Mahler Measures, Short Walks and Log-sine Integrals

A case study in hybrid computation

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THIS TALK: `http://carma.newcastle.edu.au/jon/alfcon.pdf`

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AlfCon, Newcastle, March 12–16, 2012

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Companion paper and software (*Th. Comp Sci*) : `http://carma.newcastle.edu.au/jon/wmi-paper.pdf`

Dedication from JB&AS in *J. AustMS*

Remark

We remark that it is fitting given the dedication of this article and volume that Alf van der Poorten [1942–2010] wrote the foreword to Lewin's "bible". In fact, he enthusiastically mentions the [log-sine] evaluation

$$-\operatorname{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = \frac{17}{6480}\pi^4$$

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Contents. We will cover some of the following:

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   - 9. Log-sine Integrals
   - 10. Random Walks
   - 15. Mahler Measures
   - 16. Carlson's Theorem
2. 17. Short Random Walks
   - 18. Combinatorics
   - 24. Meijer-G functions
   - 29. Hypergeometric values of W_3, W_4
   - 32. Probability and Bessel J
   - 40. Derivative values of W_3, W_4
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   - 45. Boyd's Conjectures
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The maths world is becoming hybrid: and none to soon

5529 [1967, 1015; 1968, 914]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\text{Evaluation of } \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x - a_j)}{x - a_j}\, dx,$$

with k_j, a_j, $j = 1, 2, \cdots, n$ real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the k's and be symmetric in k_j, a_j. The formula obtained in the solution

$$I = \pi \prod_{j=2}^{n} \frac{\sin k_j(a_{j-1} - a_j)}{a_{j-1} - a_j}$$

does not involve k_1 and is not symmetric as required. ($k_1 = 0$ must imply $I = 0$.)

Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

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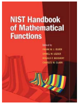
Congrats to NIST: for helping answer eternal questions

"What's fire?" "What's walking?"

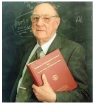
Special Functions in the 21st Century: Theory & Applications

April 6–8, 2011
Washington, DC



Objectives. The conference will provide a forum for the exchange of expertise, experience and insights among world leaders in the subject of special functions. Participants will include expert authors, editors and validators of the recently published NIST Handbook of Mathematical Functions and Digital Library of Mathematical Functions (DLMF). It will also provide an opportunity for DLMF users to interact with its creators and to explore potential areas of fruitful future developments.

Special Recognition of Professor Frank W. J. Olver. This conference is dedicated to Professor Olver in light of his seminal contributions to the advancement of special functions, especially in the area of asymptotic analysis and as Mathematics Editor of the DLMF.



F.W.J. Olver

Plenary Speakers
Richard Askey, University of Wisconsin
Michael Berry, University of Bristol
Nalini Joshi, University of Sydney, Australia
Leonard Maximon, George Washington University
William Reinhardt, University of Washington
Roderick Wong, City University of Hong Kong

Call for Contributed Talks (25 Minutes)
Abstracts may be submitted to Daniel.Lozier@nist.gov until March 15, 2011.

Registration and Financial Assistance. Registration fee: $120. Courtesy of SIAM, limited travel support is available for US-based postdoc and early career researchers. Courtesy of City University of Hong Kong and NIST, partial support is available for others in cases of need. Submit all requests for financial assistance to Daniel.Lozier@nist.gov.

Venue. Renaissance Washington Dupont Circle Hotel, 1143 New Hampshire Avenue NW, Washington, DC, 20037 USA. The conference rate is $259, available until March 15. Refreshments are supplied courtesy of University of Maryland.

Organizing Committee. Daniel Lozier, NIST, Gaithersburg, Maryland; Adri Olde Daalhuis, University of Edinburgh; Nico Temme, CWI, Amsterdam; Roderick Wong, City University of Hong Kong

To register online for the conference, and reserve a room at the conference hotel, see http://math.nist.gov/~DLozier/SF21

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SIAM Activity Group Orthogonal Polynomials and Special Functions
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Abstract

toc

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My Collaborators

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Multiple Polylogarithms:

$$\mathrm{Li}_{a_1,\ldots,a_k}(z) := \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{a_1}\cdots n_k^{a_k}}.$$

Thus, $\mathrm{Li}_{2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \sum_{j=1}^{k-1} \frac{1}{j}$. Specializing produces:

- The *polylogarithm of order* k: $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$.
- *Multiple zeta values*:

$$\zeta(a_1,\ldots,a_k) := \mathrm{Li}_{a_1,\ldots,a_k}(1).$$

- *Multiple Clausen* (Cl) and *Glaisher functions* (Gl) of depth k and weight $w := \sum a_j$:

$$\mathrm{Cl}_{a_1,\ldots,a_k}(\theta) := \begin{Bmatrix} \operatorname{Im} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \operatorname{Re} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{Bmatrix},$$

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The log-sine integrals are defined for $n = 1, 2, \ldots$ by

$$\mathrm{Ls}_n(\sigma) := -\int_0^\sigma \log^{n-1}\left|2\sin\frac{\theta}{2}\right| \mathrm{d}\theta \tag{1}$$

and their moments for $k \geq 0$ given by

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$$\mathrm{Ls}_2(\sigma) = \mathrm{Cl}_2(\sigma) := \sum_{n=1}^\infty \frac{\sin(n\sigma)}{n^2} \tag{3}$$

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Moments of Uniform Random Walks

Definition (Moments)

For complex s the n-th moment function is

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

Thus, $W_n := W_n(1)$ is the *expectation*.

- The integral for W_n is analytic precisely for $\mathrm{Re}\, s > -2$.

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Clearly $W_1 = 1$. What about $W_2(1)$?

$$W_2 = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \, ?$$

- *Mathematica* 7 and *Maple* 14 *think* the answer is 0.

- There is always a 1-dimension reduction

$$W_n(s) = \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

$$= \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1})$$

- So

$$W_2 = 4 \int_0^{1/4} \cos(\pi x) \, \mathrm{d}x = \frac{4}{\pi}.$$

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$n \geq 3$ highly nontrivial and $n \geq 5$ not well understood.

- Similar problems get *much* more difficult in five or more dimensions — e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).
  - In fact, $W_5 \approx 2.0081618$ was the best estimate we could compute *directly*, on **256** cores at Lawrence Berkeley National Laboratory.
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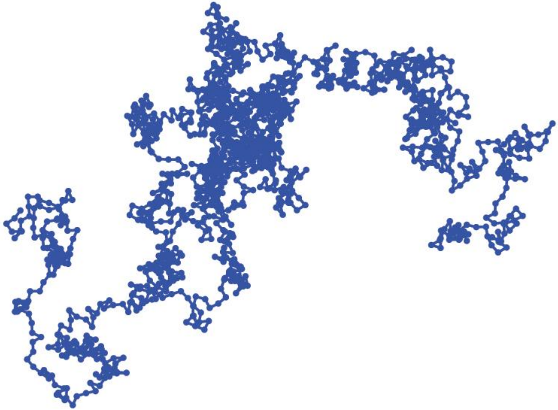
One 1500-step Ramble: a familiar picture



2D and 3D lattice walks are different:

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- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
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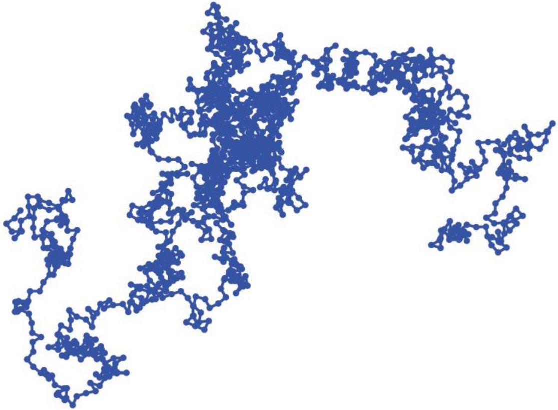

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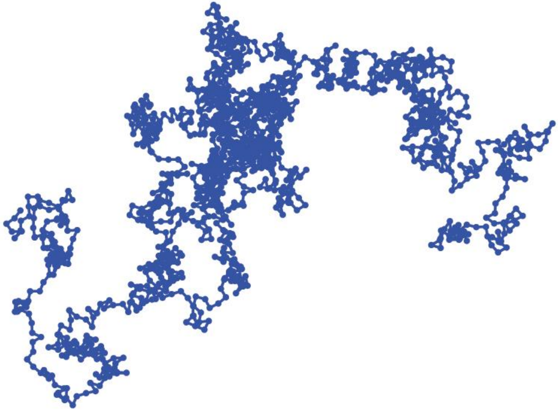

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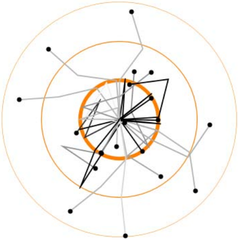
1000 three-step Rambles: a less familiar picture?



Mahler Measures (1923) in several variables

The logarithmic *Mahler measure* of a (Laurent) polynomial P:

$$\mu(P) := \int_0^1 \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n.$$

- $M_1 := P \mapsto \exp(\mu(P))$ is multiplicative.
- $n = 1$: P is a product of cyclotomics $\Leftrightarrow M_1(P) = 1$. Lehmer's conjecture (1931) is: otherwise $M_1(P) \geq M_1(1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10})$.
- $\mu(P)$ turns out to be an example of a period.
- When $n = 1$ and P has integer coefficients $M_1(P)$ is an algebraic integer.
- In several dimensions life is harder.
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Carlson's Theorem: from discrete to continuous

Theorem (Carlson (1914, PhD))

If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geq 0$, its growth on the imaginary axis is bounded by $e^{cy}, |c| < \pi$, and

$$0 = f(0) = f(1) = f(2) = \ldots$$

then $f(z) = 0$ identically.

- $\sin(\pi z)$ does not satisfy the conditions of the theorem, as it grows like $e^{\pi y}$ on the imaginary axis.
- $W_n(s)$ satisfies the conditions of the theorem (and is in fact analytic for $\mathrm{Re}\,(s) > -2$ when $n > 2$).
  - There is a lovely **1941** proof by Selberg of the bounded case.
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CARMA

$W_n(k)$ at even values

Even values are easier (combinatorial – no square roots).

k	0	2	4	6	8	10
$W_2(k)$	1	2	6	20	70	252
$W_3(k)$	1	3	15	93	639	4653
$W_4(k)$	1	4	28	256	2716	31504
$W_5(k)$	1	5	45	545	**7885**	**127905**

- Can get started by *rapidly* computing many values *naively* as symbolic integrals.
  - Observe that $W_2(s) = \binom{s}{s/2}$ for $s > -1$.
  - Entering $1, 5, 45, 545$ in the *OIES* now gives *"The function $W_5(2n)$ (see Borwein et al. reference for definition)."*

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$W_n(k)$ at odd integers

n	$k=1$	$k=3$	$k=5$	$k=7$	$k=9$
2	1.27324	3.39531	10.8650	37.2514	132.449
3	**1.57460**	6.45168	36.7052	241.544	1714.62
4	1.79909	10.1207	82.6515	822.273	9169.62
5	2.00816	14.2896	152.316	2037.14	31393.1
6	2.19386	18.9133	248.759	4186.19	82718.9

Please, memorize this number!

During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. —

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Resolution at even values

- General even formula counts n-letter abelian squares $x\pi(x)$ of length $2k$.
  - Shallit and Richmond (2008) give asymptotics:

$$W_n(2k) = \sum_{a_1+\ldots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2. \tag{4}$$

- Known to satisfy convolutions:

$$W_{n_1+n_2}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n_1}(2j) W_{n_2}(2(k-j)).$$

- Has recursions such as:

$$\begin{aligned}(k+2)^2 W_3(2k+4) - (10k^2+30k+23) W_3(2k+2)&\\ +9(k+1)^2 W_3(2k) &= 0.\end{aligned}$$

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CARMA

Analytic continuation: From Carlson's Theorem

- So integer recurrences yield complex functional equations. Viz

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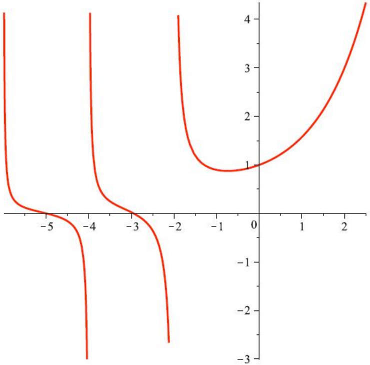
Odd dimensions look like 3

$W_3(s)$ on $[-6, \frac{5}{2}]$



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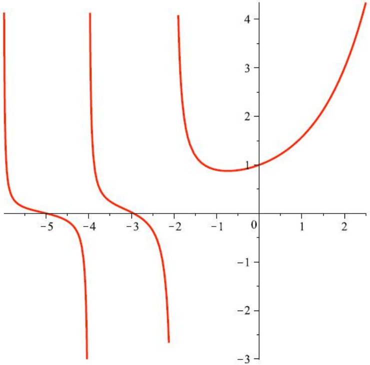

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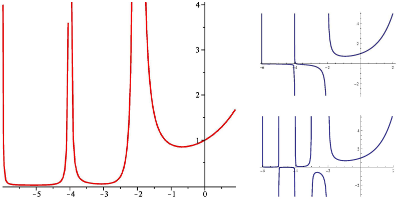
Some even dimensions look more like 4



L: $W_4(s)$ on $[-6, 1/2]$. **R**: W_5 on $[-6, 2]$ (T), W_6 on $[-6, 2]$ (B).

- The functional equation (with double poles) for $n = 4$ is

$$\begin{aligned}(s+4)^3 W_4(s+4) \quad &- \quad 4(s+3)(5s^2+30s+48)W_4(s+2) \\ &+ \quad 64(s+2)^3 W_4(s) = 0\end{aligned}$$

- There are (infinitely many) multiple poles if and only if $4|n$.
- Why is W_4 positive on $\mathbb{R}$?

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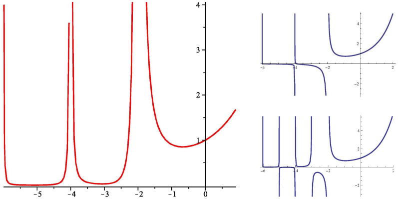

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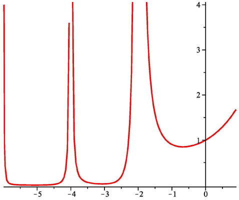

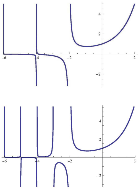

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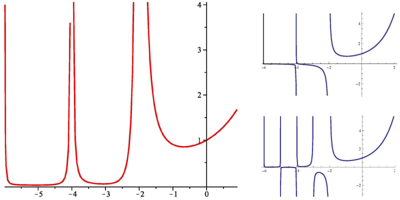

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Meijer-G functions (1936–)

Definition

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| x\right) := \frac{1}{2\pi i}\times$$

$$\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{j=m+1}^{q}\Gamma(1-b_j+s)} x^s \mathrm{d}s.$$

- Contour $\mathcal{L}$ lies between poles of $\Gamma(1-a_i-s)$ and of $\Gamma(b_i+s)$.
  - A broad generalization of hypergeometric functions — capturing Bessel Y, K and much more.
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Meijer-G forms for W_3

Theorem (Meijer form for W_3)

For s not an odd integer

$$W_3(s) = \frac{\Gamma(1+\frac{s}{2})}{\sqrt{\pi}\,\Gamma(-\frac{s}{2})}\, G^{21}_{33}\left(\begin{matrix}1,1,1\\ \frac{1}{2},-\frac{s}{2},-\frac{s}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right).$$

- First found by Crandall via CAS.
- Proved using residue calculus methods.
- $W_3(s)$ is among few non-trivial Meijer-G with a closed form.

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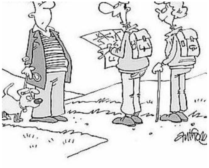
- Not helpful for odd integers. We must again look elsewhere ...



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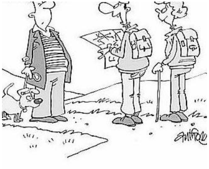
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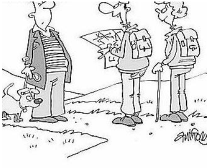

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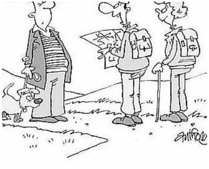
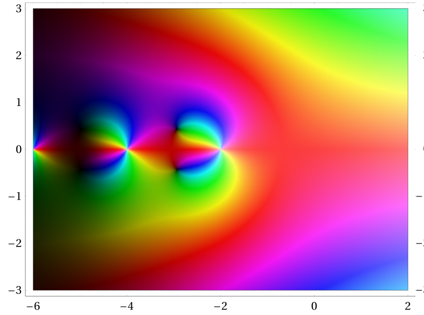
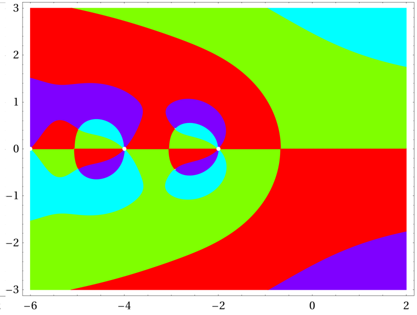


Visualizing W_4 in the complex plane





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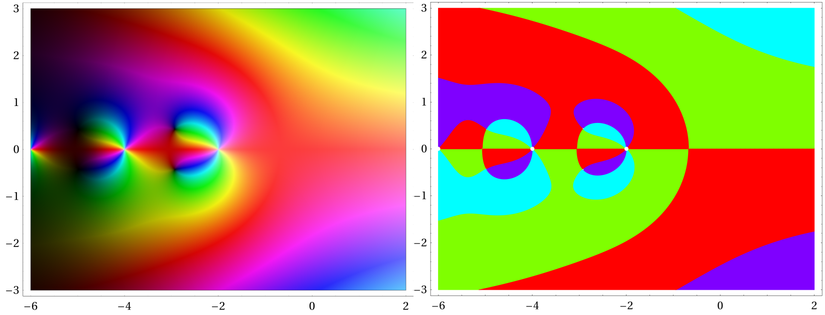

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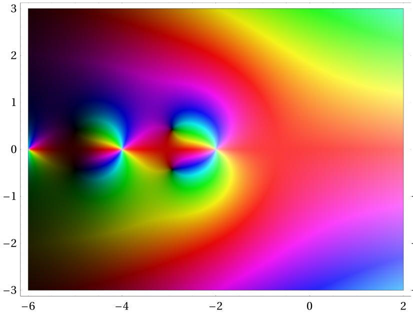

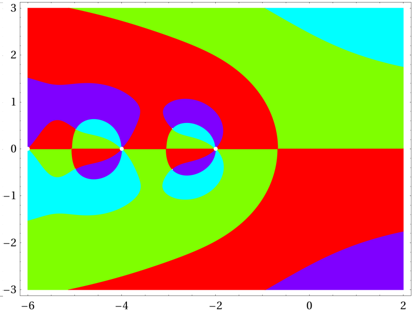

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Simplifying the Meijer integral

Corollary (Hypergeometric forms for noninteger $s > -2$)

$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| \frac{1}{4}\right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, -\frac{s-1}{2} \end{matrix} \middle| \frac{1}{4}\right),$$

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- We (humans) were able to provably take the limit:

$$W_4(-1) = \frac{\pi}{4}\, {}_7F_6\left(\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right) = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}$$

$$= \frac{\pi}{4}\, {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right) + \frac{\pi}{64}\, {}_6F_5\left(\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2 \end{matrix} \middle| 1\right).$$

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$$W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^3 {}_4F_3\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s}{2}+1 \\ \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2} \end{matrix}\right| 1\right) + \binom{s}{\frac{s}{2}} {}_4F_3\left(\left.\begin{matrix} \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, 1, -\frac{s-1}{2} \end{matrix}\right| 1\right).$$

- We (humans) were able to provably take the limit:

$$W_4(-1) = \frac{\pi}{4} {}_7F_6\left(\left.\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix}\right| 1\right) = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}$$
$$= \frac{\pi}{4} {}_6F_5\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix}\right| 1\right) + \frac{\pi}{64} {}_6F_5\left(\left.\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2 \end{matrix}\right| 1\right).$$

- We have proven the corresponding result for $W_4(1)$ ....

CARMA

Simplifying the Meijer integral

Corollary (Hypergeometric forms for noninteger $s > -2$)

$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \left(\frac{s}{\frac{s-1}{2}}\right)^2 {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| \frac{1}{4}\right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, -\frac{s-1}{2} \end{matrix} \middle| \frac{1}{4}\right),$$

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- We have proven the corresponding result for $W_4(1)$ ....

CARMA

Hypergeometric values of W_3, W_4: from Meijer-G values.

With much work involving moments of elliptic integrals we finally obtain:

Theorem (Tractable hypergeometric form for W_3)

(a) *For $s \neq -3, -5, -7, \ldots$, we have*

$$W_3(s) = \frac{3^{s+3/2}}{2\pi}\,\beta\left(s+\frac{1}{2}, s+\frac{1}{2}\right)\,{}_3F_2\left(\left.{\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}}\right| \frac{1}{4}\right). \tag{6}$$

(b) *For every natural number $k = 1, 2, \ldots$,*

$$W_3(-2k-1) = \frac{\sqrt{3}\,\binom{2k}{k}^2}{2^{4k+1}3^{2k}}\,{}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop k+1, k+1}\right| \frac{1}{4}\right).$$

A Discovery Demystified: on piecing all this together

We first proved that:

$$W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = \underbrace{{}_3F_2\left(\begin{matrix} 1/2, -k, -k \\ 1, 1 \end{matrix} \middle| 4\right)}_{=:V_3(2k)}.$$

We discovered *numerically* that: $V_3(1) = 1.57459 - .12602652i$

Theorem (Real part)

For all integers k we have $W_3(k) = \mathrm{Re}\,(V_3(k))$.

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first. ...So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work. — Richard Feynman (Nobel acceptance 1966)

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Closed Forms for W_3

- We then *confirmed* 175 digits of

$$W_3(1) \approx 1.57459723755189365749\ldots$$

- Armed with a knowledge of elliptic integrals:

$$W_3(1) = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4} = W3(-1) + \frac{6/\pi^2}{W3(-1)}, \quad (7)$$

$$W_3(-1) = \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4} = \frac{2^{\frac{1}{3}}}{4\pi^2}\beta^2\left(\frac{1}{3}\right). \quad (8)$$

Here $\beta(s) := B(s,s) = \frac{\Gamma(s)^2}{\Gamma(2s)}$.

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Probability: Bessel function representations

1906. J.C. Kluyver (1860-1932) derived the cumulative radial distribution function (P_n) and density (p_n) of the n-step distance:

$$P_n(t) = t \int_0^\infty J_1(xt)\, J_0^n(x)\, \mathrm{d}x$$

$$p_n(t) = t \int_0^\infty J_0(xt)\, J_0^n(x) x\, \mathrm{d}x \quad (n \geq 4) \tag{9}$$

where $J_n(x)$ is a Bessel function of the first kind

- See also Watson (1932, §49) – 3-dim walks are *elementary*.
  - From (11) below, we find

$$p_n(1) = \mathrm{Res}_{-2}\left(W_{n+1}\right) \qquad (n \neq 4). \tag{10}$$

  - As $p_2(\alpha) = \frac{2}{\pi\sqrt{4-\alpha^2}}$, we check in *Maple* that the following code returns $R = 2/(\sqrt{3}\pi)$ *symbolically*:

```
R:=identify(evalf[20](int(BesselJ(0,x)^3*x,x=0..infinity)))
```

A Bessel Integral for W_n

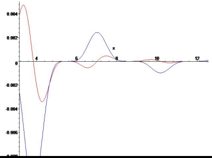
- Now $P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}$ (Pearson's original question).
- Broadhurst used (9) for $2k > s > -\frac{n}{2}$ to write

$$W_n(s) = 2^{s+1-k}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(k-\frac{s}{2})}\int_0^\infty x^{2k-s-1}\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x)\mathrm{d}x, \tag{11}$$

a useful oscillatory 1-dim integral (used below).

- Thence

$$W_n(-1) = \int_0^\infty J_0^n(x)\mathrm{d}x, \;\; W_n(1) = n\int_0^\infty J_1(x)J_0(x)^{n-1}\frac{\mathrm{d}x}{x}. \tag{12}$$



Integrands for $W_4(-1)$ (blue) and $W_4(1)$ (red) on $[\pi, 4\pi]$ from (12).

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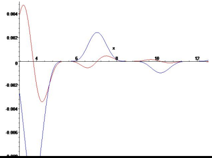

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CARMA

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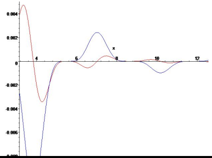

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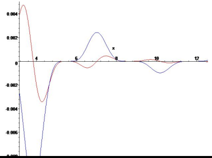

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CARMA

The Densities for $n = 3, 4$ are Modular

Let $\sigma(x) := \frac{3-x}{1+x}$. Then σ is an involution on $[0,3]$ sending $[0,1]$ to $[1,3]$:

$$p_3(x) = \frac{4x}{(3-x)(x+1)}\, p_3(\sigma(x)).$$

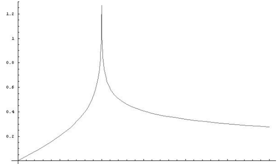
So $\frac{3}{4}p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}, p(1) = \infty$. We found:

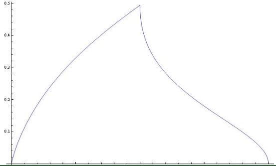
$$p_3(\alpha) = \frac{2\sqrt{3}\alpha}{\pi\,(3+\alpha^2)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\right| \frac{\alpha^2\left(9-\alpha^2\right)^2}{(3+\alpha^2)^3}\right) = \frac{2\sqrt{3}}{\pi}\,\frac{\alpha}{\mathrm{AG}_3(3+\alpha^2, 3\,(1-\alpha^2)^{2/3})}$$

where AG_3 is the *cubically convergent* mean iteration (1991):

$$\mathrm{AG}_3(a,b) := \frac{a+2b}{3} \bigotimes \left(b \cdot \frac{a^2+ab+b^2}{3}\right)^{1/3}$$

The densities p_3 (L) and p_4 (R)





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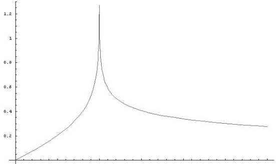
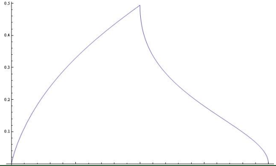
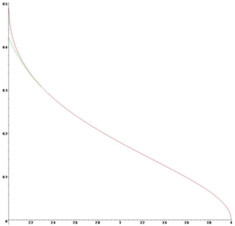




Formula for the 'shark-fin' p_4

We ultimately deduce on $2 \le \alpha \le 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2}\frac{\sqrt{16-\alpha^2}}{\alpha}\,{}_3F_2\left(\left.{\frac12,\frac12,\frac12 \atop \frac56,\frac76}\right|\frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4}\right). \qquad (13)$$



← p_4 from (13) vs 18-terms of series

✓ Proves $p_4(2) = \frac{2^{7/3}\pi}{3\sqrt{3}}\,\Gamma\left(\frac23\right)^{-6} = \frac{\sqrt{3}}{\pi}\,W_3(-1) \approx 0.494233 < \frac12$

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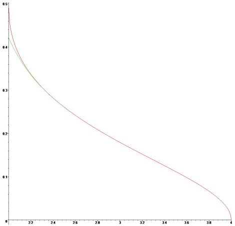

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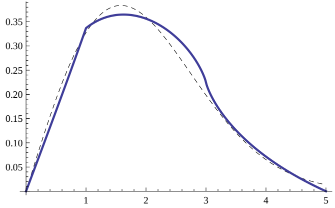
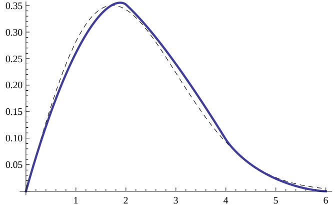
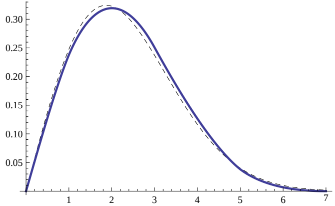
CARMA

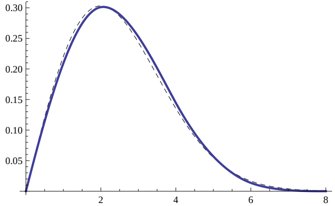
Densities for $5 \le n \le 8$ (and large n approximation)



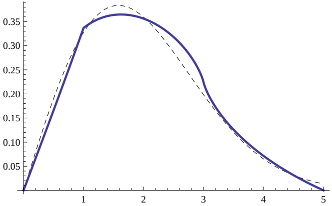


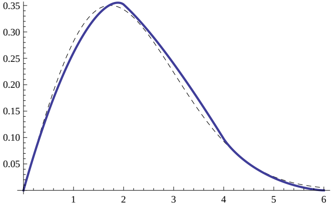
Both p_{2n+4}, p_{2n+5} are n-times continuously differentiable for $x > 0$ $\left(p_n(x) \sim \frac{2x}{n} e^{-x^2/n}\right)$. So "four is small" *but* "eight is large."



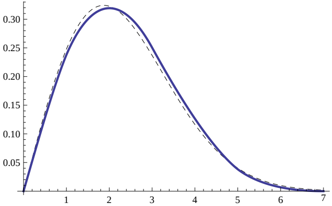


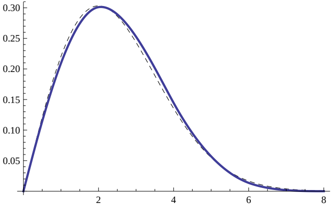
Densities for $5 \le n \le 8$ (and large n approximation)





Both p_{2n+4}, p_{2n+5} are n-times continuously differentiable for $x > 0$ $\left(p_n(x) \sim \frac{2x}{n} e^{-x^2/n}\right)$. So "four is small" *but* "eight is large."





The Five Step Walk

- The functional equation for W_5 is:

$$\begin{aligned} 225(s+4)^2(s+2)^2 W_5(s) &= -(35(s+5)^4+42(s+5)^2+3)W_5(s+4) \\ + \quad & (s+6)^4 W_5(s+6)+(s+4)^2(259(s+4)^2+104)W_5(s+2). \end{aligned}$$

- We deduce *the first two poles* — and so all — *are simple* since

$$\lim_{s\to-2}(s+2)^2 W_5(s) = \frac{4}{225}\left(285\,W_5(0)-201\,W_5(2)+16\,W_5(4)\right)=0$$

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- We *stumbled* upon

$$p_4(1) = \mathrm{Res}_{-2}(W_5) = \frac{\sqrt{15}}{3\,\pi}\,{}_3F_2\left(\left.\begin{matrix}\frac13,\frac23,\frac12\\1,1\end{matrix}\right|-4\right).$$

??? Is there a hyper-closed form for $W_5(\mp1)$???

CARMA

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CARMA

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W_5 and p_5: Bessel integrals are hard

- We only knew $\mathrm{Res}_{-4}(W_5)$ *numerically* — but to 500 digits: (Bailey in about **5.5hrs** on 1 MacPro core).
  - Sidi-"mW" method used: i.e., Gaussian quadrature on intervals of $[n\pi, (n+1)\pi]$ plus Richardson-like extrapolation.
  - **July 2011**. $r_5(2)$ was identified (with help from QFT)!

$$r_5(2) \stackrel{?}{=} \frac{13}{225} r_5(1) - \frac{2}{5\pi^4} \frac{1}{r_5(1)}. \tag{14}$$

- Here $r_5(k) := \mathrm{Res}_{(-2k)}(W_5)$. Other residues are then combinations as follows:
- From the W_5-recursion: given $r_5(0) = 0, r_5(1)$ and $r_5(2)$ we have

$$r_5(k+3) = \frac{k^4 r_5(k) - \left(5 + 28\,k + 63\,k^2 + 70\,k^3 + 35\,k^4\right) r_5(k+1)}{225(k+1)^2(k+2)^2} + \frac{\left(285 + 518\,k + 259\,k^2\right) r_5(k+2)}{225(k+2)^2}.$$

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CARMA

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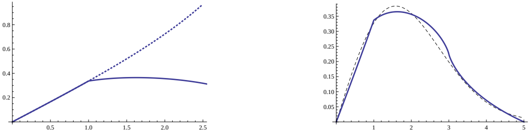

Figure: The series at zero and p_5.

- **1963**. Fettis first rigorously established nonlinearity. A few more residues yield $p_5(x) = 0.329934\,x + 0.00661673\,x^3 + 0.000262333\,x^5 + 0.0000141185\,x^7 + O(x^9)$

Hence the strikingly straight shape of $p_5(x)$ on $[0,1]$:

> "the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a *straight* line. . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines." — Karl Pearson (1906)

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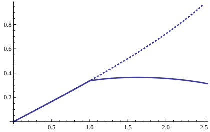

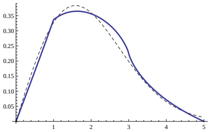

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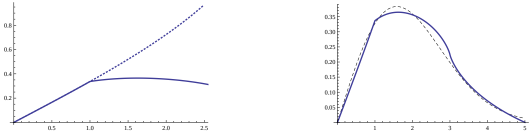

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Short Random Walks: Derivatives of W_3, W_4

From the hypergeometric forms above we get:

$$W_3'(0) = \frac{1}{\pi}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\right|\frac{1}{4}\right) = \frac{1}{\pi}\,\mathrm{Cl}\left(\frac{\pi}{3}\right). \tag{15}$$

The last equality follows from setting $\theta = \pi/6$ in the identity

$$2\sin(\theta)\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\right|\sin^2\theta\right) = \mathrm{Cl}\,(2\,\theta) + 2\,\theta\log\left(2\sin\theta\right).$$

Also

$$W_4'(0) = \frac{4}{\pi^2}\,{}_4F_3\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\end{matrix}\right|1\right) = \frac{7\zeta(3)}{2\pi^2}. \tag{16}$$

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$$W_3'(2) = \frac{3}{\pi}\mathrm{Cl}\left(\frac{\pi}{3}\right) - \frac{3\sqrt{3}}{2\pi} + 2\ldots$$

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CARMA

Multiple Mahler Measures: for $P_1, P_2, \ldots, P_m$

toc

$$\mu(P_1, P_2, \ldots, P_m) := \int_0^1 \cdots \int_0^1 \prod_{k=1}^{m} \log \left| P_k \left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n} \right) \right| \, d\theta_1 \cdots d\theta_n,$$

was introduced by Sasaki (2010); while

$$\mu_m(P) := \mu(P, P, \ldots, P), \qquad (\mu_1(P) = \mu(P))$$

is a higher Mahler measure as in Kurokawa, Lalín and Ochiai (2008). Also

$$\mu_m \left(1 + \sum_{k=1}^{n-1} x_k \right) = W_n^{(m)}(\mathbf{0}), \tag{17}$$

was evaluated in (15), (16) for $n = 3$ and $n = 4$ and $m = 1$:

1. $\mu(1 + x + y) = L_3'(-1) = \frac{1}{\pi} \operatorname{Cl}\left(\frac{\pi}{3}\right)$ (Smyth)
2. $\mu(1 + x + y + z) = 14\,\zeta'(-2) = \frac{7}{2} \frac{\zeta(3)}{\pi^2}$ (Smyth)

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2. $\mu(1 + x + y + z) = 14\,\zeta'(-2) = \frac{7}{2} \frac{\zeta(3)}{\pi^2}$ (Smyth)

– So (17) recaptured both Smyth's results.

CARMA

Relations to Dedekind's η

Denninger's **1997** conjecture, proven recently by Rogers and Zudilin (**2011**), is

$$\mu(1+x+y+1/x+1/y) \stackrel{?}{=} \frac{15}{4\,\pi^2}\, L_E(2)$$

– an L-series value for an elliptic curve E with conductor 15.

- For (17) with $n=5,6$ conjectures of Villegas become:

$$W_5'(0) \stackrel{?}{=} \left(\frac{15}{4\pi^2}\right)^{5/2} \int_0^\infty \{\eta^3(e^{-3t})\eta^3(e^{-5t}) + \eta^3(e^{-t})\eta^3(e^{-15t})\}\, t^3\, \mathrm{d}t$$

$$W_6'(0) \stackrel{?}{=} \left(\frac{3}{\pi^2}\right)^{3} \int_0^\infty \eta^2(e^{-t})\eta^2(e^{-2t})\eta^2(e^{-3t})\eta^2(e^{-6t})\, t^4\, \mathrm{d}t$$

where Dedekind's η is $\eta(q) := q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/4}$.

- Confirmed to 600 (Sidi) and to 80 digits respectively.

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CARMA

$\mu(1+x+y)$ and $\mu(1+x+y+z)$ revisited

We recall:

Lemma (Jensen's formula)

$$\int_0^1 \log\left|\alpha + e^{2\pi i\, t}\right|\, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right). \tag{18}$$

We use (18) to reduce to a one dimensional integral:

$$\mu(1+x+y) = \int_{1/6}^{5/6} \log(2\sin(\pi y))\, \mathrm{d}y = \frac{1}{\pi}\, \mathrm{Ls}_2\left(\frac{\pi}{3}\right) = \frac{1}{\pi}\, \mathrm{Cl}_2\left(\frac{\pi}{3}\right),$$

which is (15).

CARMA

$\mu(1+x+y)$ and $\mu(1+x+y+z)$ revisited

Following Boyd, on applying Jensen's formula, for complex a and b we have $\mu(ax+b) = \log|a| \vee \log|b|$. Let $w := y/z$. We now write

$$\begin{aligned}
\mu(1+x+y+z) &= \mu(1+x+z(1+w)) = \mu(\log|1+w| \vee \log|1+x|) \\
&= \frac{1}{\pi^2}\int_0^\pi \mathrm{d}\theta \int_0^\pi \max\left\{\log\left(2\sin\frac{\theta}{2}\right), \log 2\left(\sin\frac{t}{2}\right)\right\} \mathrm{d}t \\
&= \frac{2}{\pi^2}\int_0^\pi \mathrm{d}\theta \int_0^\theta \log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}t \\
&= \frac{2}{\pi^2}\int_0^\pi \theta\log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}\theta \\
&= -\frac{2}{\pi^2}\,\mathrm{Ls}_3^{(1)}(\pi) = \frac{7}{2}\frac{\zeta(3)}{\pi^2},
\end{aligned}$$

which is (16).

CARMA

Boyd's 1998 Conjectures

Theorem (Two quadratic evaluations)

Below L_{-n} *is a primitive* L*-series and* G *is Catalan's constant.*

$$\begin{aligned}
\mu_3 := \mu(y^2(x+1)^2 + y(x^2 + \mathbf{6}x + 1) + (x+1)^2) &= \frac{16}{3\pi}\,\mathrm{L}_{-4}(2) \\
&= \frac{16}{3\pi}\,\mathrm{G}, \\
\mu_{-5} := \mu(y^2(x+1)^2 + y(x^2 - \mathbf{10}x + 1) + (x+1)^2) &= \frac{5\sqrt{3}}{\pi}\,\mathrm{L}_{-3}(2) \\
&= \frac{20}{3\,\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right).
\end{aligned}$$

Log-sine Integrals are Again Inside

First proven in **2008** using Bloch-Wigner logarithms, we used a variant of Jensen's formula and slick trigonometry to arrive at:

$$\begin{aligned}\mu_3 &= \frac{1}{\pi}\int_0^{\pi} \log(1+4|\cos\theta|+4|\cos^2\theta|)\,\mathrm{d}\theta \\ &= \frac{4}{\pi}\int_0^{\pi/2} \log(1+2\cos\theta)\,\mathrm{d}\theta \\ &= \frac{4}{\pi}\int_0^{\pi/2} \log\left(\frac{2\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}}\right)\mathrm{d}\theta \\ &= \frac{4}{3\,\pi}\left(\mathrm{Ls}_2\left(\frac{3\pi}{2}\right) - 3\,\mathrm{Ls}_2\left(\frac{\pi}{2}\right)\right) = \frac{16}{3}\,\frac{\mathrm{L}_{-4}(2)}{\pi}\end{aligned}$$

as needed, since $\mathrm{Ls}_2\left(\frac{3\pi}{2}\right) = -\,\mathrm{Ls}_2\left(\frac{\pi}{2}\right) = \mathrm{L}_{-4}(2)$, which is Catalan's G.

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And More

Much the same techniques work for μ_5 and there is one bonus case:

Theorem

$$\begin{aligned}\pi\,\mu_{-1} &= \pi\,\mu\left((x+1)^2(y^2+y+1)-2xy\right)\\ &= \frac{1}{2}B\left(\frac{1}{4},\frac{1}{4}\right){}_3F_2\left(\left.\begin{matrix}\frac{1}{4},\frac{1}{4},1\\ \frac{3}{4},\frac{5}{4}\end{matrix}\right|\frac{1}{4}\right)-\frac{1}{6}B\left(\frac{3}{4},\frac{3}{4}\right){}_3F_2\left(\left.\begin{matrix}\frac{3}{4},\frac{3}{4},1\\ \frac{5}{4},\frac{7}{4}\end{matrix}\right|\frac{1}{4}\right)\end{aligned}$$

An alternative form of μ_{-1} is given by

$$\mu_{-1}=\mu\left(\left(x+1/x+2\sqrt{1/x}\right)(y+1/y+1)-2\right).$$

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Sasaki's Multiple Mahler Measures

toc

$$\mu_k(1+x+y_*) := \mu(1+x+y_1, 1+x+y_2, \ldots, 1+x+y_k)$$

was studied by Sasaki (2010). He used (18) to observe that

$$\mu_k(1+x+y_*) = -\int_{1/6}^{5/6} \log^k \left|1+e^{2\pi i\, t}\right| \, \mathrm{d}t \qquad (19)$$

and so provides an evaluation of $\mu_2(1+x+y_*)$. Immediately from (19) and the definition of the log-sine integrals we have:

Theorem (For $k = 1, 2, \ldots$)

$$\mu_k(1+x+y_*) = \frac{1}{\pi}\left\{\mathrm{Ls}_{k+1}\left(\frac{\pi}{3}\right) - \mathrm{Ls}_{k+1}(\pi)\right\}, \qquad (20)$$

where Ls_{k+1} is as given by (1).

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where Ls_{k+1} *is as given by* (1).

$\mathrm{Ls}_k(\pi)$ and $\mathrm{Ls}_n^{(k)}(\pi)$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\mathrm{Ls}_{m+1}(\pi)\,\frac{u^m}{m!}=\frac{\Gamma(1+u)}{\Gamma^2\left(1+\frac{u}{2}\right)}=\binom{u}{u/2}. \tag{21}$$

Example (Values of $\mathrm{Ls}_n(\pi)$)

For instance, we have $\mathrm{Ls}_2(\pi)=0$ as well as

$$-\mathrm{Ls}_3(\pi)=\frac{1}{12}\pi^3 \qquad \mathrm{Ls}_4(\pi)=\frac{3}{2}\pi\,\zeta(3)$$

$$-\mathrm{Ls}_5(\pi)=\frac{19}{240}\pi^5 \qquad \mathrm{Ls}_6(\pi)=\frac{45}{2}\pi\,\zeta(5)+\frac{5}{4}\pi^3\zeta(3)$$

$$-\mathrm{Ls}_7(\pi)=\frac{275}{1344}\pi^7+\frac{45}{2}\pi\,\zeta^2(3)$$

$\mathrm{Ls}_n(\pi)$ and $\mathrm{Ls}_n^{(k)}(\pi)$

Equation (21) is made for a CAS (Mma, Sage or Maple):

```
for k to 7 do
  simplify(subs(x=0,diff(Pi*binomial(x,x/2),x$k))) od
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We studied general log-sine evaluations with an emphasis on automatic provable evaluations. For example:

Theorem (Borwein-Straub)

For $2|\mu| < \lambda < 1$ we have

$$-\sum_{n,k\geq 0} \mathrm{Ls}_{n+k+1}^{(k)}(\pi)\frac{\lambda^n}{n!}\frac{(i\mu)^k}{k!} = i\sum_{n\geq 0}\binom{\lambda}{n}\frac{(-1)^n e^{i\pi\frac{\lambda}{2}} - e^{i\pi\mu}}{\mu - \frac{\lambda}{2} + n}.$$

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CARMA

Log-sine-cosine integrals

toc

The log-sine-cosine integrals

$$\mathrm{Lsc}_{m,n}(\sigma) := -\int_0^\sigma \log^{m-1}\left|2\sin\frac{\theta}{2}\right| \log^{n-1}\left|2\cos\frac{\theta}{2}\right| \mathrm{d}\theta \qquad (22)$$

appear in QFT/physical applications as well. Lewin sketches how values at $\sigma = \pi$ may be obtained much as for log-sine integrals.

- Lewin's ideas lead to:

$$-\frac{1}{\pi}\sum_{m,n=0}^{\infty} \mathrm{Lsc}_{m+1,n+1}(\pi)\frac{x^m}{m!}\frac{y^n}{n!} = \frac{2^{x+y}}{\pi}\frac{\Gamma\left(\frac{1+x}{2}\right)\Gamma\left(\frac{1+y}{2}\right)}{\Gamma\left(1+\frac{x+y}{2}\right)}$$

$$= \binom{x}{x/2}\binom{y}{y/2}\frac{\Gamma\left(1+\frac{x}{2}\right)\Gamma\left(1+\frac{y}{2}\right)}{\Gamma\left(1+\frac{x+y}{2}\right)}.$$

- The last form makes it clear that this is an extension of (21).

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$\mathrm{Ls}_n^{(k)}(\tau)$ is Made of Sterner Stuff.

- Contour integration and "polylogarithmics" yield:

Theorem (Reduction Theorem for $0 \le \tau \le 2\pi$)

For n, k such that $n-k \ge 2$, we have

$$\zeta(k,\{1\}^n) - \sum_{j=0}^{k-2} \frac{(-i\tau)^j}{j!} \mathrm{Li}_{k-j,\{1\}^n}(\mathrm{e}^{i\tau})$$

$$= \frac{(-i)^{k-1}}{(k-2)!} \frac{(-1)^n}{(n+1)!} \sum_{r=0}^{n+1} \sum_{m=0}^{r} \binom{n+1}{r} \binom{r}{m} \left(\frac{i}{2}\right)^r (-\pi)^{r-m} \mathrm{Ls}_{n+k-(r-m)}^{(k+m-2)}(\tau).$$

where $\mathrm{Li}_{2+k-j,\{1\}^{n-k-2}}(\mathrm{e}^{i\tau})$ is a harmonic polylogarithm and $\zeta(n-k,\{1\}^k)$ is an Euler-Zagier sum.

CARMA

$\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$: A small miracle occurs: $e^{-i\frac{\pi}{3}} = \overline{e^{i\frac{\pi}{3}}}$.

The Reduction Theorem now lets us find all values of $\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$ and so of $\mu_k(1+x+y_*)$:

Example (Values of $\mathrm{Ls}_n(\pi/3)$)

$$\mathrm{Ls}_2\left(\frac{\pi}{3}\right) = \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \qquad -\mathrm{Ls}_3\left(\frac{\pi}{3}\right) = \frac{7}{108}\,\pi^3$$

$$\mathrm{Ls}_4\left(\frac{\pi}{3}\right) = \frac{1}{2}\pi\,\zeta(3) + \frac{9}{2}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)$$

$$-\mathrm{Ls}_5\left(\frac{\pi}{3}\right) = \frac{1543}{19440}\pi^5 - 6\,\mathrm{Gl}_{4,1}\left(\frac{\pi}{3}\right)$$

$$\mathrm{Ls}_6\left(\frac{\pi}{3}\right) = \frac{15}{2}\pi\,\zeta(5) + \frac{35}{36}\,\pi^3\zeta(3) + \frac{135}{2}\,\mathrm{Cl}_6\left(\frac{\pi}{3}\right)$$

$$-\mathrm{Ls}_7\left(\frac{\pi}{3}\right) = \frac{74369}{326592}\pi^7 + \frac{15}{2}\,\pi\zeta(3)^2 - 135\,\mathrm{Gl}_{6,1}\left(\frac{\pi}{3}\right)$$

A Result for General τ

- An illustration of results produced by our programs:

Example (For $0 \le \tau \le 2\pi$)

$$\begin{aligned}
\mathrm{Ls}_4^{(1)}(\tau) &= 2\zeta(3,1) - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) \\
&+ \frac{1}{4}\mathrm{Ls}_4^{(3)}(\tau) - \frac{1}{2}\pi\,\mathrm{Ls}_3^{(2)}(\tau) + \frac{1}{4}\pi^2\,\mathrm{Ls}_2^{(1)}(\tau) \\
&= \frac{1}{180}\pi^4 - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) \\
&- \frac{1}{16}\tau^4 + \frac{1}{6}\pi\tau^3 - \frac{1}{8}\pi^2\tau^2.
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&= \frac{1}{180}\pi^4 - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) \\
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Irreducibility and Binomial Sums

Example (The first presumably irreducible value for $\pi/3$)

$$\mathrm{Gl}_{4,1}\left(\frac{\pi}{3}\right) = \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n-1} \frac{1}{k}}{n^4} \sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{3341}{1632960}\pi^5 - \frac{1}{\pi}\zeta^2(3) - \frac{3}{4\pi}\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^6}$$

while always

$$\mathrm{Ls}_{n+2}^{(1)}\left(\frac{\pi}{3}\right) = \frac{n!(-1)^{n+1}}{2^n} \sum_{k=1}^{\infty} \frac{1}{k^{n+2}\binom{2k}{k}}.$$

- Alternating binomial sums come from imaginary values of τ via $\log\sinh$ integrals at $\rho = \frac{1+\sqrt{5}}{2}$.

CARMA

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- Alternating binomial sums come from imaginary values of τ via $\log \sinh$ integrals at $\rho = \frac{1+\sqrt{5}}{2}$.

First Evaluation

Let

$$\mu_k(1+x+y_*+z_*) := \mu(1+x+y_1+z_1,\ldots,1+x+y_k+z_k). \tag{23}$$

Theorem

For all positive integers k, we have

$$\mu_k(1+x+y_*+z_*) = -\frac{1}{\pi^{k+1}}\int_0^\pi \left(\theta\log\left(2\sin\frac{\theta}{2}\right) - \mathrm{Cl}_2\left(\theta\right)\right)^k \mathrm{d}\theta$$

Then

$$\mu_1(1+x+y_*+z_*) = -\frac{2}{\pi^2}\,\mathrm{Ls}_3^{(1)}\left(\pi\right) = \frac{7}{2}\frac{\zeta(3)}{\pi^2},$$

$$\mu_2(1+x+y_*+z_*) = -\frac{1}{\pi^3}\,\mathrm{Ls}_5^{(2)}\left(\pi\right) + \frac{\pi^2}{90} = \frac{4}{\pi^2}\,\mathrm{Li}_{3,1}(-1) + \frac{7}{360}\pi^2.$$

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Two More Evaluations: with Kummer-type logarithms

Let

$$\lambda_n(x) := (n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \operatorname{Li}_{n-k}(x) \log^k |x| + \frac{(-1)^n}{n} \log^n |x|,$$

so that

$$\lambda_1\left(\tfrac{1}{2}\right) = \log 2, \quad \lambda_2\left(\tfrac{1}{2}\right) = \frac{1}{2}\zeta(2), \quad \lambda_3\left(\tfrac{1}{2}\right) = \frac{7}{8}\zeta(3),$$

and $\lambda_4\left(\frac{1}{2}\right)$ is the first to reveal the presence of $\operatorname{Li}_n\left(\frac{1}{2}\right)$. From the value of $W_4''(0)$ we derive:

Theorem

$$\mu_2(1+x+y+z) = \frac{12}{\pi^2}\lambda_4\left(\tfrac{1}{2}\right) - \frac{\pi^2}{5}$$

$$\mu(1+x, 1+x, 1+x+y+z) = \frac{4}{3\pi^2}\lambda_5\left(\tfrac{1}{2}\right) - \frac{3}{4}\zeta(3) + \frac{31}{16\pi^2}\zeta(5).$$

KLO's Mahler Measures

Theorem (Hypergeometric forms for $\mu_n(1+x+y)$)

For complex $|s|<2$, we may write

$$\sum_{n=0}^{\infty} \mu_n(1+x+y)\frac{s^n}{n!} = \frac{\sqrt{3}}{2\pi}\, 3^{s+1}\, \frac{\Gamma(1+\frac{s}{2})^2}{\Gamma(s+2)}\, {}_3F_2\left(\left.{\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}}\right| \frac{1}{4}\right) \tag{24}$$

$$= \frac{\sqrt{3}}{\pi}\left(\frac{3}{2}\right)^{s+1} \int_0^1 \frac{z^{1+s}\, {}_2F_1\left(\left.{1+\frac{s}{2}, 1+\frac{s}{2} \atop 1}\right| \frac{z^2}{4}\right)}{\sqrt{1-z^2}}\, \mathrm{d}z.$$

Evaluation of $\mu_n(1+x+y)$ Requires a Taylor Expansion

Consider

$$ {}_3F_2\left(\left.\begin{matrix} \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2} \end{matrix}\right| \frac{1}{4}\right) = \sum_{n=0}^{\infty} \alpha_n \varepsilon^n. \tag{25} $$

Indeed, from (24) and Leibnitz' rule we have

$$ \mu_n(1+x+y) = \frac{\sqrt{3}}{2\pi} \sum_{k=0}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \tag{26} $$

where β_k is defined by

$$ 3^{\varepsilon+1} \frac{\Gamma(1+\frac{\varepsilon}{2})^2}{\Gamma(\varepsilon+2)} = \sum_{n=0}^{\infty} \beta_n \varepsilon^n. $$

Note, as above, that β_k is easy to compute.

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Faà di Bruno's Formula

We can now read off the terms α_n of the ε-expansion:

Theorem (For $n = 0, 1, 2, \ldots$)

Let $A_{k,j} := \sum_{m=2}^{2j-1} \frac{2(-1)^{m+1}-1}{m^k}$. *Then*

$$[\varepsilon^n]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) = (-1)^n \sum_{j=1}^{\infty} \frac{2}{j} \frac{1}{\binom{2j}{j}} \sum \prod_{k=1}^{n} \frac{A_{k,j}^{m_k}}{m_k! k^{m_k}} \tag{27}$$

where we sum over all $m_1, \ldots, m_n$ *with* $m_1 + 2m_2 + \ldots + nm_n = n$.

Proof.

Equation (27) follows from (24) on using Faà di Bruno's formula for the n-th derivative of the composition on two functions via Pochhammer notation. □

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Davydychev and Kalmykov's Binomial Sums Yield:

Example

$$\begin{aligned}
\mu_1(1+x+y) &= \frac{3}{2\pi}\,\mathrm{Ls}_2\left(\frac{2\pi}{3}\right)\\
\mu_2(1+x+y) &= \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{\pi^2}{4}\\
\mu_3(1+x+y) &\stackrel{?}{=} \frac{6}{\pi}\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-\frac{9}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{\pi}{4}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{1}{2}\zeta(3).
\end{aligned}$$

As we had obtained by other methods. Also PSLQ then finds:

$$\begin{aligned}
\pi\mu_4(1+x+y) \stackrel{?}{=}\ & 12\,\mathrm{Ls}_5\left(\frac{2\pi}{3}\right)-\frac{49}{3}\,\mathrm{Ls}_5\left(\frac{\pi}{3}\right)+81\,\mathrm{Gl}_{4,1}\left(\frac{2\pi}{3}\right)\\
&+3\pi^2\,\mathrm{Gl}_{2,1}\left(\frac{2\pi}{3}\right)+2\zeta(3)\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\pi\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)^2-\frac{29}{90}\pi^5.
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Conclusion

We also have generalized arctangent forms, such as:

$$\mu_2(1+x+y) = \frac{24}{5\pi}\mathrm{Ti}_3\left(\frac{1}{\sqrt{3}}\right) + \frac{2\log 3}{\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\log^2 3}{10} - \frac{19\pi^2}{180}.$$

1. We are still hunting for a complete accounting of $\mu_n(1+x+y)$.
2. Our log-sine and MZV algorithms uncovered many errors and gaps (e.g., values of Euler sums such as $\zeta(\overline{2n+1}1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$) in the literature.
3. Automated simplification, validation and correction tools are more and more important.
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Postscript: π^2 base 2

Base-64 digits of π^2 beginning at position 10 trillion. The first run, which produced base-64 digits starting from position $10^{12} - 1$, required an average of 253,529 seconds per thread, subdivided into seven partitions of 2048 threads each, so the total cost was $7 \cdot 2048 \cdot 253529 = 3.6 \times 10^9$ CPU-seconds. Each rack of the IBM Blue Gene P system features 4096 cores, so the total cost is **10.3** "rack-days." The second run, which produced base-64 digits starting from position 10^{12}, took the same time (within a few minutes). The two resulting base-8 digit strings are

$$75|60114505303236475724500005743262754530363052416350634|573227604$$

$$xx|60114505303236475724500005743262754530363052416350634|220210566$$

(each pair of base-8 digits corresponds to a base-64 digit). Here the digits in agreement are delimited by $|$. Note that 53 consecutive base-8 digits (159 binary digits) are in agreement.

π^2 base three

Base-729 digits of π^2 beginning at position 10 trillion. Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was 6.5×10^9 CPU-seconds, or **18.4** "rack-days" for each run.
The two resulting base-9 digit strings are

$$001|12264485064548583177111135210162856048323453468|10565567635862$$

$$xxx|12264485064548583177111135210162856048323453468|04744867134524$$

(each triplet of base-9 digits corresponds to one base-729 digit). Note here that 47 consecutive base-9 digits (94 base-3 digits) are in agreement.

CARMA