ON THE MINIMUM OF POSITIVE DEFINITE HERMITIAN FORMS

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Let $K = K(\sqrt{-D})$ be an imaginary quadratic field, and let

$$f(x, y) = ax\bar{x} + bx\bar{y} + \bar{b}\bar{x}y + cy\bar{y} \tag{1}$$

be a positive definite Hermitian form of determinant

$$ac - b\bar{b} = +1; \tag{2}$$

as usual, $\bar{x}$ is the complex number conjugate to x.

The problem is to find the upper bound $M(D)$ of $m(f)$, where $m(f)$ is the minimum of $f(x, y)$ for all sets of integers x, y in K other than $(0, 0)$, and also the upper bound $M_1(D)$ of $m_1(f)$, where $m_1(f)$ is the minimum of $f(x, y)$ for integers x, y in K, which are relatively prime. Obviously

$$m(f) \leqslant m_1(f), \quad M(D) \leqslant M_1(D); \tag{3}$$

when the class number h of K is unity, then†

$$m(f) = m_1(f), \quad M(D) = M_1(D).$$

In some recent papers, A. Speiser‡, O. Perron§, A. Oppenheim||, and H. Oberseider¶ have determined $M(D)$ in a great number of cases, viz. Speiser for $D = 1$, and 3; Perron for $D = 1, 2, 3, 7, 11$, and 19 (these are all the $D \leqslant 20$ for which $h = 1$); Oberseider for $D = 43, 67, 163$ $(h = 1)$, 5

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† For if $h = 1$, then the minimum of $f(x, y)$ is attained for relatively prime numbers x and y.

‡ *Journal für Math.*, 167 (1932), 88–97.

§ *Math. Zeitschrift*, 36 (1932), 148–160.

|| *Math. Zeitschrift*, 38 (1934), 538–545.

¶ *Math. Zeitschrift*, 38 (1934), 591–632.

6, 10, 13, 15 ($h=2$), 14, 17 ($h=4$); and Oppenheim for all D for which

$$x^2 \equiv -2 \pmod{D}$$

has a solution. All four authors seem to have overlooked the classical papers of L. Bianchi* and G. Humbert† on the fundamental domain of the generalized Picard group $G(D)$ of all substitutions

$$x \to \alpha x + \beta y, \quad y \to \gamma x + \delta y \quad (\alpha\delta - \beta\gamma = 1), \tag{4}$$

where α, β, γ, δ are integers in K‡§. Since the reviews of these papers in the *Jahrbuch über die Fortschritte der Mathematik* do not mention their connection with the problem of $M(D)$, it may be of interest to give a short abstract of their results.

Using Picard's method||, we represent the form $f(x, y)$ by the point (ξ, η, ζ) with coordinates given by

$$\xi + \eta i = -\frac{\bar{b}}{a}, \quad \xi - \eta i = -\frac{b}{a}, \quad \zeta = \frac{1}{a} \tag{5}$$

in the upper half space H of the three dimensional Euclidian space; then, conversely, to the point (ξ, η, ζ) in H corresponds the form

$$f(x, y) = \frac{1}{\zeta}\{x\bar{x} - (\zeta - \eta i)x\bar{y} - (\xi + \eta i)\bar{x}y + (\xi^2 + \eta^2 + \zeta^2)y\bar{y}\}. \tag{6}$$

The substitutions (4) of $G(D)$ generate a group of transformations of H into itself, which is isomorphic to $G(D)$; these transformations are conformal, and they leave invariant the set of all spheres and planes perpendicular to the plane $\zeta = 0$.

Two forms $f(x, y)$ and $f'(x, y)$, which are changed one into the other by transformations (4), are considered as equivalent with respect to $G(D)$; this equivalence is extended to the points in H which represent these forms. In the set of all forms equivalent to a given form, one form $f(x, y)$ is chosen as the reduced form and similarly one point (ξ, η, ζ) is chosen as

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* *Math. Annalen*, 40 (1892), 332–412, and *Math. Annalen*, 42 (1893), 30–57; these papers are quoted below as B1 and B2.

† *Comptes Rendus, Paris*, 161 (1915), 189–196 and 227–234; quoted in the text as H1 and H2.

‡ More generally, $\alpha\delta - \beta\gamma$ may be taken as a unit in K, and we may also consider transformations like

$$x \to \alpha\bar{x} + \beta\bar{y}, \quad y \to \gamma\bar{x} + \delta\bar{y}.$$

The Picard group $G(D)$ is usually considered in the inhomogeneous form with x/y as variable.

§ For the original Picard group $G(1)$ see the account in the first volume of Klein-Fricke, *Automorphe Funktionen* (1897).

|| *Loc. cit.*, and B1, H1.

a reduced point from the set equivalent to a given point. We say that the form $f(x, y)$ is semi-reduced if the following conditions are satisfied:

$$(7) \qquad f(x, y) \geqslant \begin{cases} a & \text{for all relatively prime integers } x, y \text{ in } K, \\ c & \text{for all integers } x \text{ in } K, \text{ and } y = 1. \end{cases}$$

It is easily proved that to any given form there is at least one equivalent semi-reduced form (1). For, if $f'(x, y)$ is the given form, take two relatively prime integers α and γ in K, such that

$$f'(\alpha, \gamma) = a$$

is the minimum $m_1(f')$; further determine two other integers β and δ, such that $\alpha\delta - \beta\gamma = 1$, and such that $f'(\beta, \delta) = c$ is as small as possible. Then*

$$f(x, y) = f'(\alpha x + \beta y, \gamma x + \delta y)$$

is semi-reduced and equivalent to $f'(x, y)$.

To the set of all semi-reduced forms corresponds a certain domain Π in H. This domain need not be the fundamental domain of $G(D)$, since it allows a finite number of transformations of this group into itself. We can, however, find a part of it, say the domain P, which is the fundamental domain; P is obviously bounded by spheres or planes perpendicular to $\zeta = 0$. In their papers, Bianchi in a number of special cases, and Humbert in general, prove that P can be chosen in such a way that it is bounded by only a finite number of spheres and planes. More exactly, P is defined by a finite number of inequalities

$$\lambda_j x + \mu_j y + \nu_j \geqslant 0$$

and a finite number of inequalities

$$(\zeta - \lambda_j)^2 + (\eta - \mu_j)^2 + \zeta^2 \geqslant \nu_j^2.$$

Except for a finite number of "singular vertices", P does not meet the plane $\zeta = 0$; there are exactly h families of singular vertices which are non-equivalent with respect to $G(D)$, if the point at infinity is also counted as a singular vertex of P.

Thus, if $h = 1$, then P lies entirely above the plane $\zeta = 0$; there is only a finite number of points in P, say the points

$$(\xi_j, \eta_j, Z) \quad (j = 1, 2, \ldots, r),$$

* In the simplest case $D = 1$, this method of reduction was given by Ch. Hermite, *Œuvres*, 1, 251–253.

for which the ordinate Z is a minimum. By (5), for every form $f(x, y)$,

$$a \leqslant \frac{1}{Z},$$

and therefore $$m(f) = m_1(f) \leqslant \frac{1}{Z};$$

and so we find that

$$M(D) = M_1(D) = \frac{1}{Z}, \tag{8}$$

if $h = 1$, since in particular, $m_1(f_j) = m(f_j) = Z$ for the special forms

$$\frac{1}{Z}\{x\bar{x} - (\xi_j - \eta_j i)\, x\bar{y} - (\xi_j + \eta_j i)\, \bar{x}y + (\xi_j^2 + \eta_j^2 + Z^2)\, y\bar{y}\}.$$

For example, if $D = 19$, then P is defined by the seven inequalities*

$$\xi \geqslant 0, \quad \xi \leqslant \tfrac{1}{2}, \quad \eta \geqslant 0, \quad \eta \leqslant \frac{\sqrt{19}}{2}, \quad \xi^2 + \eta^2 + \zeta^2 \geqslant 1,$$

$$(\xi - \tfrac{1}{2})^2 + \left(\eta - \frac{\sqrt{19}}{2}\right)^2 + \zeta^2 \geqslant 1, \quad (\xi - \tfrac{1}{4})^2 + \left(\eta - \frac{\sqrt{19}}{4}\right)^2 \geqslant \tfrac{1}{4}.$$

Hence, as is easily verified,

$$\zeta \geqslant Z = \sqrt{\tfrac{2}{19}},$$

and therefore $$M(19) = M_1(19) = \sqrt{\tfrac{19}{2}},$$

this maximum being attained for

$$\sqrt{\tfrac{19}{2}}\left\{x\bar{x} + \frac{6i}{\sqrt{(19)}}\, x\bar{y} - \frac{6i}{\sqrt{(19)}}\, \bar{x}y + 2y\bar{y}\right\}$$

and $$\sqrt{\tfrac{19}{2}}\left\{x\bar{x} - \frac{\sqrt{(19)} - 7i}{2\sqrt{(19)}}\, x\bar{y} - \frac{\sqrt{(19)} + 7i}{2\sqrt{(19)}}\, \bar{x}y + 4y\bar{y}\right\},$$

corresponding to the points

$$\left(0, \ \frac{6}{\sqrt{(19)}}, \ \sqrt{\tfrac{2}{19}}\right) \quad \text{and} \quad \left(\tfrac{1}{2}, \ \frac{7}{2\sqrt{(19)}}, \ \sqrt{\tfrac{2}{19}}\right)$$

of minimum ordinate. Similarly for the other values of $D < 20$ with $h = 1$, Bianchi finds that†

$$Z = \sqrt{\tfrac{1}{2}}, \quad \tfrac{1}{2}, \quad \sqrt{\tfrac{2}{3}}, \quad \sqrt{\tfrac{3}{7}}, \quad \sqrt{\tfrac{2}{11}},$$

for $$D = 1, \quad 2, \quad 3, \quad 7, \quad 11.$$

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* B1, 381–382.

† B1, 361–65, 369–72.

Hence $\qquad M(D) = M_1(D) = \sqrt{2}, \quad 2, \quad \sqrt{\tfrac{3}{2}}, \quad \sqrt{\tfrac{7}{3}}, \quad \sqrt{\tfrac{11}{2}},$

for $\qquad\qquad D = 1, \quad 2, \quad 3, \quad 7, \quad 11.$

These are precisely the results of Speiser and Perron; $M(1) = \sqrt{2}$ was previously proved by Hermite*.

If $h \geqslant 2$, then P has singular vertices in $\zeta = 0$, and

$$(9) \qquad M_1(D) = +\infty.$$

It is then easy to construct reduced forms $f(x, y)$, for which $a = m_1(f)$ is arbitrarily large. Now, in general,

$$m(f) < m_1(f).$$

Humbert† shows, however, how to form the greatest partial domain Q of P, such that points in Q and only these points correspond to forms for which

$$m(f) = m_1(f).$$

He gives the following example of a form for which this equation does not hold. For $D = 5$,

$$f(x, y) = 5x\bar{x} - 2(1-\sqrt{-5})\,x\bar{y} - 2(1+\sqrt{-5})\,\bar{x}y + 5y\bar{y}$$

is reduced, but its true minimum is

$$f(1+\sqrt{-5},\ 2) = 2 < 5,$$

where $\qquad x = 1+\sqrt{-5}, \quad y = 2$

are not relatively prime‡.

From formula (9), only the maximum $M(D)$ is of interest when $h \geqslant 2$. The simple method based on the value of Z fails in this case. By using additional results of Bianchi, however, we can determine $M(D)$ in special cases, as I prove here for $D = 5$.

For this value of D, P is defined by the inequalities§

$$0 \leqslant \zeta \leqslant \tfrac{1}{2}, \quad 0 \leqslant \eta \leqslant \frac{\sqrt{5}}{2}, \quad \xi^2+\eta^2+\zeta^2 \geqslant 1,$$

$$\xi^2 + \left(\eta - \frac{\sqrt{5}}{2}\right)^2 + \zeta^2 \geqslant \tfrac{1}{4}, \quad (\xi - \tfrac{1}{2})^2 + \left(\eta - \frac{2}{\sqrt{5}}\right)^2 + \zeta^2 \geqslant \tfrac{1}{20}.$$

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* *Loc. cit.* Footnote on page 139.

† H2, 232–234.

‡ Compare also Mahler, *Math. Zeitschrift*, 38 (1934), 541, 605f., and the end of this paper.

§ B1, 366–368. More exactly, the domain given in the text is only half the fundamental domain, the other half being obtained by reflection in the plane $\xi = 0$, *i.e.* by the substitution

$$x \to \bar{x}, \quad y \to \bar{y}.$$

It is sufficient to consider this partial domain.

Let P' be the partial domain of all those points of P for which

$$(10) \qquad \zeta \leqslant \sqrt{\tfrac{11}{80}}.$$

Bianchi has proved* that P remains invariant under the inversion with respect to the sphere

$$(11) \qquad (\xi-\tfrac{1}{2})^2+\left(\eta-\frac{\sqrt{5}}{2}\right)^2+\zeta^2=\tfrac{1}{2};$$

in the variables x, y, this corresponds to the substitution

$$(12) \qquad x \to \frac{1+\sqrt{-5}}{\sqrt{2}}\,\bar{x}-\sqrt{2}\,\bar{y}, \quad y \to \sqrt{2}\,\bar{x}+\frac{-1+\sqrt{-5}}{\sqrt{2}}\,\bar{y},$$

of determinant -1. By the same inversion, P' changes into the domain P'' given by

$$(13) \qquad \begin{cases} 0 \leqslant \xi \leqslant \tfrac{1}{2}, \quad 0 \leqslant \eta \leqslant \dfrac{\sqrt{5}}{2}, \quad \xi^2+\eta^2+\zeta^2 \geqslant 1, \\ (\xi-\tfrac{1}{2})^2+(\eta-\tfrac{1}{2}\sqrt{5})^2+(\zeta-\sqrt{\tfrac{5}{11}})^2 \geqslant \tfrac{5}{10}. \end{cases}$$

It is easily verified that, for all points in P'',

$$(14) \qquad \zeta \geqslant \sqrt{\tfrac{11}{20}}.$$

In particular, the points

$$\left(\tfrac{1}{2},\ \frac{7}{4\sqrt{5}}\ \sqrt{\tfrac{11}{80}}\right) \text{ in } P' \quad \text{and} \quad \left(\tfrac{1}{2},\ \frac{1}{\sqrt{5}},\ \sqrt{\tfrac{11}{20}}\right) \text{ in } P''$$

correspond to each other in the inversion; hence (14) cannot be improved.

We can now prove that

$$(15) \qquad M(5)=\sqrt{\tfrac{80}{11}}.$$

For let $f(x, y)$ be a reduced form and let (ξ, η, ζ) be the corresponding point in P. If this point does not lie in P', then

$$m_1(f)=\frac{1}{\zeta} \leqslant \sqrt{\tfrac{80}{11}},$$

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* B1, 367–68, and B2.

and the assertion is proved. If, however, this point belongs to P', then the representative point of the new form

$$f'(x) = f\left(\frac{1+\sqrt{-5}}{\sqrt{2}}\,\bar{x} - \sqrt{2}\,\bar{y},\quad \sqrt{2}\,\bar{x} + \frac{-1+\sqrt{-5}}{\sqrt{2}}\,\bar{y}\right)$$

belongs to P''; hence

$$m_1(f') \leqslant \sqrt{\tfrac{20}{11}}$$

and therefore

$$f\left(\frac{1+\sqrt{-5}}{\sqrt{2}},\ \sqrt{2}\right) \leqslant \sqrt{\tfrac{20}{11}},\quad \textit{i.e.}\quad f(1+\sqrt{-5},\ 2) \leqslant 2\sqrt{\tfrac{20}{11}} = \sqrt{\tfrac{80}{11}},$$

so that the theorem is true also in this case. That the relation (15) is the best possible one follows immediately from the special form

$$f(x,\ y) = \sqrt{\tfrac{80}{11}}\left\{x\bar{x} - \left(\tfrac{1}{2} - \frac{7i}{4\sqrt{5}}\right)x\bar{y} - \left(\tfrac{1}{2} + \frac{7i}{4\sqrt{5}}\right)\bar{x}y + y\bar{y}\right\},$$

which belongs to the point

$$\left(\tfrac{1}{2},\ \frac{7}{4\sqrt{5}},\ \sqrt{\tfrac{11}{80}}\right)$$

mentioned before*.

A similar method can be applied when $D = 6$, and probably in all cases in which the class number is a power of 2. Possibly the class field of K will have to be used in the general case.

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* Though this proof is not given explicitly in Bianchi's paper, the idea of using the reflection in the sphere (11) was applied by Bianchi himself to show that the class number of positive definite Hermitian forms with integer coefficients in K of given determinant is finite.

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