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K. Mahler

Obituary of Kurt Mahler

by

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Mahler's life

Kurt Mahler was born on July 26, 1903 into a middle-class Jewish family in Krefeld, a town of some 100,000 inhabitants in a predominantly Catholic part of the Prussian Rhineland. Kurt and his twin sister Hilde (1903–1934) were the youngest children of Hermann Mahler (1858–1941) and his wife Henriette, *née* Stern (1860–1942). There was an older sister Lydia (died 1984) and brother Josef (died ? 1945); four other children had died in infancy. For several generations his ancestors had lived in the Rhineland. His father and several uncles worked in printing and bookbinding, his father having worked up from apprentice to owning his own small business. One uncle, apparently the only well-to-do member of the family, owned a silk tie factory. Mahler recalls that "In those days (but not after the first world war) our home was still run on strictly orthodox Jewish lines and we were also good German patriots ... there was even then a certain amount of anti-Semitism, although more as a reminder of the past".

The children acquired a love of reading from their father. "When I was not playing as a child with some mechanical toy", he recalled "then I was certainly reading". In particular, there was an elementary book on geometry which he would not then understand, but of which he liked to copy the figures.

He was a sickly child and at the age of about 5 developed tuberculosis of the right knee. Consequently, until the age of 14 he had only about four years of formal schooling, but his parents arranged for him to have tuition at home. His first intention was to become a *Feinmechaniker* (precision tool and instrument maker), and so from Easter 1917 for two years he attended elementary technical schools. Here he became fascinated with mathematics and soon was buying books on trigonometry, analytical geometry and the calculus which he studied, of course without a teacher. Soon he moved to the theory of numbers and function theory.

With the aim of studying at a polytechnic (*Technische Hochschule*) he became an apprentice in a machine factory in Krefeld, where he stayed for not quite three years. Here he continued his mathematical self-education. Among the books he acquired and studied were the tomes of Klein and Fricke *Modulfunktionen*, *Automorphe Funktionen* and Hilbert's *Grundlagen der Geometrie*. There was no one to help his reading, but he recalls that a useful guide was the advertisements which Teubner inserted in their mathematical books.

Although they could not understand them, his parents encouraged his mathematical interests. He had started to write small articles about what he had read and what it implied. Without consulting him, his father sent some of them to the headmaster of the local modern grammar school (*Realschule*) Dr Junker, who happened to be a mathematician. He sent some of these articles to the now ageing Felix Klein, who passed them on to his young *Assistent* Carl Siegel. Siegel was impressed and suggested that Mahler be helped to pass the university entrance examination (*Abitur*) and to go to university.

So Mahler left the factory and for the next two years Dr Junker arranged for him to be coached for the *Abitur* by teachers from his school while he continued his private mathematical studies, which were now at university level. Later, he dedicated his doctoral thesis [2] to Dr Junker. An essay from this period survives [215] and makes interesting reading. His German teacher found his essays boring and proposed the topic "Why I have a special preference for mathematics".

The Prussian authorities allowed Mahler to take the *Abitur* under special arrangements: he says he "just scraped through". In the meantime Siegel became professor at Frankfurt/Main and at his invitation Mahler enrolled there at the age of 20. Although a recent foundation, Frankfurt had a strong mathematical department and he attended lectures of Dehn, Hellinger, Madelung and Szász as well as of Siegel, who continued to act as mentor. In the spring of 1925, when Siegel went on leave, he migrated to Göttingen, where he stayed until he left Germany in 1933. Here he was most influenced by Emmy Noether, Courant, Landau, but he attended lectures of Born, Heisenberg, Herglotz, Hilbert, Ostrowski ... and was unpaid assistant to the visiting Wiener (which explains [1]).

Mahler's doctoral thesis [2], approved in late 1927, was on the zeros of the incomplete gamma function. Hitherto he had been supported by his parents and other members of the Jewish community in Krefeld; for the next four years he held a research fellowship of the *Notgemeinschaft der Deutschen Wissenschaft*. In this period he developed a new method in transcendence theory, found his celebrated classification of transcendental numbers and pioneered diophantine approximation in p-adic fields.

With the advent of Hitler to power in 1933, Mahler realized that he had no future in Germany. In the summer of 1933 he spent 6 weeks in Amsterdam with van der Corput and his pupils Koksma and Popken. Then for the academic year 33–34 Mordell secured a small research fellowship for him in

Manchester. The next two academic years were spent in Groningen in the Netherlands supported by a stipend obtained by van der Corput from a Dutch Jewish group. At this period a new theme showed itself in his research, the geometry of numbers.

In 1936 the tuberculosis in his right knee was reactivated as the result of an encounter with a cyclist. He returned to Krefeld where he underwent several operations culminating in the removal of the kneecap. There was a long and painful convalescence. In the course of the next few years the infection disappeared, but he was left with a permanent limp.

In 1937 he returned to Manchester. For some of the time he was supported by small fellowships or a temporary assistant lectureship, but for substantial periods he lived on his savings. Under the leadership of Mordell, Manchester was a lively centre of mathematics, and of number theory in particular. In 1939 he planned to join Chao Ko in China: the onset of war prevented this, but the Chinese language became a major hobby (the other was photography). He was interned for some months in 1940 as an "enemy alien" first in a tented camp on the Welsh border, then on the Isle of Man. While interned he was awarded an ScD by the University. In 1941 he was appointed to his first regular post, an assistant lectureship. In the next few years he developed a geometry of numbers of general sets in n-dimensional space, including his compactness theorem.

His future was now assured. He was promoted to Lecturer (1944), Senior Lecturer (1947), Reader (1949) and in 1952 the first personal Chair in the history of the University was created for him. In 1948 he was elected to the Royal Society, having become a British subject in 1946. The London Mathematical Society awarded him its Senior Berwick Prize in 1950.

Although we had met earlier, my acquaintance with Mahler dates from the year 1949–50 I spent in Manchester. He was a fine draftsman and some years earlier he had drawn a large number of diagrams depicting the analogue of Farey section for the Gaussian and Eisenstein integers. The consideration of the regularities they revealed gave rise to the joint paper [114] with Walter Ledermann and myself. His handwriting, as his accent, retained a strongly teutonic character. In the first part of 1950 he was severely ill with diptheria and complications, and I had to correct the proofs of a paper he had written for a French periodical. In the MS he had typed in casual occurrences of the variables but had written the displays in his usual hand. There were also greek capitals (Alpha, Beta, ...) and these he had written in a straightforward manner. The French compositor took the greek capitals for latin ones and sought out his swashest type to set the handwritten letters, and I had to sort out the mess.

At this time Mahler lived in Donner House, a hostel where some 25–30 single university staff lived in bedsitting rooms and dined communally. Paul Cohn, who went there in 1952 as a beginning Assistant Lecturer recalls. "The only other mathematician I found there was Professor Mahler, so we saw a good deal of each other for the next six years. He was without any pretensions and one could discuss anything under the sun with him, though preferably mathematics, photography or Chinese (in that order). For his holidays he always went to the island of Herm, saying that it suited him because he could not walk far. The island could be reached only by a small boat from Guernsey; it had no motor traffic and could be crossed in half an hour, walking slowly." ...

"His attitude to mathematics was like his outlook on life: he liked things as simple as possible and usually eschewed abstraction, but with his direct methods was often able to go surprisingly far. He told me that his favourite way of starting a seminar talk was with the words 'If I have a convex body ...', but he had given it up regretfully because of the mirth it produced (he was short and tubby, with a rather peculiar shape; this and his limp were due to an early bout of tuberculosis). He was generally very precise and punctual, rising and going to bed early. At his request seminars at Manchester were held at 2 p.m. If a speaker was still on his feet at 3.01, Mahler (who always sat in the front row) would open and shut his little attaché-case repeatedly with a loud click". Walter Ledermann recalls when he and his wife were invited by Mahler to supper at Donner House together with David Rees and some others. When his regular bedtime of 9.30 arrived, he said that Rees would continue to entertain them, shook hands all round, and went off.

In 1958 Donner House was shut down to make way for student residences and Mahler bought himself a small house. He was increasingly isolated as colleagues left to take up appointments elsewhere. In 1962 he visited the Institute for Advanced Study, Canberra at the invitation of one of them, Bernhard Neumann, who had become head of the department of mathematics. He was impressed by what he saw and when offered a professorship accepted with alacrity.

In Canberra he lived in University House, which provided the comforts and friendly atmosphere he appreciated and which was conveniently near to the Institute. The appointment at the Institute was a purely research one, but he offered a course on number theory in the undergraduate school of the Australian National University, as none existed. One of the undergraduates, John Coates, asked to be initiated into research.

In 1968 he reached the retirement age at the Institute and accepted an invitation to Ohio State University. But in 1972 returned to Canberra, where he continued to work in the Institute as an Emeritus Professor. In November 1977 at a special ceremony in Frankfurt he received a diploma to mark the golden jubilee of his doctorate. As he grew older, deteriorating health made it more and more difficult for him to travel, but he remained mathematically active until the end. He died on Thursday 25 February 1988 in his 85th year.

For a mathematician of his distinction, Mahler had comparatively few doctoral pupils. They included K. Ollerenshaw (subsequently Dame Kathleen and Lord Mayor of Manchester), L. J. Parry, J. V. Whitworth, H. Turkstra, N. Mullineux, A. C. Woods and P. W. Aitchison.

He became a Fellow of the Royal Society in 1948. The London Mathematical Society awarded him the Senior Berwick Prize in 1950 and the De Morgan Medal in 1971. The Australian Academy of Science elected him a Fellow in 1965 and awarded him its Lyle medal in 1977. The Wiskundig Genootschap (Dutch Mathematical Society) made him an honorary member in 1957, as did the Australian Mathematical Society in 1986.

Sources. Mahler himself gives accounts of his life and work in [192, 210, 215]. His papers include a manuscript dated 1971 which he clearly used for [192, 210], but which is fuller than both. The "Personal Record" which he made for the Royal Society is particularly full and up-to-date. The archive of the Society for the Protection of Science and Learning (formerly: Academic Assistance Council) in the Bodleian Library, Oxford contains material(¹) relating to the period 1934–1945. The papers of Professor H. Davenport in the Library of Trinity College, Cambridge contain a number of letters from Mahler. Professors A. Baker, P. M. Cohn, J. H. Davenport, W. Ledermann, B. H. Neumann, R. Rado and A. Schinzel have given help and advice. Professors J. H. Coates and A. J. van der Poorten, who are writing the Biographical Memoir for the Royal Society, have put their materials at my disposal: in particular, the list of Mahler's publications given here is based on theirs.

The photograph was supplied by the Australian National University, Canberra.

Mahler's mathematics

It is convenient to discuss Mahler's work under separate themes, most of which occupied him at various times throughout his life. He himself gave an account of his work in [210].

Transcendence

Mahler was a leading pioneer of transcendence theory. It was his most long-standing interest and there is no doubt that his unflagging enthusiasm played a major role in its development.

The early papers [4, 7, 8] contain an elegant argument to show that the values taken at algebraic values of functions satisfying certain functional equations are transcendental. One particular case is the Fredholm series $\sum z^n$, where n runs through the powers of 2 and another is that in which the expression of n in the scale of some integer g omits a given digit [96, 115]. The argument is similar in style to the later celebrated work of Siegel on the Bessel functions and in fact for many years it was completely neglected while Siegel's

(¹) Unfortunately I have been unable to go to Oxford to consult it.

seemingly more powerful technique was greatly extended by Shidlovsky and others. But when some of these results were rediscovered, he recalled his own work [170] and "Mahler's method" became the object of vigorous study (cf. Loxton & van der Poorten (1978, 1982) and Loxton (1984, 1988)). There are now numerous applications to finite automata, topological dynamics, harmonic analysis, fractals and other topics.

Mahler was particularly expert on Hermite approximations to the exponential and logarithmic functions, and a recurring theme is the refinement of the classical Hermite–Lindemann theory relating to e and π ([11, 13, 118, 164]; the book [200] is especially illuminating). Some earlier results had been obtained by Borel and Popken, but Mahler established new basic integral formulae and obtained measures of transcendence for the function values at algebraic points. A particularly striking example [119] is that

$$|\pi - p/q| > q^{-42}$$

for integers p and $q \geqslant 2$. The exponent was improved to 20.6 by Mignotte (1974) by a refinement of the method, and to 19.9, 15.62 by Chudnovsky (1982, 1988). For a generalization of Mahler's formulae and applications, see Baker (1966, 1967).

In [11] Mahler introduced his famous classification of real or complex numbers in terms of the approximation to zero of the values taken by polynomials with rational integer coefficients. There are three main classes $\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U}$ of transcendental numbers in addition to the algebraic numbers $\boldsymbol{A}$. Two numbers can be algebraically dependent only if they are in the same class. Thus, as he had already shown [9], e is algebraically independent of Liouville numbers, for the latter are $\boldsymbol{U}$-numbers while Popken's estimates showed e to be a $\boldsymbol{S}$-number. That $\boldsymbol{T}$-numbers actually exist was shown only later by Schmidt (1968). Mahler showed [12] that almost all real and almost all complex numbers are $\boldsymbol{S}$-numbers. There is a finer classification of $\boldsymbol{S}$-numbers in terms of a parameter and he conjectured that almost all real numbers are of type 1 and almost all complex numbers of type 1/2. This conjecture was proved by Sprindzhuk (1965). A different but closely related classification in terms of the approximation by algebraic numbers was introduced by Koksma (1939). Many intricate questions remain: for example while π is almost certainly an $\boldsymbol{S}$-number, all that has been proved is that it is not an $\boldsymbol{A}$- or $\boldsymbol{U}$-number.

In [27] Mahler gives an analogue of his classification of transcendental numbers for the p-adics. Indeed much of his work on transcendence and diophantine approximation involve the p-adics. Especially well known are his analogue [14] of the Hermite–Lindemann theorem that the p-adic exponential function is transcendental at nonzero algebraic values of the argument and the p-adic analogue [30] of the Gel'fond–Schneider theorem that α^β is transcendental for algebraic α and irrational algebraic β. The p-adic generalizations of Thue–Siegel theory will be described in the next section.

Not all his work is p-adic, however. In [46] he establishes the transcendence of 0.12345678910111213 14... (the concatenation of the integers in decimal notation). Champernowne (1933) had proved it normal: Baker (1964) showed that it is not a U-number.

p-adics

Hensel had introduced p-adic numbers at the beginning of the century and Hasse made revolutionary use of them for quadratic forms in the early 20's. The p-adics were a major theme of Mahler, and his early work helped them to become part of the general mathematical culture in the early 30's.

Mahler extended results in diophantine approximation to include p-adic valuations as well as the ordinary absolute value. These are sometimes spoken of a "p-adic analogues", but the absolute value must occur in the enunciation as well as one or finitely many p-adic valuations. In [24] this is done for Minkowski's linear forms theorem. A deeper and extremely fruitful example of such an extension was to Siegel's then recent improvement of Thue's theorem about the approximation of algebraic numbers by rationals. This permits interesting new applications to diophantine equations, e.g. [17, 20] on the greatest prime factor of integers represented by a binary form or [21] which shows that there are only finitely many S-integral points on a curve of genus 1. [An S-integer is a rational number whose denominator contains only primes from a given finite set S.] Roth's definitive version of the Thue–Siegel theorem can similarly be generalized. As an application Mahler [135] obtained lower bounds for the fractional parts of the powers of a non-integral rational. The value of $g(k)$ in Waring's problem is given by one of two formulae depending on the fractional part of $(3/2)^k$: Mahler's result shows that one of the two formulae holds for all sufficiently large k (but "sufficiently large" is non-constructive, and remains so to this day!). For a reformulation of this circle of ideas and later work, see e.g. Lang (1962, 1983).

Skolem (1933) deduced results about the number of solutions of certain diophantine problems from a new kind of series expansion. In his review in the Zentralblatt Mahler noted that the method was essentially p-adic, though the paper did not use p-adic language. Mahler used similar methods on the coefficients of the expansions of rational functions of a single variable over an algebraic number field. He showed that, up to a finite number of exceptions, the zero coefficients are precisely those whose index lies in one of a finite number of arithmetic progressions [25, 31]. There is an equivalent formulation in terms of vanishing elements of sequences satisfying a recurrence relation with polynomial coefficients. Simultaneously and independently Skolem (1934) gave a similar result, in some ways weaker, in others more precise. Lech (1954) generalized Mahler's result to any field of characteristic 0, reducing it to the p-adic case by a specialization argument. Although Mahler reviewed Lech's paper for the Zentralblatt, he forgot about it and gave a proof on different lines [131, 131a]. For a simplified treatment of much of this material and a further discussion of the literature, see Cassels (1986). The treatment in Hansel (1986) claims not to use p-adic methods but is clearly p-adic in inspiration. For a weaker generalization to nonzero characteristic see Bézivin (1987), and for an interesting generalization to the coefficients of the solutions of a wide class of differential equations see Bézivin (1989).

In [139, 145] Mahler proves a criterion for a function on the positive integers to be extendible continuously to a function on the p-adic integers, and gives an elegant formula for the extension if it exists. This result is basic for the theory of p-adic L-functions, which are obtained by interpolation from values at the negative integers, and so is a foundation of what is now a booming industry ("Iwasawa theory"): but Mahler himself did not take part. His book [205] (a much extended second edition of [184]) contains an extensive study of p-adic functions defined in this way, their differentiability properties and so on.

A sequence of early papers [34, 37, 38, 39] is devoted to pseudo-valuations. These are similar to non-archimedean valuations, except that the equality for the value of a product is replaced by an inequality. Frequently a pseudo-valuation is a function of a number of valuations, and Mahler discusses when this is the case. This work was carried further in a joint paper [122] with P. M. Cohn. Cohn (1954) gives conditions for a pseudo-valuation to be a supremum of valuations and Bergman (1971) showed this to be an analogue of the theorem describing radical ideals as intersections of prime ideals. In expositions Mahler usually starts with pseudo-valuations rather than valuations. He regards p-adic numbers as a special case of what he calls g-adic numbers, where g is any positive integer. The g-adic pseudo-valuation is defined rather as is the p-adic valuation. The g-adic completion is the product of the p-adic completions over the p dividing g.

Geometry of numbers

The use of spatial intuition in the theory of numbers goes far back, but it was Minkowski who coined the term "Geometry of Numbers" and used it to describe arguments based in particular on considerations of packing and covering. In fairly simple situations this leads to elegant theorems and striking proofs. In more complicated situations Minkowski's approach still gives valuable intuitions; but it may be difficult to be sure that they are a reliable guide, and even more difficult to convey those intuitions to others as a convincing formal proof. In 1946 Mahler published his compactness theorem which systematized and simplified many of these intuitive considerations. The theorem states that the set of lattices in n-dimensional space satisfying a couple of clearly necessary conditions is compact with respect to a natural topology.

Mahler himself explored the consequences of his compactness theorem in [82, 83, 84, 87, 88]. The geometry of numbers of some nonconvex bodies had been considered already by Davenport, Mordell and others, but Mahler gave a systematic general theory, especially for star bodies. He made a special study of the consequences of the existence of an infinite group of automorphisms of a star body. Many of the concepts and techniques introduced have been basic for much later work. The compactness theorem was used by Davenport (1955) to reduce certain problems in simultaneous approximation to simpler problems in the geometry of numbers. For a recent striking result where Mahler's compactness theorem is an essential ingredient of the proof, see Margulis (1987).

Chabauty (1950) has given a treatment of Mahler's compactness theorem which is independent of successive minima and which generalizes to certain algebraic groups: see also Mumford (1971).

Independently and almost simultaneously with Hlawka, Mahler [79] proved a slightly weaker form of the Minkowski–Hlawka theorem. He was the first to give a strengthening [91], but this has been superseded by later estimates.

In [126, 127, 128, 129] Mahler introduces the notion of the pth compound of a convex body. Any choice of p points from the body determines a point on the appropriate grassmannian. The pth compound is the convex closure. Mahler's results about compounds were a tool in the generalization by Schmidt (1970) of Roth's theorem to simultaneous approximations.

For more on the topics of this section, see Cassels (1959), Rogers (1964), and Gruber & Lekkerkerker (1987).

Transference theorems

Khintchine (1926) appears to have been the first to observe that there are connections between the diophantine behaviour of a set of linear forms with real coefficients, the behaviour of the dual set of forms and also the corresponding inhomogeneous problems. He called such results *Übertragungssätze* (transference theorems). At first this circle of ideas appeared somewhat mysterious and was dealt with by a variety of *ad hoc* techniques. After [44], [53] and [56] dealing with various aspects of these questions, Mahler systematized this body of relationships in [57]. He showed that the natural setting is that of convex bodies, their distance-functions and their duals. There are then inequalities connecting the successive minima of a lattice with respect to a convex body and those of the dual lattice with respect to the dual body. Transference theorems follow in a simple and transparent way. This does not give the best possible constants in the theorems, but clarifies their logical structure. In a note Mahler acknowledges that these ideas may have been to some extent anticipated by Marcel Riesz (1936).

The arguments of [57] require estimates for the product of the volumes of a pair of dual convex bodies. In [101] he obtains the best constants for 2 dimensions.

This may be the appropriate place to mention [105]. Rogers and Chabauty had independently found an inequality for the product of the successive minima of a general star body. In [105] Mahler constructs an example to show that it is best possible. A similar example was constructed independently by Chabauty.

Quadratic forms

Minkowski developed his theory of the reduction of definite quadratic forms in the context of the geometry of numbers. The ordinary euclidean distance provides an exemplar for his general distance-functions for convex bodies, though it is appreciable simpler to handle. It seems likely that [55] led Mahler to his work on transference theorems and also to his compactness theorem.

For a (Minkowski-)reduced definite quadratic form

$$\sum a_{ij} X_i X_j$$

in n variables of determinant D, Minkowski [*Werke II*, 63] showed that there is a constant λ_n depending only on n such that

$$\lambda_n a_{11} \ldots a_{nn} \leqslant D.$$

He did not, however, give any estimate for λ_n. Bieberbach and Schur (1928) gave the first but very weak estimate. It was improved by Remak (1938) in a long and difficult paper. In [55] Mahler gives an estimate which applies to all convex distance-functions, but which is rather worse than Remak's for the special case of quadratic forms.

The best possible value of λ_3 was already known to Gauss. In [70] Mahler gives another derivation and in [92] he obtains the best possible λ_4.

For background and discussion see van der Waerden (1956): in particular it is shown how Remak's estimate for λ_n can be obtained by a modification of Mahler's argument.

Polynomials

Several of Mahler's papers are concerned with measure for the size of a polynomial

$$f(X) = f_0 X^n + \ldots + f_n.$$

In [143] he introduces

$$M(f) = |f_0| \prod \max(1, |\alpha_j|),$$

where α_j runs through the roots of $f(X)$. He notes that $M(f)$ is given by Jensen's formula applied to $\log(f(X))$ and deduces that

$$M(f) < L(f) < 2^n M(f),$$

where

$$L(f) = |f_0| + \ldots + |f_n|.$$

In [144] it is shown that

$$M(f') \leqslant nM(f).$$

In [150, 153] there is an investigation of the best possible constants in the ratios of $M(f)$ to other measures of the size of $f(X)$ such as $L(f)$. The paper [154] discusses, in effect, what information can be derived about the separation of the roots of $f(X)$ in terms of $M(f)$ and the discriminant. In [148] there are some generalizations to functions of several variables.

Mahler appears to have been the first to study $M(f)$ systematically, but it does occur earlier. Landau (1905) gives an estimate for the zeros of analytic functions which, for polynomials, implies that $M(f)$ is bounded above by

$$L_2(f) = \{(|f_0|)^2 + \ldots + (|f_n|)^2\}^{1/2}.$$

Lehmer (1933) notes that $M(f) \geqslant 1$ for irreducible polynomials with integer coefficients and with $f_0 = 1$, and that there is equality only for cyclotomic polynomials. He raises the question whether $M(f)$ is uniformly bounded away from 1 if f is not cyclotomic. This question gave rise to many papers and is still unresolved, but Mahler does not seem to have discussed it. The closely related function

$$\prod (1 + |\alpha_j|)$$

was used by Siegel (1921) [Hilfssatz 1] and figures in one of the exercises which Schur contributed to Pólya & Szegö (1924) [Aufgabe V196].

In the estimation of the sizes of polynomials the naive measures of size such as $L(f)$ give information about behaviour under addition, whereas $M(f)$ satisfies the obvious multiplicative identity

$$M(fg) = M(f) M(g).$$

There follow estimates for $L(fg)$ in terms of $L(f) L(g)$. These and similar estimates have proved useful in transcendence theory, which was doubtless Mahler's purpose. More recently, they have become important in complexity theory, see Mignotte (1981). In this context the results of [154] have been extended by J. H. Davenport (1985).

Miscellaneous

In their work on Waring's problem Hardy and Littlewood found it desirable to suppose that certain diophantine equations could not have too many solutions ("Hypothesis K"). In [40] Mahler disproved this hypothesis for cubic forms: he displayed a parametric representation of a 12th power as a sum of 3 cubes which implies that m^{12} for positive integer m has at least m representations as the sum of 3 nonnegative cubes. This is still the only known exception to Hypothesis K. Earlier he [23] used the identity

$$x^4+y^4+(x+y)^4 = 2(x^2+xy+y^2)^2$$

to show that the number of representations of an integer as the sum of 3 fourth powers is unbounded.

In [28] he proved the following. Let $f(x, y)$ be a cubic form with integral coefficients and non-zero discriminant, and denote by $N(m)$ the number of (not necessarily primitive) representations of an integer m by f. Then there is a $c>0$, depending only on f, such that $N(m) > c(\log m)^{1/4}$ for infinitely many m. The exponent 1/4 was improved to 1/3 (and sometimes better) by Silverman (1983).

In another direction, there are the effective estimates [20, 32] for the greatest prime factor of Ax^2+E, where $E = \pm 1, \pm 2$, which were somewhat improved by Schinzel (1967).

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