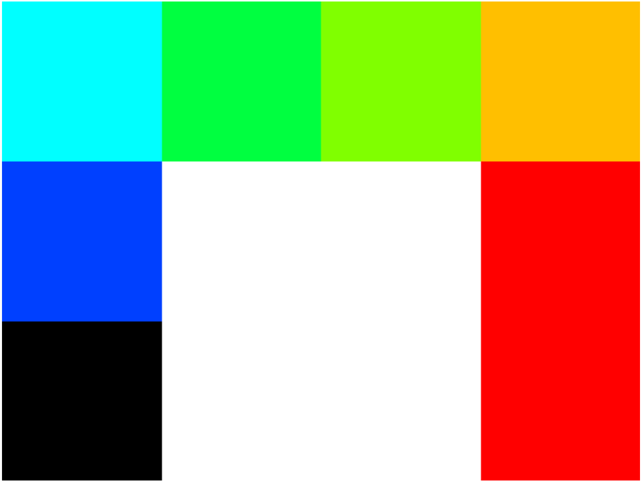

$$0 = \rightarrow$$

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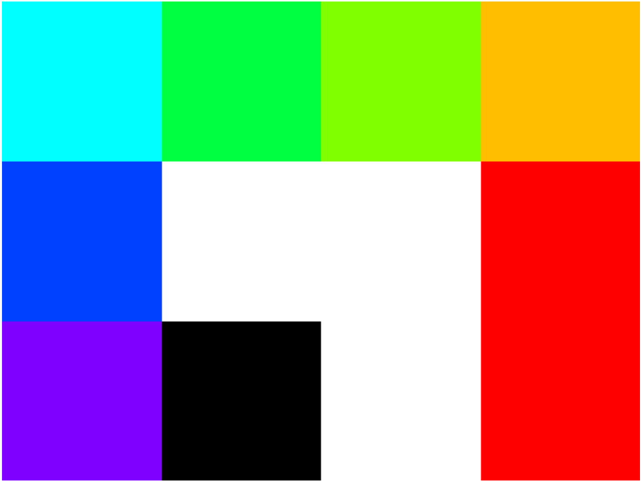

11222330

What is a random walk (base 4)?

Pick a random number in $\{0,1,2,3\}$ and move $0=\rightarrow$, $1=\uparrow$, $2=\leftarrow$, $3=\downarrow$ ANIMATION

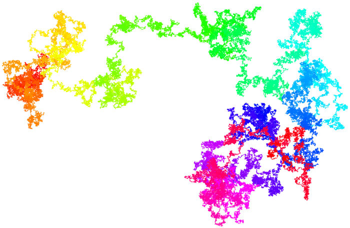

Figure : A million step base-4 pseudorandom walk. We use the spectrum to show when we visited each point (ROYGBIV and R).

Random walks look similarish

Chaos theory (order in disorder)

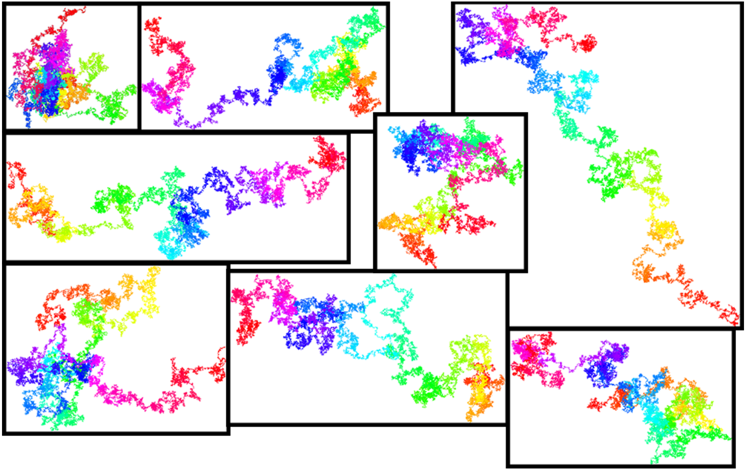

Figure : Eight different base-4 (pseudo)random[2] walks of one million steps.

[2] Python uses the *Mersenne Twister* as the core generator. It has a period of $2^{19937} - 1 \approx 10^{6002}$.

Base-b random walks: Our direction choice

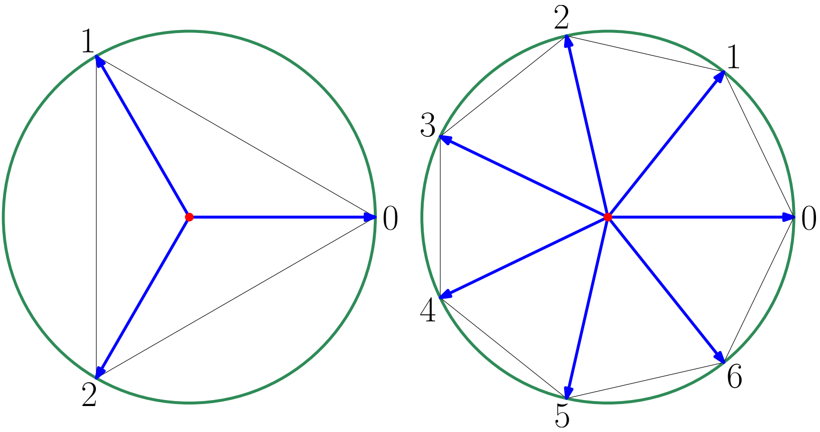

Figure : Directions for base-3 and base-7 random walks.

Contents

Two rational numbers

ANIMATION

The base-4 digit expansion of $Q1$ and $Q2$:

```
Q1=
0.221221012232121200122101223121001222100011232123121000122210001222
1000122210001222100001222100012220110301012201001201031103333333333
3333333333333333011111111111111111111111111110010000000003003003 20032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232223000322230032221330023321233023213232112112121222323233
33303000001000323003230032203032030110333011103301103101111011332333
3232322321221211211121122322222122...
```

```
Q2=
0.221221012232121200122101223121001222100011232123121000122210001222
1000122210001222100001222100012220110301012201001201031103333333333
333333333333333301111111111111111111111111111001000000000300300320032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232223000322230032221330023321233023213232112112121222323233
33303000001000323003230032203032030110333011103301103101111011000000
000000...
```

Two rational numbers

ANIMATION

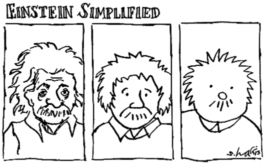

Figure : Self-referent walks on the rational numbers $Q1$ (top) and $Q2$ (bottom).

Two more rationals

Hard to tell from their decimal expansions

The following relatively small rational numbers [G. Marsaglia, 2010]

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{1000000000061},$$

have base-10 periods with huge length of **1,750,000,000** digits and **1,000,000,000,060** digits, respectively.

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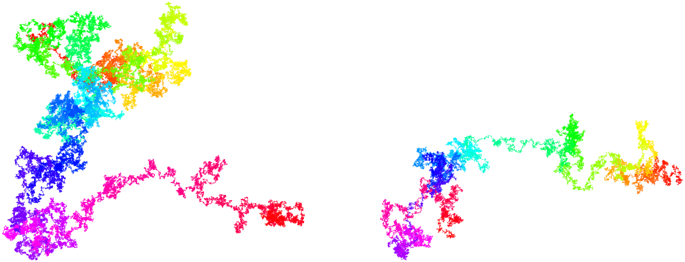

(a) $Q3$ (b) $Q4$

Figure : Walks on the first million base-10 digits of the rationals $Q3$ and $Q4$.

Walks on the digits of numbers

ANIMATION

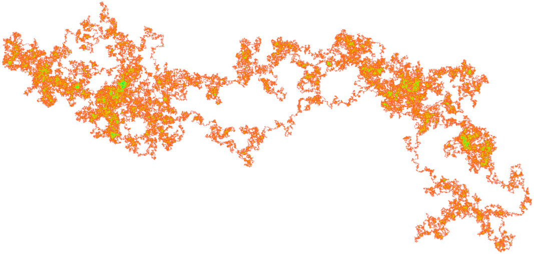

Figure : A walk on the first 10 million base-4 digits of π.

Walks on the digits of numbers

Coloured by hits (more pink is more hits)

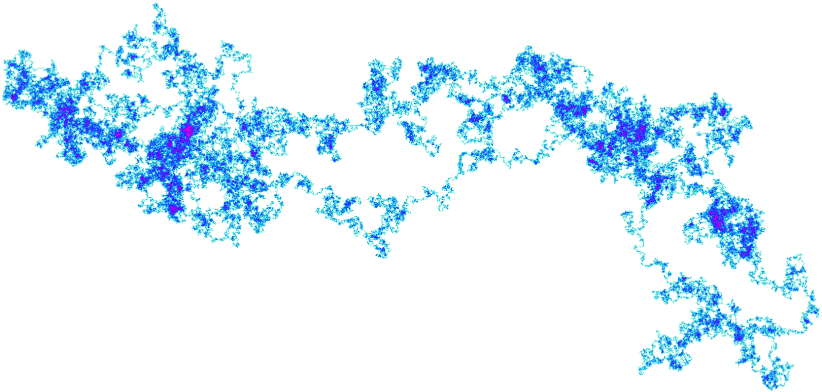

Figure : 100 million base-4 digits of π coloured by number of returns to points.

Contents

The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

1973 Richard Stoneham proved some of the following (nearly 'natural') constants are b-normal for relatively prime integers b, c:

$$\alpha_{b,c} := \frac{1}{cb^c} + \frac{1}{c^2 b^{c^2}} + \frac{1}{c^3 b^{c^3}} + \ldots.$$

Such super-geometric sums are Stoneham constants. To 10 places

$$\alpha_{2,3} = \frac{1}{24} + \frac{1}{3608} + \frac{1}{3623878656} + \ldots$$

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- Since $3 < 2^{3-1} = 4$, $\alpha_{2,3}$ is 2-normal and 6-nonnormal !

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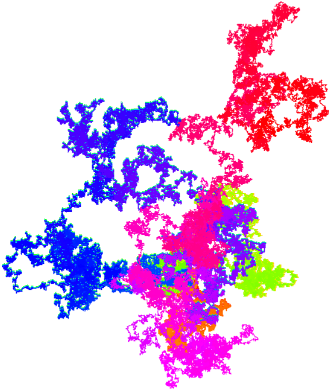

Figure : $\alpha_{2,3}$ is 2-normal (top) but 6-nonnormal (bottom). Is seeing believing?

The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

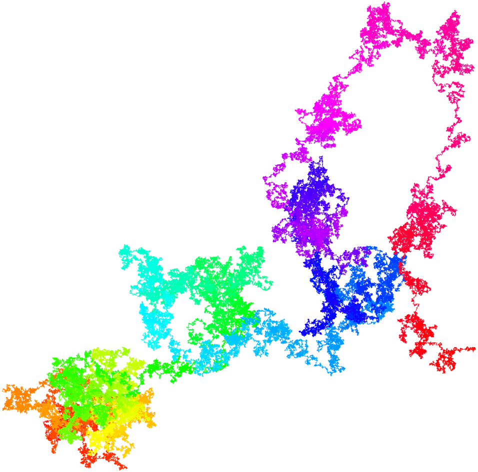

Figure : Is $\alpha_{2,3}$ 3-normal or not?

Contents

The expected distance to the origin $\frac{\sqrt{\pi N}}{2\,d_N} \to 1$

Theorem

The expected distance d_N to the origin of a base-b *random walk* of N steps behaves like to $\sqrt{\pi N}/2$.

The expected distance to the origin

$\frac{\sqrt{\pi N}}{2\, d_N} \to 1$

Theorem

The expected distance d_N to the origin of a base-b *random walk* of N steps behaves like to $\sqrt{\pi N}/2$.

Number	Base	Steps	Average normalized dist. to the origin: $\frac{1}{\text{Steps}} \sum_{N=2}^{\text{Steps}} \frac{\text{dist}_N}{\frac{\sqrt{\pi N}}{2}}$	Normal
Mean of 10,000 random walks	4	1,000,000	**1.00315**	Yes
Mean of 10,000 walks on the digits of π	4	1,000,000	**1.00083**	?
$\alpha_{2,3}$	3	1,000,000	0.89275	?
$\alpha_{2,3}$	4	1,000,000	0.25901	Yes
$\alpha_{2,3}$	6	1,000,000	108.02218	No
π	4	1,000,000	0.84366	?
π	6	1,000,000	0.96458	?
π	10	1,000,000	0.82167	?
π	10	1,000,000,000	0.59824	?
$\sqrt{2}$	4	1,000,000	0.72260	?
Champernowne C_{10}	10	1,000,000	59.91143	Yes

Contents

Number of points visited

For a 2D lattice

- The expected number of distinct points visited by an N-step random walk on a two-dimensional lattice behaves for large N like $\pi N/\log(N)$ (Dvoretzky–Erdős, **1951**).

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- For example, for $N=10^6$ these bounds are $(199256.1, 203059.5)$, while $\pi N/\log(N)=227396$, which overestimates the expectation.

Catalan's constant

$G = 1 + 1/4 + 1/9 + 1/16 + \cdots$

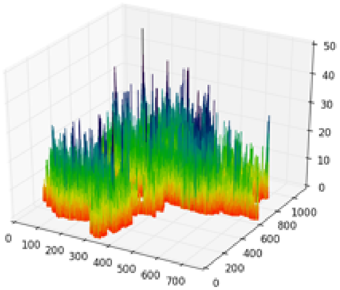


Figure : A walk on one million quad-bits of G with height showing frequency

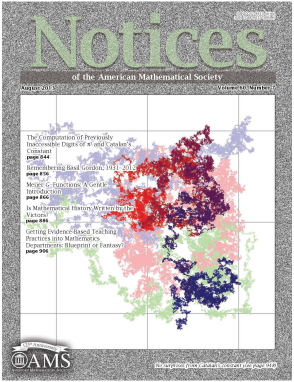
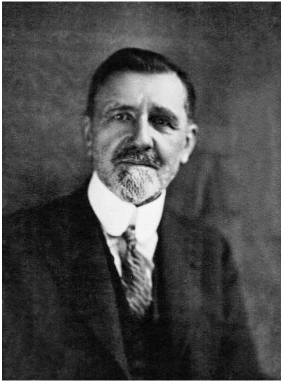


Figure : http://www.ams.org/notices/201307/

Paul Erdős (1913-1996) “My brain is open”

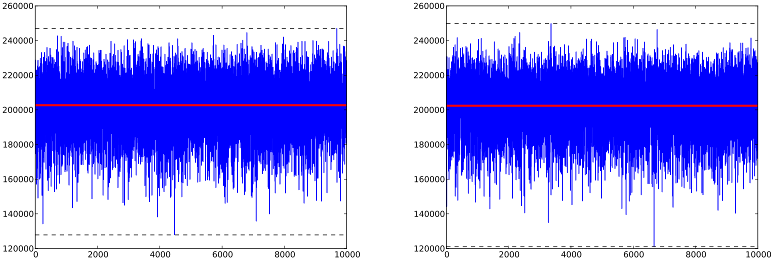
(a) Paul Erdős (Banff 1981. I was there)



(b) Émile Borel (1871–1956)

Figure : Two of my favourites. Consult MacTutor.

Number of points visited: Again π looks random



(a) (Pseudo)random walks.

(b) Walks built by chopping up 10 billion digits of π.

Figure : Number of points visited by $10,000$ million-steps base-4 walks.

Points visited by various base-4 walks

Number	Steps	Sites visited	Bounds on the expectation of sites visited by a random walk	
			Lower bound	Upper bound
Mean of 10,000 random walks	1,000,000	**202,684**	199,256	203,060
Mean of 10,000 walks on the digits of π	1,000,000	**202,385**	199,256	203,060
$\alpha_{2,3}$	1,000,000	95,817	199,256	203,060
$\alpha_{3,2}$	1,000,000	195,585	199,256	203,060
π	1,000,000	**204,148**	199,256	203,060
π	10,000,000	1,933,903	1,738,645	1,767,533
π	100,000,000	16,109,429	15,421,296	15,648,132
π	1,000,000,000	138,107,050	138,552,612	140,380,926
e	1,000,000	176,350	199,256	203,060
$\sqrt{2}$	1,000,000	**200,733**	199,256	203,060
$\log 2$	1,000,000	**214,508**	199,256	203,060
Champernowne C_4	1,000,000	548,746	199,256	203,060
Rational number Q_1	1,000,000	378	199,256	203,060
Rational number Q_2	1,000,000	939,322	199,256	203,060

Normal numbers need not be so "random" ...

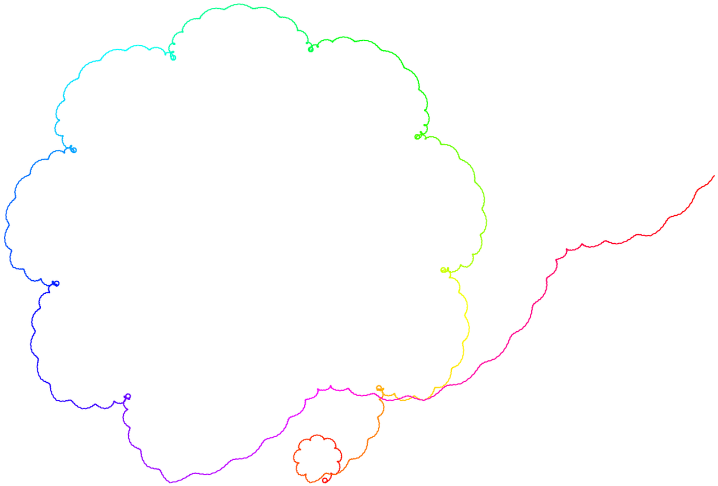


Figure : Champernowne $C_{10} = 0.123456789101112\ldots$ (normal).
Normalized distance to the origin: 15.9 (50,000 steps).

Normal numbers need not be so "random" ...

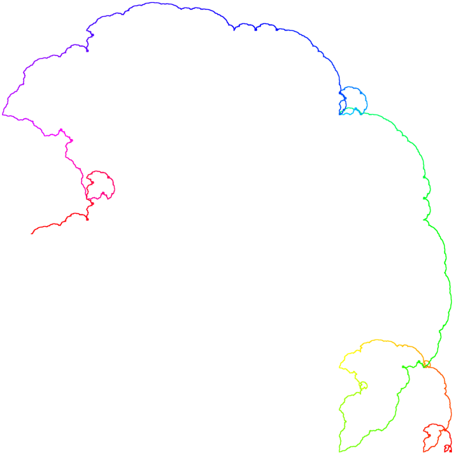


Figure : Champernowne $C_4 = 0.123101112132021\ldots$ (normal).
Normalized distance to the origin: 18.1 (100,000 steps).
Points visited: 52760. Expectation: (23333, 23857).

Normal numbers need not be so “random” ...

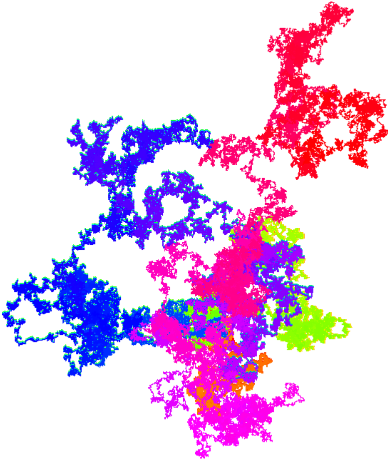


Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

Normal numbers need not be so "random" ...

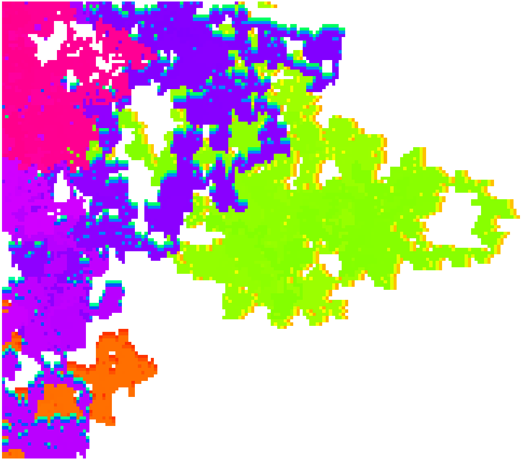
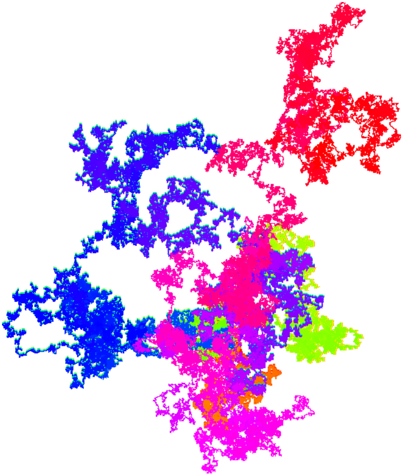


Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

$\alpha_{2,3}$ is 4-normal but not so "random" ANIMATION



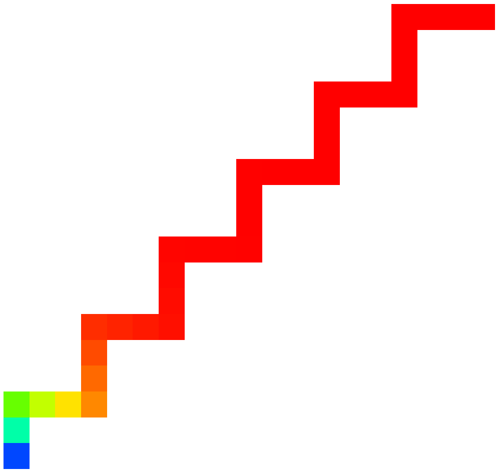


Figure : A pattern in the digits of $\alpha_{2,3}$ base 4. We show only positions of the walk after $\frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2\cdot 3^n$ steps, $n = 0, 1, \ldots, 11$.

Experimental conjecture

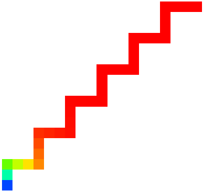
Proven 12-12-12 by Coons

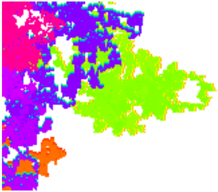
Theorem (Base-4 structure of Stoneham $\alpha_{2,3}$)

Denote by a_k the k^{th} digit of $\alpha_{2,3}$ in its base 4 expansion: $\alpha_{2,3} = \sum_{k=1}^{\infty} a_k/4^k$, with $a_k \in \{0,1,2,3\}$ for all k. Then, for all $n = 0,1,2,\dots$ one has:

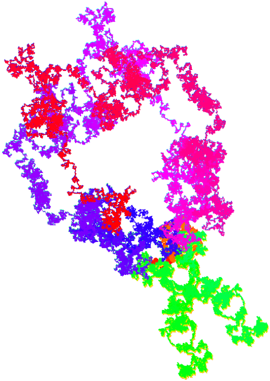
(i) $$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n} e^{a_k \pi i/2} = \begin{cases} -i, & \text{n odd} \\ -1, & \text{n even} \end{cases};$$

(ii) $a_k = a_{k+3^n} = a_{k+2\cdot 3^n}$ if $k = \dfrac{3(3^n+1)}{2}, \dfrac{3(3^n+1)}{2}+1, \dots, \dfrac{3(3^n+1)}{2}+3^n-1.$





Likewise, $\alpha_{3,5}$ is 3-normal ... but not very “random” ANIMATION



Contents

Box-dimension:

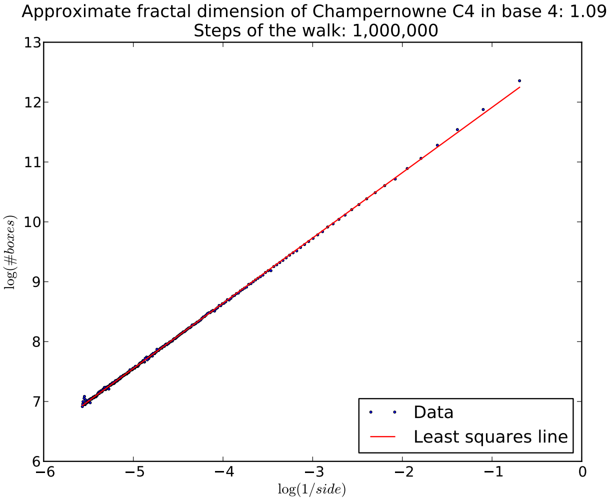
Tends to ‘2’ for a planar random walk

$$\text{Box-dimension} = \lim_{\text{side}\to 0} \frac{\log(\#\text{ boxes})}{\log(1/\text{side})}$$

Norway is “frillier” — *Hitchhiker’s Guide to the Galaxy*

Box-dimension:

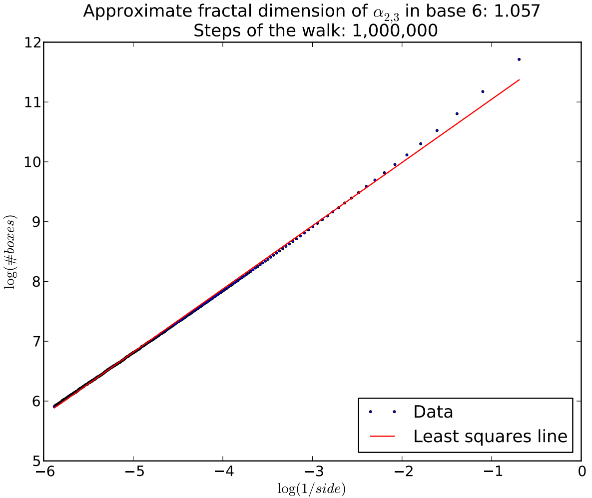
Tends to '2' for a planar random walk



Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

Box-dimension:

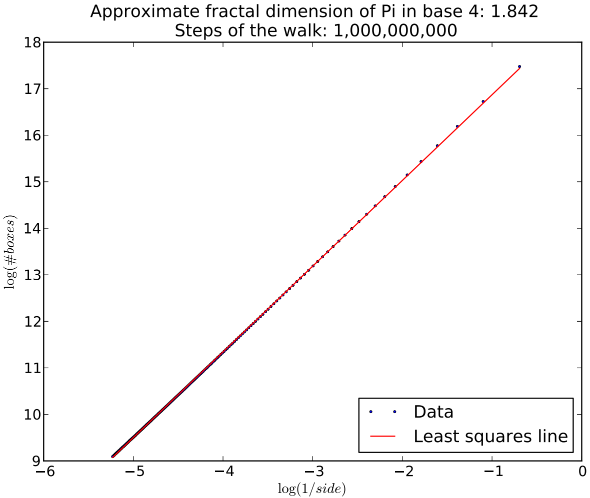
Tends to '2' for a planar random walk



Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

Box-dimension:

Tends to '2' for a planar random walk



Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

Contents

Three dimensional walks: Using base six — soon on 3D screen

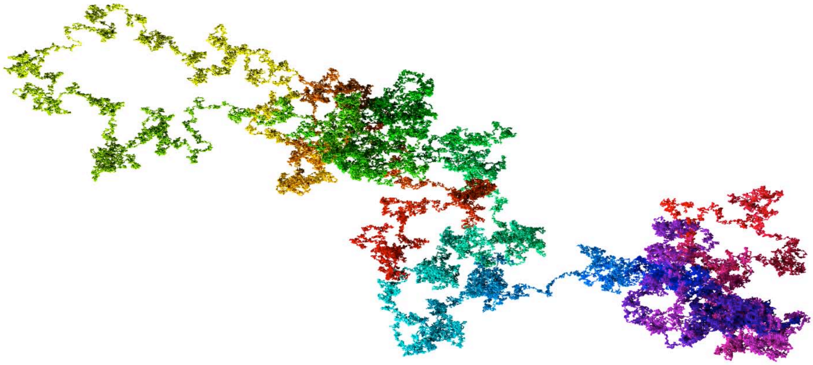


Figure : Matt Skerritt's 3D walk on π (base 6), showing one million steps. But 3D random walks are not recurrent.

Three dimensional walks: Using base six — soon on 3D screen

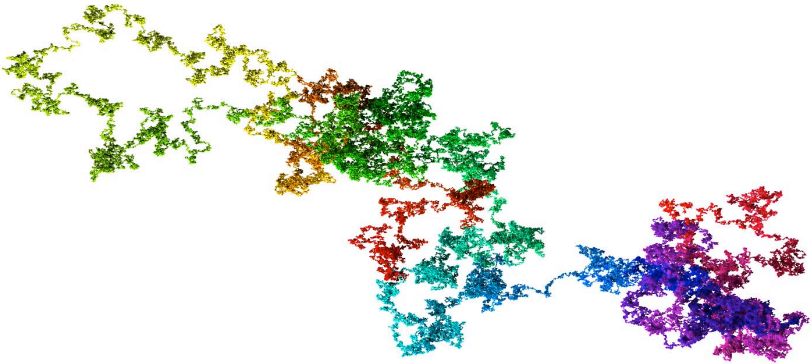
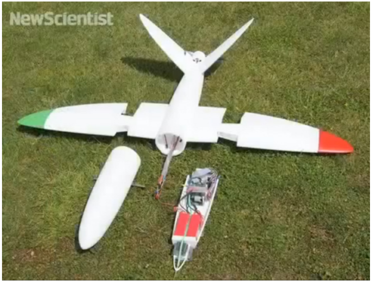


Figure : Matt Skerritt's 3D walk on π (base 6), showing one million steps. But 3D random walks are not recurrent.

"A drunken man will find his way home, a drunken bird will get lost forever." (Kakutani)

Three dimensional printing:

3D everywhere



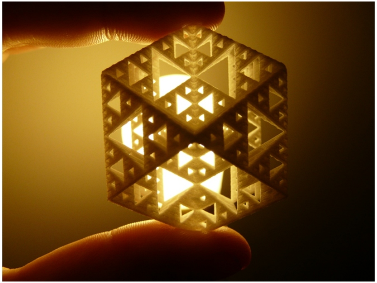


Figure : The future is here ...

www.digitaltrends.com/cool-tech/the-worlds-first-plane-created-entirely-by-3d-printing-takes-flight/

www.shapeways.com/shops/3Dfractals

Contents

Chaos games: Move half-way to a (random) corner

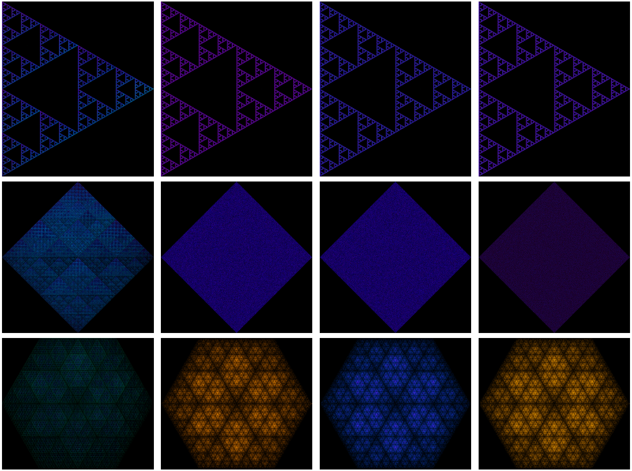


Figure : Coloured by frequency — leads to random fractals.
Row 1: Champernowne C_3, $\alpha_{3,5}$, random, $\alpha_{2,3}$. Row 2: Champernowne C_4, π, random, $\alpha_{2,3}$. Row 3: Champernowne C_6, $\alpha_{3,2}$, random, $\alpha_{2,3}$.

Contents

Automatic numbers: Thue–Morse and Paper-folding

Automatic numbers are never normal. They are given by simple but fascinating rules...giving structured/boring walks:

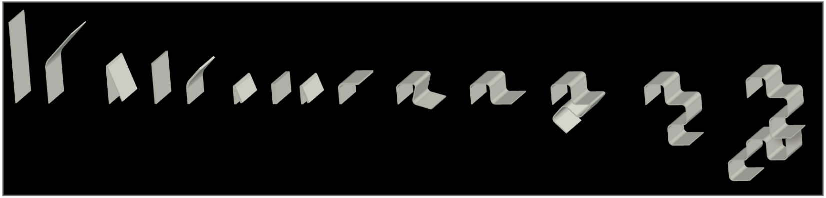


Figure : **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

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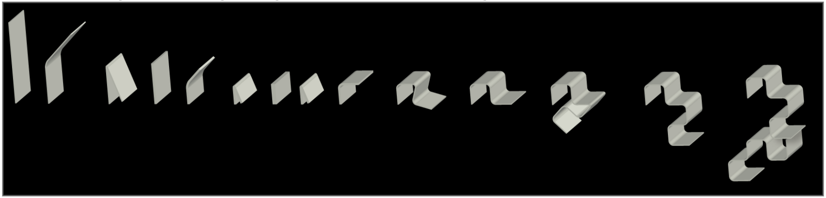

Figure : **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

Thue–Morse constant (transcendental; 2-automatic, hence nonnormal):

$$TM_2 = \sum_{n=1}^{\infty} \frac{1}{2^{t(n)}} \text{ where } t(0)=0, \text{ while } t(2n)=t(n) \text{ and } t(2n+1)=1-t(n)$$

$$0.01101001100101101001011001101001\ldots$$

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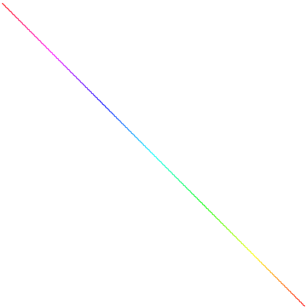
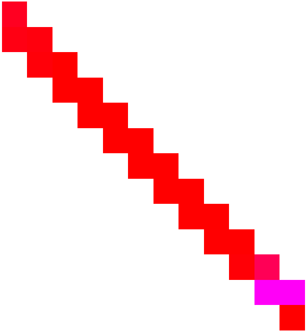


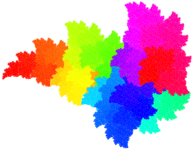
(a) 1,000 bits of Thue–Morse sequence.



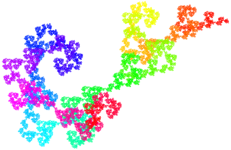
(b) 10 million bits of paper-folding sequence.

Figure : Walks on two automatic and so nonnormal numbers.

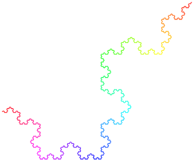
Automatic numbers: Turtle plots look great!



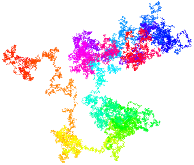
(a) Ten million digits of the paper-folding sequence, rotating 60°.



(b) One million digits of the paper-folding sequence, rotating 120° (a dragon curve).



(c) 100,000 digits of the Thue–Morse sequence, rotating 60° (a Koch snowflake).



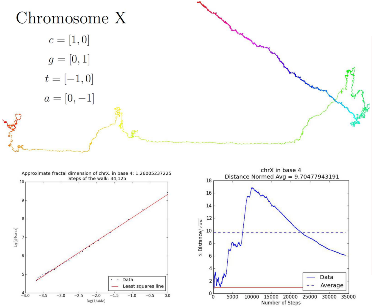
(d) One million digits of π, rotating 60°.

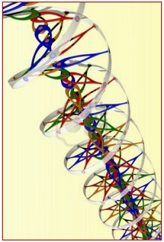
Figure : Turtle plots on various constants with different rotating angles in base 2—where '**0**' yields forward motion and '**1**' rotation by a fixed angle.

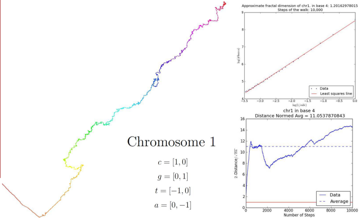
Contents

Genomes as walks:

... we are all base 4 numbers (ACGT/U)







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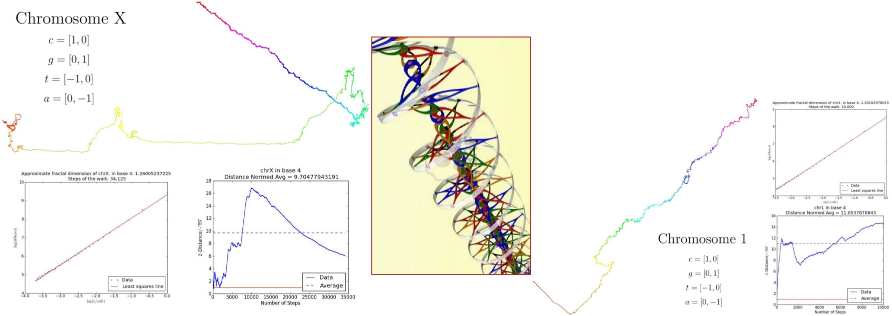

The X Chromosome (34K) and Chromosome One (10K).

Genomes as walks:

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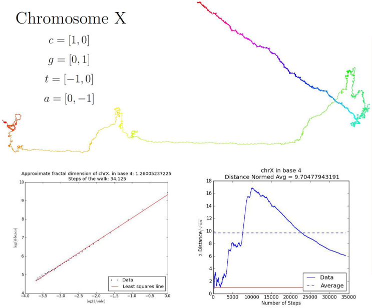

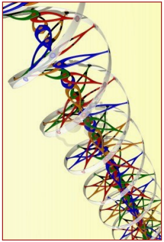

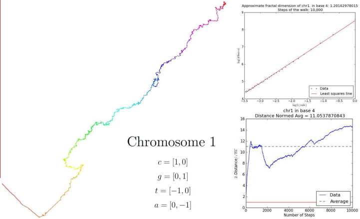

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® Chromosomes look less like π and more like concatenation numbers?

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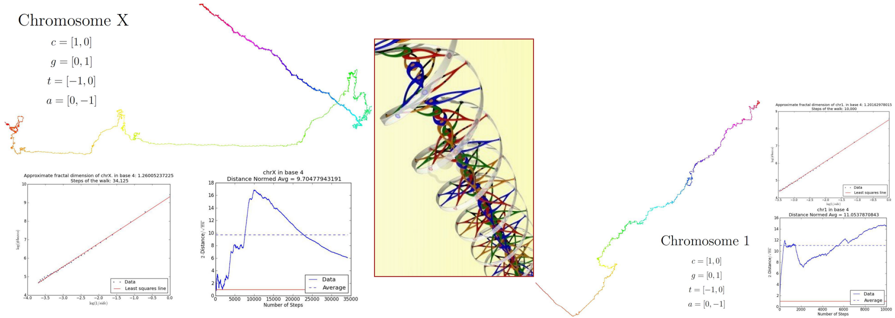

The X Chromosome (34K) and Chromosome One (10K).

- ® Chromosomes look less like π and more like concatenation numbers?
- **Thank you**!

Main References

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