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On the solutions of algebraic differential equations

BY

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Let

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots \qquad (1)$$

be a convergent or divergent power series with coefficients in a finite algebraic field K, which formally satisfies an algebraic differential equation; i.e. there is a polynomial $F(z, y_0, y_1, \dots, y_m) \not\equiv 0$ in K, such that identically in z [1])

$$F(z, f(z), f'(z), \dots, f^{(m)}(z)) = 0. \qquad (2)$$

In his Groningen Thesis [2]), J. POPKEN proved the following

Theorem 1: There is a positive number c independent of n, such that for all sufficiently large indices, either

$$\alpha_n = 0, \quad or \quad |\alpha_n| \geq exp\left(-c\,n\,(log\ n)^2\right).$$

The proof then given was rather complicated. In this note, I give a simpler proof, which depends on the following results of G. PÒLYA [3]):

Theorem 2: There is an infinite sequence $a_0, a_1, a_2, \dots$ of positive integers, such that all numbers $a_n \alpha_n$ $(n = 0, 1, 2, \dots)$ are algebraic integers, and such that

$$\frac{log\ a_n}{n\,(log\ n)^2} = O(1).$$

Theorem 3: There is a positive number c_1, which does not depend on n, such that for all sufficiently large indices n,

$$|\alpha_n| \leq n!^{c_1}.$$

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1) It suffices to suppose that the TAYLOR coefficients α_n are algebraic and that $f(z)$ satisfies an equation (2). For then, without loss of generality, the coefficients of the polynomial F may be assumed to be algebraic, and therefore the α's can be expressed as rational functions with rational coefficients in a finite number of the α's and in the coefficients of F.

2) Amsterdam 1935, N.V. Noord-Hollandsche Uitgeversmaatschappij, Satz 12.

3) C. R. **201** (1935), p. 444, first two theorems. I need these theorems only in the special case of rational coefficients α_n.

Proof of Theorem 1: Without loss of generality, the coefficients of the polynomial F may be assumed to be rational numbers [4]). Let the field K be of degree N, and θ be a generating number of this field; hence

$$a_n = \sum_{h=0}^{N-1} A_{hn}\, \theta^h \qquad (n = 0, 1, 2, \ldots), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the A_{hn} are rational numbers. Put

$$f_h(z) = \sum_{n=0}^{\infty} A_{hn}\, z^n \qquad (h = 0, 1, \ldots, N-1) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and for arbitrary t

$$f(z \mid t) = \sum_{h=0}^{N-1} t^h f_h(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that

$$f(z) = f(z \mid \theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in F, we get

$$F\left(z, f(z \mid t), \frac{\partial f(z \mid t)}{\partial z}, \ldots, \frac{\partial^m f(z \mid t)}{\partial z^m}\right) = \sum_{n=0}^{\infty} P_n(t)\, z^n, \quad . \quad . \quad (7)$$

where the $P_n(t)$ are polynomials in t with rational coefficients.

Suppose now that $\theta_0, \theta_1, \ldots, \theta_{N-1}$ are the N different conjugates of θ in the field of all complex numbers. Since by (2)

$$P_n(\theta) = 0 \qquad (n = 0, 1, 2, \ldots),$$

obviously also for $h = 0, 1, \ldots, N-1$

$$P_n(\theta_h) = 0 \qquad (n = 0, 1, 2, \ldots).$$

Therefore by (7), the N power series

$$f(z \mid \theta_h) \qquad (h = 0, 1, \ldots, N-1) \quad . \quad . \quad (8)$$

all satisfy algebraic differential equations, viz. the same equation (2).

Now it is easily shown that if $g_1(z), \ldots, g_r(z)$ are power series which satisfy algebraic differential equations, and if $\lambda_1, \ldots, \lambda_r$ are constants, then the series $\lambda_1 g_1(z) + \ldots + \lambda_r g_r(z)$ also satisfies a certain algebraic differential equation [5]). Therefore the N functions (4) must be solutions

[4]) If necessary, multiply F by its conjugate polynomials with respect to K.

[5]) Put $\lambda_1 g_1(z) + \ldots + \lambda_r g_r(z) = g(z)$ and suppose that $F_1 = 0, \ldots, F_r = 0$ are the differential equations for $g_1(z), \ldots, g_r(z)$. By differentiating these equations a sufficient number of times and by considering $g(z)$ as known, we obtain so many equations for the functions $g_1(z), \ldots, g_r(z)$ and their differential coefficients, that we can eliminate them.

of algebraic differential equations; for by (5) they can be expressed linearly with constant coefficients by means of the functions (8).

Write the rational numbers A_{hn} as

$$A_{hn} = \frac{P_{hn}}{Q_{hn}} \qquad (h = 0, 1, \ldots, N-1;\, n = 0, 1, 2, \ldots), \quad (9)$$

where P_{hn} and $Q_{hn} \geqslant 1$ are relatively prime integers. By the Theorems 2 and 3, there is a positive constant c_2, such that for $h = 0, 1, \ldots, N-1$ and for all sufficiently large n

$$max\,(|P_{hn}|, |Q_{hn}|) \leqslant exp\,(c_2\, n\,(log\, n)^2). \quad (10)$$

Put

$$q_n = \prod_{h=0}^{N-1} Q_{hn},\; p_{hn} = A_{hn}\, q_n \qquad (h = 0, 1, \ldots, N-1;\, n = 0, 1, 2, \ldots), \quad (11)$$

such that all p_{hn} and $q_n \geqslant 1$ are integers, and that

$$A_{hn} = \frac{p_{hn}}{q_n} \qquad (h = 0, 1, \ldots, N-1;\, n = 0, 1, 2, \ldots). \quad (12)$$

Then by (10) there is a positive constant c_3, such that for sufficiently large n

$$max\,(q_n, |p_{0n}|, |p_{1n}|, \ldots, |p_{N-1\,n}|) \leqslant exp\,(c_3\, n\,(log\, n)^2). \quad (13)$$

Now by (3)

$$\alpha_n = \frac{1}{q_n} \sum_{h=0}^{N-1} p_{hn}\, \theta^h.$$

Hence Theorem 1 follows immediately from (13) and from the well known [6])

Theorem 4: To every real or complex algebraic number θ of degree N, there is a positive constant c_4, such that, if $A_0, A_1, \ldots, A_{N-1}$ are N integers which do not vanish simultaneously, then

$$\left| \sum_{h=0}^{N-1} A_h\, \theta^h \right| \geqslant c_4 \{max\,(|A_0|, |A_1|, \ldots, |A_{N-1}|)\}^{-(N-1)}.$$

Manchester, 8 November 1938.

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[6]) See J. F. KOKSMA, Diophantische Approximationen, Satz 6, p. 55.