NOTE ON HYPOTHESIS K OF HARDY AND LITTLEWOOD

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1. Some weeks ago, Prof. Mordell asked me in a letter whether the Diophantine equation

(1) $$x^3+y^3+z^3=1$$

has an infinity of integer solutions other than the trivial ones such as $x=1$, $y=-z$. I found the following answer to this question.

As is well known†,

$$x^3+y^3+z^3=u^3$$

identically, if

$$x=\rho^2-\sigma\rho', \quad y=\sigma'\rho'-\rho^2, \quad z=\rho'^2-\rho\sigma', \quad u=\rho'^2-\rho\sigma,$$

where

$$\rho=f^2+3g^2, \quad \rho'=f'^2+3g'^2, \quad \sigma=ff'+3gg'+3fg'-3f'g,$$
$$\sigma'=ff'+3gg'-3fg'+3f'g.$$

Here, obviously, $u=1$, if $\rho'=1$ and $\sigma=0$; and this requires $f'=1$, $g'=0$, $f-3g=0$, whence $\rho=12g^2$, $\sigma'=6g$. Then

$$x=12^2g^4, \quad y=6g-12^2g^4, \quad z=1-6.12g^3, \quad u=1,$$

or, with $2g=\xi$,

(2) $$(9\xi^4)^3+(3\xi-9\xi^4)^3+(1-9\xi^3)^3=1.$$

Thus, for every integer ξ, we find a non-trivial solution of (1).

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† L. E. Dickson, *History of the theory of numbers*, 2, 555.

More generally,

(3) $$(9d^3\,\xi^4)^3+(3d\xi-9d^3\,\xi^4)^3+d(1-9d^2\,\xi^3)^3=d,$$

and therefore both the equations

$$x^3+y^3+dz^3=d, \quad d^2(x^3+y^3)+z^3=1$$

have an infinity of integer solutions. Since

(4) $$(6d^2\,\xi^3+1)^3+(-6d^2\,\xi^3+1)^3+d(-6d\xi^2)^3=2$$

identically in ξ, there are also infinitely many integers x, y, z for which

$$x^3+y^3+dz^3=2,$$

or $$d^2(x^3+y^3)+z^3=d^2.$$

This last identity is a special case of

(5) $$\lambda_1(\xi^n+a_1)^n+\lambda_2(\xi^n+a_2)^n+\ldots+\lambda_{n-1}(\xi^n+a_{n-1})^n+\lambda_n(\xi^2)^n=\mu.$$

Here $n\geqslant 3$, $a_1, a_2, \ldots, a_{n-1}$ are $n-1$ different integers with $a_1+\ldots+a_{n-1}=0$, and

$$\lambda_\nu=(-\lambda)^{\nu-\lambda}\prod_{\substack{\lambda=1\\ \lambda\neq\nu}}^{n-1}\prod_{\substack{\kappa=1\\ \kappa\neq\nu}}^{\lambda-1}(a_\kappa-a_\lambda) \quad (\nu=1,\,2,\,\ldots,\,n-1);$$

$$\lambda_n=-\binom{n}{2}\prod_{\lambda=1}^{n-1}\prod_{\kappa=1}^{\lambda-1}(a_\kappa-a_\lambda); \quad \mu=\sum_{\nu=1}^{n-1}a_\nu{}^n\,\lambda_\nu.$$

Hence, when $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu$ have these values, there are an infinity of integers $x_1, x_2, \ldots, x_n$ for which

$$\lambda_1 x_1{}^n+\lambda_2 x_2{}^n+\ldots+\lambda_n x_n{}^n=\mu.$$

2. About ten years ago, in a paper on Waring's problem, Hardy and Littlewood* gave consequences of the following

HYPOTHESIS K. *If $n\geqslant 2$ is an integer, then the number of solutions of*

$$x_1{}^n+x_2{}^n+\ldots+x_n{}^n=N,$$

in non-negative integers $x_1, x_2, \ldots, x_n$, is $O(N^\epsilon)$ for every positive ϵ and large N.

The hypothesis is known to be true for $n=2$, but the number of solutions is not bounded, and indeed is sometimes larger than

$$\exp\left(c\,\frac{\log N}{\log\log N}\right),$$

* *Math. Zeitschrift*, 23 (1925), 1–37.

where $c > 0$ is an appropriate constant. Recent theorems of Chowla* and Erdös† show that these last results are also true for larger values of n; but the truth or falsity of the hypothesis itself, for $n \geqslant 3$, has remained undecided.

I can now prove that Hypothesis K is false for $n = 3$. If we replace ξ by ξ/η in (2), it becomes

$$(2') \qquad (g\xi^4)^3 + (3\xi\eta^3 - 9\xi^4)^3 + (\eta^4 - 9\xi^3\eta)^3 = \eta^{12},$$

and here all three cubes will be positive for

$$\eta > 0, \quad 0 < \xi < 9^{-\frac{1}{3}}\eta.$$

Hence, for all large N which are 12th powers, the equation

$$x^3 + y^3 + z^3 = N \quad (N = \eta^{12})$$

has at least
$$9^{-\frac{1}{3}} N^{\frac{1}{12}}$$

solutions in non-negative integers.

An analogous result holds for the more general equations

$$x^3 + y^3 + dz^3 = N, \quad d^2(x^3 + y^3) + z^3 = N \quad (d = 1, 2, 3, \ldots),$$

as follows in the same way from (3). The identities (4) and (5) are of smaller value, since the terms of their left-hand sides are not all of the same sign. They only lead to the following result. Suppose that $n \geqslant 3$. Then there are integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive constants $A_1, A_2, \ldots, A_n, C$ such that

$$\lambda_1 x_1^n + \lambda_2 x_2^n + \ldots + \lambda_n x_n^n = N,$$

$$|x_1^n| < A_1 N, \quad |x_2^n| < A_2 N, \quad \ldots, \quad |x_n^n| < A_n N$$

for an infinity of N and more than

$$CN^{n-2}$$

sets of integers $x_1, x_2, \ldots, x_n$. This result suggests that Hypothesis K is probably false generally for $n \geqslant 3$.

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* *Indian Physico-Mathematical Journal*, 6 (1935), 65–68.

† *Journal London Math. Soc.*, 11 (1936), 133–136.

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