ON HECKE'S THEOREM ON THE REAL ZEROS OF THE L-FUNCTIONS AND THE CLASS NUMBER OF QUADRATIC FIELDS

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Prof. Mordell, by generalizing a method of Deuring†, proves‡ that Riemann's hypothesis for $\zeta(s)$ is true if an infinity of different imaginary quadratic fields $K(\sqrt{-d})$ exists with the same value of the class number $h(-d)$. The proof depends on an asymptotic formula for $\zeta(s)L_d(s)$ for $d \to \infty$, where $L_d(s)$ denotes the L-series belonging to the field.

I find that the same asymptotic formula may be applied to the study of $L_d(s)$ instead of $\zeta(s)$, and that it leads to relations between the class number $h(-d)$ and the real zeros of $L_d(s)$. The following result, including that due to Hecke§, will be proved in this note:

Suppose that

$$a_n x^2 + b_n xy + c_n y^2 \quad (n = 1, 2, \ldots, h)$$

is the system of all reduced positive quadratic forms of fundamental discriminant $-d = b_n^2 - 4a_n c_n < 0$, and that h' denotes the quotient

$$h' = \frac{h}{\sum_{n=1}^{h} \frac{1}{a_n}}.$$

* Received 15 January, 1934; read 18 January, 1934.

† M. Deuring, "Imaginäre quadratische Zahlkörper mit der Klassenzahl 1", *Math. Zeitschrift*, 37 (1933), 405–415.

‡ L. J. Mordell, "On the Riemann hypothesis and imaginary quadratic fields with a given class number", *Journal London Math. Soc.*, 9 (1934), 289–298.

§ E. Hecke, in the paper by E. Landau, "Über die Klassenzahl imaginärquadratischer Zahlkörper", *Göttinger Nachrichten* (1918), 285–295.

Then, when d is sufficiently large, corresponding to given constants $\gamma > 0$, $g > 0$, there exist constants $\Gamma = \Gamma(\gamma) > 0$, $G = G(g) > 0$, such that if $L_d(s)$ has at least one zero in the interval

$$1 - \frac{\gamma}{\log d} \leqslant s \leqslant 1,$$

then

$$h' \leqslant \Gamma \frac{\sqrt{d}}{\log d};$$

and if $L_d(s)$ has no zero in the interval

$$1 - \frac{g}{\log d} \leqslant s \leqslant 1,$$

then

$$h' \geqslant G \frac{\sqrt{d}}{\log d}.$$

1. Suppose that

$$Q = ax^2 + bxy + cy^2$$

is a reduced quadratic form with integer coefficients and negative discriminant

$$-d = b^2 - 4ac < 0,$$

so that

$$0 < a \leqslant \sqrt{(\tfrac{1}{3}d)}.$$

From Mordell's paper, formulae (4), (11), for $d < -4$,

$$\begin{aligned} 2\zeta_Q(s) &= \sum_{\substack{x=-\infty \\ x^2+y^2>0}}^{+\infty} \sum_{y=-\infty}^{+\infty} Q^{-s} \\ &= 2a^{-s}\zeta(2s) + 2d^{\frac{1}{2}-s} a^{s-1} \left(\frac{\zeta(2s-1)\,\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)} + O(1) \right) \end{aligned}$$

uniformly in the interval

$$\tfrac{1}{2} \leqslant s \leqslant 1$$

as $d \to \infty$.

The zeta function $\zeta_d(s)$ of the imaginary quadratic field $K(\sqrt{-d})$ satisfies the equations

$$\zeta_d(s) = \zeta(s)\, L_d(s) = \sum_Q \zeta_Q(s),$$

where

$$L_d(s) = \sum_{n=1}^{\infty} \left(\frac{-d}{n} \right) n^{-s}$$

is the corresponding L-series, and the summation in Q refers to all different reduced forms Q. Hence

$$\zeta(s)\,L_d(s)=f_d(s)\,\zeta(2s)+d^{\frac{1}{2}-s}f_d(1-s)\left(\frac{\zeta(2s-1)\,\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}+a_d(s)\right),$$

where $f_d(s)$ is the finite Dirichlet series

$$f_d(s)=\sum_Q a^{-s},$$

and where $a_d(s)$ denotes a real function of s, which is uniformly bounded in the interval

$$\tfrac{1}{2}\leqslant s\leqslant 1$$

when d tends to infinity.

Now take an arbitrary constant σ_0 with

$$\tfrac{1}{2}<\sigma_0<1.$$

Then there exist five positive numbers $c_1, \ldots, c_5$, which depend only on σ_0, such that, for sufficiently large d,

$$c_1\leqslant\zeta(2s)\leqslant c_2,\qquad c_3\leqslant\frac{\zeta(2s-1)\,\Gamma(s-\frac{1}{2})\sqrt{\pi}(s-1)}{\Gamma(s)}\leqslant c_4,\qquad |a_d(s)|\leqslant c_5,$$

uniformly in s in the interval

$$\sigma_0\leqslant s\leqslant 1.$$

Hence, if σ_1 denotes the number

$$\sigma_1=\max\left(\sigma_0,\ 1-\frac{c_3}{2c_5}\right),$$

the inequality

$$c_6=\frac{c_3}{2}\leqslant\left(\frac{\zeta(2s-1)\,\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}+a_d(s)\right)(s-1)\leqslant c_4+\frac{c_3}{2}=c_7$$

holds uniformly for s in

$$\sigma_1\leqslant s\leqslant 1,$$

as d tends to infinity. We write

$$A_d(s)=\zeta(2s),\qquad B_d(s)=\left(\frac{\zeta(2s-1)\,\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}+a_d(s)\right)(s-1),$$

and then have the result:

"There exist a constant σ_1, with $\frac{1}{2}<\sigma_1<1$, and also four positive constants c_1, c_2, c_6, c_7, such that

$$\zeta(s)\,L_d(s)=A_d(s)\,f_d(s)-\frac{B_d(s)}{1-s}\,d^{\frac{1}{2}-s}f_d(1-s),$$

where $$c_1 \leqslant A_d(s) \leqslant c_2, \qquad c_6 \leqslant B_d(s) \leqslant c_7$$
uniformly in s for the interval J or

$$\sigma_1 \leqslant s \leqslant 1,$$

for all sufficiently large positive integers d."

2. We write $$h = f_d(0), \qquad \eta = f_d(1).$$
Since $$0 < a \leqslant \sqrt{(\tfrac{1}{3}d)} \leqslant \sqrt{d}$$
for reduced forms, we have obviously

$$hd^{-\frac{1}{2}(1-s)} \leqslant f_d(1-s) \leqslant h,$$

and $$\eta \leqslant f_d(s) \leqslant \eta d^{\frac{1}{2}(1-s)}.$$

Therefore $$h' d^{-(1-s)} \leqslant \frac{f_d(1-s)}{f_d(s)} \leqslant h',$$

where $$h' = h/\eta.$$

Hence the two functions

$$X_d(s) = \frac{(1-s)\zeta(s) L_d(s)}{B_d(s) f_d(s)}, \quad Y_d(s) = \frac{(1-s)\zeta(s) L_d(s)}{B_d(s) f_d(s) d^{1-s}},$$

which in J have the same sign as $\zeta(s)L_d(s)$ and hence the opposite sign to $L_d(s)$, satisfy the inequalities

$$X_d(s) = \frac{A_d(s)}{B_d(s)}(1-s) - \frac{f_d(1-s)}{f_d(s)} d^{\frac{1}{2}-s} \leqslant \frac{c_2}{c_6}(1-s) - \frac{h'}{\sqrt{d}} \tag{1}$$

and

$$Y_d(s) = \frac{A_d(s)}{B_d(s)}(1-s) d^{-(1-s)} - \frac{f_d(1-s)}{f_d(s)} d^{-\frac{1}{2}} \geqslant \frac{c_1}{c_7}(1-s) d^{-(1-s)} - \frac{h'}{\sqrt{d}}. \tag{2}$$

When s lies in the interval J and is sufficiently near to $s=1$, then obviously both functions are negative.

3. It is clear from (1) that $-X_d(s)$, and therefore also $L_d(s)$, is always positive in the interval

$$\max\left(\sigma_1,\ 1 - \frac{c_6}{c_2}\frac{h'}{\sqrt{d}}\right) < s \leqslant 1.$$

We obtain as a special case

THEOREM 1. *Suppose that γ is a positive constant and that the integer $d > 0$ is greater than a certain number d_0 which depends only on γ. Then,* *if the L-series $L_d(s)$ has at least one zero in the interval*

$$1-\frac{\gamma}{\log d}\leqslant s\leqslant 1,$$

there exists a number $\Gamma>0$, depending only on γ, such that

$$h'\leqslant \Gamma\frac{\sqrt{d}}{\log d}.$$

Next, in the inequality (2), the term

$$\frac{c_1}{c_7}(1-s)\,d^{-(1-s)}$$

has its maximum at $s=1-(\log d)^{-1}$, and is then equal to

$$\frac{c_1}{c_7}\frac{e^{-1}}{\log d};$$

in the interval

$$1-\frac{1}{\log d}\leqslant s\leqslant 1$$

it assumes every value between this maximum and zero. Hence, when

$$h'\leqslant G\frac{\sqrt{d}}{\log d},$$

where G is a positive number with

$$G<\frac{c_1}{c_7}\frac{e^{-1}}{\log d},$$

there exists a second positive number g with $g\leqslant 1$, such that $Y_d(s)$, and so also $L_d(s)$, changes its sign at least once in the interval

$$1-\frac{g}{\log d}\leqslant s\leqslant 1.$$

This result proves

THEOREM 2. *Suppose that g is a positive constant and that the integer $d>0$ is greater than a certain number d_1 which depends only on g. Then, if the L-series $L_d(s)$ has no zero in the interval*

$$1-\frac{g}{\log d}\leqslant s\leqslant 1,$$

there exists a number $G>0$, depending only on g, such that

$$h'\geqslant G\frac{\sqrt{d}}{\log d}.$$

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