ON THE PRODUCT OF TWO COMPLEX LINEAR POLYNOMIALS IN TWO VARIABLES

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Some time ago, E. Hlawka (*Monatshefte für Mathematik und Physik*, 46, 324–334) proved the following

THEOREM. *Let α, β, γ, δ be four complex numbers such that*

$$\alpha\delta - \beta\gamma = 1,$$

and ξ, η two other complex numbers. Then there are two integers x and y in the Gaussian field K(i) such that

$$(a) \qquad \left|(\alpha x + \beta y + \xi)(\gamma x + \delta y + \eta)\right| \leqslant \tfrac{1}{2}.$$

The sign of equality is necessary if, and only if, the product can be written as

$$\left(ax + by + g + \frac{1+i}{2}\right)\left(cx + dy + h + \frac{1+i}{2}\right),$$

where a, b, c, d, g, h are integers in K(i) such that ad − bc = 1. If α/β is not an element of K(i), then for every ε > 0 there exists a solution of (a) for which

$$|\alpha x + \beta y + \xi| < \epsilon.$$

Hlawka's method depends on Ford's theorem on the approximation of complex numbers by elements of $K(i)$, and on a lemma on quadratic polynomials, of which the proof is somewhat complicated.

In this paper‡ I give another proof, which is based on the theory of Hermitian forms and on a simple geometrical idea. By means of the same method I prove two analogous theorems in the quadratic fields $K(\sqrt{-2})$ and $K(\sqrt{-3})$; here the constants on the right-hand side of (a) are $\tfrac{3}{4}$ and $\tfrac{1}{3}$ instead of $\tfrac{1}{2}$. The method can also be used to prove the theorems of Minkowski and Remak on the product of two or three real linear polynomials, and can probably be applied to other problems as well.

1. *Reduction of Hermitian forms.*

Let D be one of the numbers 1, 2, or 3, and let $K = K(\sqrt{-D})$ be the imaginary quadratic field generated by $\sqrt{-D}$. The ring $J = J(\sqrt{-D})$ of all integers in K has the basis

$$1, \omega = \sqrt{-D} \qquad \text{for} \quad D = 1 \text{ or } 2,$$
$$1, \omega = \tfrac{1}{2}(1 + \sqrt{-D}) \quad \text{for} \quad D = 3.$$

If a is complex, then let $\bar{a}$, as usual, denote its conjugate complex; if a lies in K, then $\bar{a}$ is also the conjugate of a with respect to this quadratic field.

By Γ we denote the group of all linear transformations

$$(1) \qquad \begin{cases} x = \alpha x' + \beta y', & \bar{x} = \bar{\alpha}\bar{x}' + \bar{\beta}\bar{y}', \\ y = \gamma x' + \delta y', & \bar{y} = \bar{\gamma}\bar{x}' + \bar{\delta}\bar{y}', \end{cases}$$

where α, β, γ, δ are elements of J for which

$$\alpha\delta - \beta\gamma = 1.$$

Let
$$f(x, y) = ax\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + cy\bar{y}$$

be a positive definite Hermitian form of determinant

$$ac - b\bar{b} = 1,$$

with arbitrary real coefficients a and c, and arbitrary complex coefficient b. By the transformation (1), $f(x, y)$ changes into a new form

$$F(x', y') = Ax'\bar{x}' + B\bar{x}'y' + \bar{B}x'\bar{y}' + Cy'\bar{y}'$$

of determinant 1, which is called equivalent to $f(x, y)$.

We say that $f(x, y)$ is a reduced form, if

$$(2) \qquad f(\xi, \eta) \geqslant \begin{cases} a & \text{for} \quad |\xi| + |\eta| > 0, \\ c & \text{for} \quad \eta = 1, \end{cases}$$

when ξ and η lie in J. Since $f(x, y)$ can be written in the form

$$(3) \qquad f(x, y) = a\left(x + \frac{b}{a}y\right)\left(\bar{x} + \frac{\bar{b}}{a}\bar{y}\right) + \frac{1}{a}y\bar{y},$$

these conditions imply

$$0 < a \leqslant c, \quad \left|\xi + \frac{b}{a}\right| \geqslant \left|\frac{b}{a}\right| \quad \text{for all } \xi \text{ in } J.$$

Hence it is easily verified that the following inequalities are necessary for reduced forms†:

$$(4) \quad \begin{cases} 0 < a \leqslant c, & \left| R\left(\dfrac{b}{a}\right) \right| \leqslant \tfrac{1}{2}, & \left| I\left(\dfrac{b}{a}\right) \right| \leqslant \tfrac{1}{2} & \text{for} \quad D = 1, \\[2mm] 0 < a \leqslant c, & \left| R\left(\dfrac{b}{a}\right) \right| \leqslant \tfrac{1}{2}, & \left| I\left(\dfrac{b}{a}\right) \right| \leqslant \dfrac{1}{\sqrt{2}} & \text{for} \quad D = 2, \\[2mm] 0 < a \leqslant c, & |\rho^k b + \rho^{-k} \bar{b}| \leqslant a & (\rho = e^{\frac{2}{3}\pi i}, \ k = 1, 2, 3) & \text{for} \quad D = 3. \end{cases}$$

Since $a^2 \leqslant ac = b\bar{b} + 1$, it follows that

$$(5) \qquad a^2 \leqslant ac \leqslant \begin{cases} 2 & \text{for} \quad D = 1, \\ 4 & \text{for} \quad D = 2, \\ \tfrac{3}{2} & \text{for} \quad D = 3. \end{cases}$$

(The three fields K, where $D = 1, 2,$ or 3, have the class number 1, and in them the Euclidean algorithm holds. This is no longer always the case for $D > 3$, e.g. not for $D = 5$. For this reason, the method of this paper would require modification if analogues to Hlawka's theorem in higher quadratic fields are to be obtained.)

2. The geometrical representation of $f(x, y)$.

Using Picard's method, we represent $f(x, y)$ by a point P with coordinates

$$(6) \qquad X = R\left(\frac{b}{a}\right), \qquad Y = I\left(\frac{b}{a}\right), \qquad Z = \frac{1}{a} > 0,$$

which lies in the *upper half-space* $H : Z > 0$. The group Γ generates an isomorphic group Γ^* of point transformations of H into itself. These transformations are conformal, and they change spheres into spheres‡. Points in H which are transformed into each other by elements of Γ^* belong to equivalent forms. For reduced forms, the point P lies in the

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† These conditions are also sufficient. For further literature on the reduction of Hermitian forms see my note " On the minimum of positive definite Hermitian forms ", *Journal London Math. Soc.*, 14 (1939), 137–143. As usual, $R(z)$ and $I(z)$ are the real and imaginary part of a complex number z.

‡ Planes are considered as spheres of infinite radius.

fundamental domain Π of Γ^*, which is defined by the inequalities

$$(7)\quad X^2+Y^2+Z^2\geqslant 1, \text{ and } \begin{cases} |X|\leqslant\tfrac{1}{2},\ \ |Y|\leqslant\tfrac{1}{2} & \text{for}\ \ D=1, \\[2mm] |X|\leqslant\tfrac{1}{2},\ \ |Y|\leqslant 1/\sqrt{2} & \text{for}\ \ D=2, \\[2mm] |X|\leqslant\tfrac{1}{2},\ \ |X+\sqrt{3}\,Y|\leqslant 1,\ \ |X-\sqrt{3}\,Y|\leqslant 1 \\ & \text{for}\ \ D=3. \end{cases}$$

There are only a finite number of elements of Γ^* which transform Π into itself, namely

$$X'=\epsilon\,X,\quad Y'=\epsilon\,Y\,(\epsilon=\mp 1),\quad Z'=Z\quad\text{for}\quad D=1,$$

$$X'+iY'=\rho^k(X+iY)\,(\rho=e^{\frac{1}{3}\pi i},\,k=1,\,2,\,3),\quad Z'=Z\quad\text{for}\quad D=3,$$

and only the identical transformation for $D=2$. The half-space H is filled completely without overlapping by the set S of all *different* domains Π_ν into which Π is transformed by the elements of Γ^*.

This set S has the following properties: The surface of each of its elements Π_ν is formed by 5 (for $D=1$ or 2) or 7 (for $D=3$) spheres with their centres in the plane $Z=0$. The set of all surfaces of the elements of S contains the following surface Σ, extending to infinity, as a part:

$$(X-h)^2+(Y-k)^2+Z^2=1\quad\text{for}\quad |X-h|\leqslant\tfrac{1}{2},\ \ |Y-k|\leqslant\tfrac{1}{2}$$
$$(h,\,k=0,\,\mp 1,\,\mp 2,\,\ldots)\quad\text{for}\quad D=1;$$

$$(X-h)^2+(Y-k\,\sqrt{2})^2+Z^2=1\quad\text{for}\quad |X-h|\leqslant\tfrac{1}{2},\ \ |Y-k\,\sqrt{2}|\leqslant 1/\sqrt{2}$$
$$(h,\,k=0,\,\mp 1,\,\mp 2,\,\ldots)\quad\text{for}\quad D=2;$$

$$(X-h-\tfrac{1}{2}k)^2+(Y-\tfrac{1}{2}k\,\sqrt{3})^2+Z^2=1\quad\text{for}$$
$$|X-h-\tfrac{1}{2}k|\leqslant\tfrac{1}{2},\ \ |X\mp Y\,\sqrt{3}-h-2k|\leqslant 1$$
$$(h,\,k=0,\,\mp 1,\,\mp 2,\,\ldots)\quad\text{for}\quad D=3.$$

Obviously, Σ is simply connected, and a continuous curve from a point in $Z=0$ to a point in Π intersects Σ in at least one point. If this curve begins in a point $(X,\,Y,\,0)$ of $Z=0$ for which $X+iY$ is not an element of the field K, and if it is perpendicular to the plane in this point, then it passes through an infinity of elements of S.

3. On a special Hermitian form.

Theorem 1. *Let* $\alpha,\,\beta,\,\gamma,\,\delta$ *be complex numbers of determinant* $\alpha\delta-\beta\gamma=1$, t *a positive parameter,* $f_t(x,\,y)$ *the Hermitian form*

$$f_t(x,\,y)=t\,|\,\alpha x+\beta y\,|^2+\frac{1}{t}\,|\,\gamma x+\delta y\,|^2,$$

and

$$F_t(x, y) = A_t x\bar{x} + B_t \bar{x}y + \bar{B}_t x\bar{y} + C_t y\bar{y}$$

the reduced form which is equivalent to $f_t(x, y)$. *Then*

$$A_t = C_t$$

for at least one *value of* t; *if* β/a *does not belong to* K, *then there are* arbitrarily large t *with this property*.

Proof. Since

$$tf_t(x, y) = (a\bar{a}t^2 + \gamma\bar{\gamma})x\bar{x} + (\bar{a}\beta t^2 + \bar{\gamma}\delta)\bar{x}y + (a\bar{\beta}t^2 + \gamma\bar{\delta})x\bar{y} + (\beta\bar{\beta}t^2 + \delta\bar{\delta})y\bar{y},$$

the point P_t corresponding to $f_t(x, y)$ is given by

$$X + iY = \frac{t^2 \bar{a}\beta + \bar{\gamma}\delta}{t^2 a\bar{a} + \gamma\bar{\gamma}}, \quad Z = \frac{t}{t^2 a\bar{a} + \gamma\bar{\gamma}}.$$

If t assumes all values in the interval $0 < t < \infty$, then P_t describes the semi-circle C which is perpendicular to the plane $Z = 0$ at the two points†

$$X + iY = \beta/a \quad \text{and} \quad X + iY = \delta/\gamma;$$

for $t \to \infty$, P_t tends to the point in $Z = 0$ for which $X + iY = \beta/a$. Thus, in particular, if β/a does not belong to K, then C passes through an infinity of different polyhedra Π_ν of S.

Let Π_0 be any one of the polyhedra which has a point in common with C, and let P^* be the transformation in Γ^* which changes Π_0 into Π. It also transforms C into another semi-circle C' perpendicular to $Z = 0$, but with at least one point in Π. Hence, as noted in §2, C' intersects the surface Σ in at least one point, say the point $t = t_0$. From the form of Σ, there must be a number in J, say $\zeta = \xi + i\eta$, such that the translation P^{**} of Γ^*,

$$X \to X + \xi, \quad Y \to Y + \eta, \quad Z \to Z,$$

transforms the point $t = t_0$ on C' into a point (X, Y, Z) of Π for which

$$X^2 + Y^2 + Z^2 = 1.$$

Let P be the element of Γ corresponding to the product $P^{**}P^*$; for

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† The plane $Z = 0$ is identified with the complex plane. The equations of C are

$$\begin{vmatrix} X+iY & \bar{a}\beta & \bar{\gamma}\delta \\ X-iY & a\bar{\beta} & \gamma\bar{\delta} \\ 1 & a\bar{a} & \gamma\bar{\gamma} \end{vmatrix} = 0, \qquad \begin{vmatrix} X+iY & \bar{a}\beta & \bar{\gamma}\delta \\ X-iY & a\bar{\beta} & \gamma\bar{\delta} \\ X^2+Y^2+Z^2 & \beta\bar{\beta} & \delta\bar{\delta} \end{vmatrix} = 0.$$

$t = t_0$, P changes $f_t(x, y)$ into $F_t(x, y)$, and by our construction

$$C_{t_0} = A_{t_0}(X^2 + Y^2 + Z^2) = A_{t_0},$$

as follows from (6).

In the special case in which β/α does not belong to K, there are obviously an infinity of different values $t = t_0 \to \infty$ with the required property, since C passes through an infinity of different polyhedra Π_ν of S in the neighbourhood of $t = \infty$ (*i.e.* of $X + iY = \beta/\alpha$, $Z = 0$), and the arc of the transformed semi-circle C' which lies above the surface Σ enters only a finite number of elements of S.

4. *A geometrical extremum problem.* ($D = 1$ and $D = 3$.)

Represent the elements of J as points in the complex z-plane. Then they form a lattice L generated by

squares of side 1   for $D = 1$,

rectangles of sides 1 and $\sqrt{2}$ for $D = 2$,

equilateral triangles of side 1   for $D = 3$.

Let α be a number in the interval

$$0 \leqslant \alpha \leqslant \begin{cases} \sqrt{\tfrac{1}{2}} & \text{for } D = 1, \\ \sqrt{\tfrac{3}{4}} & \text{for } D = 2, \\ \sqrt{\tfrac{1}{3}} & \text{for } D = 3, \end{cases}$$

and denote by $Q(\alpha)$

a square of side α for $D = 1$,

a rectangle of sides α and $\alpha\sqrt{2}$ for $D = 2$,

an equilateral triangle of side α for $D = 3$,

in arbitrary position in the z-plane. Let δ_Q be the shortest distance between the vertices of $Q(\alpha)$ and the points of the lattice L. This minimum distance δ_Q is a bounded continuous function of the position of $Q(\alpha)$, and therefore has a maximum value $\delta(\alpha)$, which evidently is a *continuous* function of α. In order to determine $\delta(\alpha)$, we distinguish the three values of D.

(a) $D = 1$. Place the square $Q(\alpha)$ so that its centre coincides with that of one of the squares q of L and so that its sides are parallel to the

diagonals of q; *e.g.*, if q is the square with vertices $0, 1, 1+i, i$, then the vertices of $Q(a)$ are at

$$\frac{1+i}{2} - \frac{a}{\sqrt{2}}, \quad \frac{1+i}{2} - \frac{ai}{\sqrt{2}}, \quad \frac{1+i}{2} + \frac{a}{\sqrt{2}}, \quad \frac{1+i}{2} + \frac{ai}{\sqrt{2}},$$

and therefore

$$\delta_Q = \sqrt{\left\{\left(\tfrac{1}{2} - \frac{a}{\sqrt{2}}\right)^2 + (\tfrac{1}{2})^2\right\}} = \sqrt{\left(\tfrac{1}{2} - \frac{a}{\sqrt{2}} + \tfrac{1}{2}a^2\right)}.$$

This is also the value of $\delta(a)$:

$$(8) \qquad\qquad \delta(a) = \sqrt{\left(\tfrac{1}{2} - \frac{a}{\sqrt{2}} + \tfrac{1}{2}a^2\right)}.$$

For draw circles of radius δ_Q about all points of L as their centres, and exclude the interiors of all these circles from the z-plane. The remainder R of the plane consists of an infinite set of curvilinear squares κ. If $a < \sqrt{\tfrac{1}{2}}$, then the diagonal of $Q(a)$ is less than 1 and therefore less than the diameter of each excluded circle; thus the centre of $Q(a)$ cannot lie in one of these circles. It follows that all vertices of $Q(a)$ can only belong to the set R if they coincide with those of one of the squares κ. This proves (8) for $a < \sqrt{\tfrac{1}{2}}$, and therefore, by continuity, also in the limiting case $a = \sqrt{\tfrac{1}{2}}$.

(b) $D = 3$. Place the triangle $Q(a)$ so that its centre falls into that of one of the triangles q of L and so that its sides are parallel to those of q, but lie on different sides of its centre; *e.g.* if q is the triangle with vertices $0, 1, \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}$, then the vertices of $Q(a)$ are at

$$\frac{a+1}{2} + \frac{(a+1)\sqrt{-3}}{6}, \quad \frac{1-a}{2} + \frac{(a+1)\sqrt{-3}}{6}, \quad \tfrac{1}{2} + \frac{(1-2a)\sqrt{-3}}{6},$$

so that

$$\delta_Q = \sqrt{\left\{(\tfrac{1}{2})^2 + \left(\frac{(1-2a)\sqrt{3}}{6}\right)^2\right\}} = \sqrt{\left(\frac{1-a+a^2}{3}\right)}.$$

Hence

$$(9) \qquad\qquad \delta(a) = \sqrt{\{\tfrac{1}{3}(1-a+a^2)\}},$$

as follows by the same consideration as in the last case.

5. *A geometrical extremum problem.* $(D = 2.)$

(c) $D = 2$. The result becomes more complicated since now q and $Q(a)$ are less symmetrical. We start from the remark that if $Q(a)$ lies so that $\delta_Q = \delta(a)$, then the distances $\delta_1, \delta_2, \delta_3, \delta_4$ of its four vertices from their

nearest lattice points are all equal. For if $\delta_1 \leqslant \delta_2 \leqslant \delta_3 \leqslant \delta_4$, $\delta_1 < \delta_4$, then it is evidently possible to move $Q(a)$ a little so that δ_1 increases and the inequalities $\delta_1 \leqslant \delta_2 \leqslant \delta_3 \leqslant \delta_4$ remain satisfied. It suffices therefore to determine all those positions of $Q(a)$ in which $\delta_1 = \delta_2 = \delta_3 = \delta_4$. These are given in the following table:

$$\textit{Vertices of } Q(a). \qquad\qquad \textit{Value of } \sigma = \delta_Q{}^2.$$

(1) $\qquad \mp \dfrac{a}{2} \mp \dfrac{a\sqrt{-2}}{2}, \qquad\qquad \sigma_1 = \dfrac{3a^2}{4}.$

(2) $\qquad \tfrac{1}{2} \mp \dfrac{a}{2} \mp \dfrac{a\sqrt{-2}}{2}, \qquad\qquad \sigma_2 = \tfrac{1}{4} - \dfrac{a}{2} + \dfrac{3a^2}{4}.$

(3) $\qquad \tfrac{1}{2} \mp \dfrac{a\sqrt{2}}{2} \mp \dfrac{ai}{2}, \qquad\qquad \sigma_3 = \tfrac{1}{4} - \dfrac{a}{\sqrt{2}} + \dfrac{3a^2}{4}.$

(4) $\qquad \dfrac{\sqrt{-2}}{2} \mp \dfrac{a}{2} \mp \dfrac{a\sqrt{-2}}{2}, \qquad\qquad \sigma_4 = \tfrac{1}{2} - a + \dfrac{3a^2}{4}.$

(5) $\qquad \dfrac{\sqrt{-2}}{2} \mp \dfrac{a\sqrt{2}}{2} \mp \dfrac{ai}{2}, \qquad\qquad \sigma_5 = \begin{cases} \tfrac{1}{2} - \dfrac{a}{\sqrt{2}} + \dfrac{3a^2}{4} \\ \quad \text{for } 0 \leqslant a \leqslant 1/\sqrt{2}. \\[4pt] \tfrac{3}{2} - \dfrac{3a}{\sqrt{2}} + \dfrac{3a^2}{4} \\ \quad \text{for } a \geqslant 1/\sqrt{2}, \end{cases}$

(6) $\qquad \dfrac{1+\sqrt{-2}}{2} \mp \dfrac{a}{2} \mp \dfrac{a\sqrt{-2}}{2}, \qquad\qquad \sigma_6 = \tfrac{3}{4} - \tfrac{3}{2}a + \dfrac{3a^2}{4}.$

(7) $\qquad \dfrac{1+\sqrt{-2}}{2} \mp \dfrac{a\sqrt{2}}{2} \mp \dfrac{ai}{2}, \qquad\qquad \sigma_7 = \tfrac{3}{4} - a\sqrt{2} + \dfrac{3a^2}{4}.$

(8) $\qquad \dfrac{1+\sqrt{-2}}{2} + \dfrac{1+\sqrt{-2}}{\sqrt{3}}\left(\mp \dfrac{a\sqrt{2}}{2} \mp \dfrac{ai}{2}\right), \qquad \sigma_8 = \tfrac{3}{4} - \sqrt{(\tfrac{3}{2})}\,a + \dfrac{3a^2}{4}.$

A simple calculation shows that

$$\max \sigma_k = \begin{cases} \sigma_8 & \text{for } 0 \leqslant a \leqslant a_0, \\ \sigma_5 & \text{for } a_0 \leqslant a \leqslant 1/\sqrt{2}, \\ \sigma_1 & \text{for } 1/\sqrt{2} \leqslant a \leqslant \sqrt{\tfrac{3}{4}}, \end{cases}$$

where $\qquad a_0 = \dfrac{\sqrt{24}+\sqrt{8}}{16} = 0\cdot483\ldots.$

Hence the maximum $\delta(a)$ is given by

$$(10) \qquad \delta(a) = \begin{cases} \sqrt{\{\frac{3}{4} - \sqrt{(\frac{3}{2})}\,a + \frac{3}{4}a^2\}} & \text{for} \quad 0 \leqslant a \leqslant a_0, \\ \sqrt{\{\frac{1}{2} - \sqrt{(\frac{1}{2})}\,a + \frac{3}{4}a^2\}} & \text{for} \quad a_0 \leqslant a \leqslant \sqrt{\frac{1}{2}}, \\ \sqrt{(\frac{3}{4})}\,a & \text{for} \quad \sqrt{\frac{1}{2}} \leqslant a \leqslant \sqrt{\frac{3}{4}}. \end{cases}$$

From these expressions we get

$$(11) \qquad \delta(a)^2 \leqslant \tfrac{3}{4} - \tfrac{3}{4}\sqrt{2}\,a + \tfrac{3}{4}a^2 \quad \text{for} \quad 0 \leqslant a \leqslant \sqrt{\tfrac{1}{2}},$$

a result which could also have been proved directly in a similar way to the formulae (8) and (9).

6. *The principal lemma for $D = 1$.*

From now onwards, we use the notation

$$z_1 \equiv z_2$$

if z_1 and z_2 are two complex numbers which are *congruent mod J, i.e.* whose difference $z_1 - z_2$ lies in the ring J.

Theorem 2. *Suppose that $D = 1$, and that*

$$f(x, y) = ax\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + ay\bar{y}$$

is a reduced positive definite Hermitian form of determinant 1. *Then, corresponding to any given complex numbers x_0 and y_0, there exist two other numbers x_1 and y_1, such that*

$$(12) \qquad x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad f(x_1, y_1) \leqslant 1.$$

The sign of equality is necessary if, and only if,

$$x_0 \equiv y_0 \equiv \tfrac{1}{2}(1+i), \quad a = 1, \quad f(x, y) = x\bar{x} + y\bar{y}.$$

Proof. Since x_0 and y_0 may be replaced by congruent numbers, we may assume without loss of generality that

$$|R(y_0)| \leqslant \tfrac{1}{2}, \quad |I(y_0)| \leqslant \tfrac{1}{2},$$

$$(13) \qquad \left| R\left(x_0 + \frac{b}{a}\,y_0\right) \right| \leqslant \tfrac{1}{2}, \quad \left| I\left(x_0 + \frac{b}{a}\,y_0\right) \right| \leqslant \tfrac{1}{2},$$

and therefore

$$\left(x_0 + \frac{b}{a}\,y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\,\bar{y}_0\right) \leqslant \tfrac{1}{2}, \quad y_0\bar{y}_0 \leqslant \tfrac{1}{2}.$$

Hence, if the stronger inequality†

(14) $y_0 \bar{y}_0 < \frac{1}{2}(2a-a^2)$

holds, then

$$f(x_0, y_0) = a\left(x_0 + \frac{b}{a} y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a} \bar{y}_0\right) + \frac{1}{a} y_0 \bar{y}_0 < a \cdot \frac{1}{2} + \frac{1}{a}\frac{2a-a^2}{2} = 1,$$

so that the conditions (12) are satisfied for $x_1 = x_0$, $y_1 = y_0$ with the sign "$<$" instead of "$\leqslant$".

Suppose now that (14) is not true, *i.e.* that y_0 lies in the domain G:

(15) $|R(y)| \leqslant \frac{1}{2}$, $|I(y)| \leqslant \frac{1}{2}$, $|y| \geqslant \beta$,   where $\beta = \sqrt{\{\frac{1}{2}(2a-a^2)\}}$.

Now, from (5), $1 \leqslant a \leqslant \sqrt{2}$ and therefore $\beta \geqslant \sqrt{(\sqrt{2}-1)} > \sqrt{\frac{1}{4}} = \frac{1}{2}$. Hence G consists of four separate curvilinear triangles; denote by

$$\Delta(\epsilon_1, \epsilon_2) \qquad\qquad (\epsilon_1, \epsilon_2 = \pm 1)$$

that triangle for which

$$\epsilon_1 R(y) > 0, \quad \epsilon_2 I(y) > 0.$$

By the translations of the y-plane

$$y \to y - \epsilon_1 \eta_1 - \epsilon_2 \eta_2 i \qquad\qquad (\eta_1, \eta_2 = 0 \text{ or } 1)$$

we obtain the set of 16 triangles

$$\Delta(\epsilon_1, \epsilon_2 | \eta_1, \eta_2) = \Delta(\epsilon_1, \epsilon_2) - \epsilon_1 \eta_1 - \epsilon_2 \eta_2 i$$

$$(\epsilon_1, \epsilon_2 = \pm 1; \ \eta_1, \eta_2 = 0 \text{ or } 1),$$

which together form a new domain G' consisting of four curvilinear squares with their centres at $\frac{1}{2}(\pm 1 \pm i)$ and their vertices at

$$\tfrac{1}{2}\epsilon_1 + \epsilon_2 \gamma i, \quad \epsilon_1 \gamma + \tfrac{1}{2}\epsilon_2 i, \quad \tfrac{1}{2}\epsilon_1 + \epsilon_2(1-\gamma)i, \quad \epsilon_1(1-\gamma) + \tfrac{1}{2}\epsilon_2 i.$$

Here γ is defined by

$$\gamma^2 + (\tfrac{1}{2})^2 = \beta^2, \quad i.e. \quad \gamma^2 = \tfrac{1}{4}\{1 - 2(a-1)^2\}.$$

Obviously, therefore, for all points of G',

(16) $y\bar{y} \leqslant (\tfrac{1}{2})^2 + (1-\gamma)^2 = \tfrac{1}{2}\{1-(a-1)^2\} + 1 - \sqrt{\{1-2(a-1)^2\}}.$

† For $2a - a^2 \leqslant 1$ if $1 \leqslant a \leqslant 2$.

Suppose now that y_0 lies in the triangle $\Delta(\epsilon_1{}^0, \epsilon_2{}^0)$; then the four points

$$y_0, \quad y_0 - \epsilon_1{}^0, \quad y_0 - \epsilon_2{}^0 i, \quad y_0 - \epsilon_1{}^0 - \epsilon_2{}^0 i$$

all belong to G'.   For fixed $X \equiv x_0$, the four points in the z-plane

$$Z_{\eta_1 \eta_2} = X + \frac{b}{a}\, (y_0 - \epsilon_1{}^0 \eta_1 - \epsilon_2{}^0 \eta_2 i) \qquad (\eta_1, \eta_2 = 0 \text{ or } 1)$$

form the vertices of a square $Q(\alpha)$ of side

$$\alpha = \left| \frac{b}{a} \right| = \sqrt{\left(1 - \frac{1}{a^2}\right)} \leqslant \frac{1}{\sqrt 2}.$$

Their minimum distance δ_Q from the nearest lattice point therefore satisfies by (8) the inequality

$$\delta_Q{}^2 \leqslant \delta(\alpha)^2 = \tfrac{1}{2} - \frac{\alpha}{\sqrt 2} + \frac{\alpha^2}{2} = 1 - \frac{1}{2a^2} - \sqrt{\left(\frac{a^2 - 1}{2a^2}\right)}.$$

Hence we have proved that, if $y = y_0$ is a point in G, then there exist a number $y = y_1 \equiv y_0$ in G' [which therefore satisfies (16)] *and a number $x_1 \equiv x_0$, for which*

$$(17) \qquad \left(x_1 + \frac{b}{a}\, y_1\right)\left(\bar x_1 + \frac{\bar b}{a}\, \bar y_1\right) = \delta_Q{}^2 \leqslant 1 - \frac{1}{2a^2} - \sqrt{\left(\frac{a^2 - 1}{2a^2}\right)}.$$

From the identity (3) and from (16) and (17), we get

$$f(x_1, y_1) \leqslant a\left(x_1 + \frac{b}{a}\, y_1\right)\left(\bar x_1 + \frac{\bar b}{a}\, \bar y_1\right) + \frac{1}{a}\, y_1 \bar y_1$$

$$\leqslant a\left\{1 - \frac{1}{2a^2} - \sqrt{\left(\frac{a^2 - 1}{2a^2}\right)}\right\}$$

$$+ \frac{1}{a}\left[\tfrac{1}{2}\{1 - (a-1)^2\} + 1 - \sqrt{\{1 - 2(a-1)^2\}}\right],$$

i.e.

$$f(x_1, y_1) \leqslant 1 + A, \quad \text{where} \quad A = \frac{a^2 + 1}{2a} - \sqrt{\left(\frac{a^2 - 1}{2}\right)} - \frac{1}{a}\, \sqrt{\{1 - 2(a-1)^2\}}.$$

In this inequality, A is not positive.   For put $a = 1 + t$, so that $0 \leqslant t \leqslant \sqrt 2 - 1 < \tfrac{1}{2}$.   Then A becomes

$$A = \frac{2 + 2t + t^2 - 2\sqrt{(1 - 2t^2)}}{2 + 2t} - \sqrt{\left(t + \frac{t^2}{2}\right)}.$$

Using the inequalities for t, we easily get

$$\sqrt{(1-2t^2)} \geqslant 1-\tfrac{3}{2}t^2, \quad \text{since} \quad (1-\tfrac{3}{2}t^2)^2 = 1-2t^2-t^2(1-\tfrac{9}{4}t^2) \leqslant 1-2t^2,$$

$$\frac{2+2t+t^2-2\sqrt{(1-2t^2)}}{2+2t} \leqslant \frac{t+2t^2}{1+t} \leqslant t\,\frac{1+2\cdot\tfrac{1}{2}}{1+1\cdot\tfrac{1}{2}} = \tfrac{4}{3}t = \sqrt{\left(\frac{64t^2}{36}\right)},$$

$$\sqrt{\left(t+\frac{t^2}{2}\right)} \geqslant \sqrt{\left(2t^2+\frac{t^2}{2}\right)} = \sqrt{\left(\frac{90t^2}{36}\right)}.$$

Hence finally $\qquad A \leqslant \sqrt{\left(\dfrac{64t^2}{36}\right)} - \sqrt{\left(\dfrac{90t^2}{36}\right)} \leqslant 0.$

This proof shows that (12) can always be satisfied with the sign " $<$ " instead of " $\leqslant$ ", except for $a=1$, *i.e.* for the form

$$f(x,y) = x\bar{x}+y\bar{y}.$$

In this case, if only the sign " $=$ " is to be true, then necessarily

$$x_1\bar{x}_1 \geqslant \tfrac{1}{2} \quad \text{for} \quad x_1 \equiv x_0 \quad \text{and} \quad y_1\bar{y}_1 \geqslant \tfrac{1}{2} \quad \text{for} \quad y_1 \equiv y_0,$$

which requires that $\qquad x_0 \equiv \tfrac{1}{2}(1+i), \quad y_0 \equiv \tfrac{1}{2}(1+i).$

7. *The principal lemma for $D = 3$.*

Theorem 3. *Suppose that $D = 3$, and that*

$$f(x,y) = ax\bar{x}+b\bar{x}y+\bar{b}x\bar{y}+ay\bar{y}$$

is a reduced positive definite Hermitian form of determinant 1. Then corresponding to any given complex numbers x_0 and y_0, there exist two other numbers x_1 and y_1, such that

(18) $$x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad f(x_1,y_1) \leqslant \tfrac{2}{3}.$$

The sign of equality is necessary if, and only if,

$$x_0 \equiv \mp y_0 \equiv \mp(\tfrac{1}{2}+\tfrac{1}{6}\sqrt{-3}), \quad a = 1, \quad f(x,y) = x\bar{x}+y\bar{y}.$$

Proof. Since x_0 and y_0 may be replaced by congruent numbers, we may assume without loss of generality that

$$|\rho^k y_0 + \rho^{-k}\bar{y}_0| \leqslant 1$$

(19) $$\left|\rho^k\left(x_0+\frac{b}{a}y_0\right)+\rho^{-k}\left(\bar{x}_0+\frac{\bar{b}}{a}\bar{y}_0\right)\right| \leqslant 1 \qquad (\rho = e^{\frac{1}{3}\pi i},\; k = 0,\,1,\,2),$$

and therefore

$$\left(x_0 + \frac{b}{a}\, y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\, \bar{y}_0\right) \leqslant \tfrac{1}{3}, \qquad y_0\bar{y}_0 \leqslant \tfrac{1}{3}.$$

Hence, if the stronger inequality†

(20) $$y_0\bar{y}_0 < \tfrac{1}{3}(2a - a^2)$$

holds, then

$$f(x_0,\, y_0) = a\left(x_0 + \frac{b}{a}\, y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\, \bar{y}_0\right) + \frac{1}{a}\, y_0\bar{y}_0 < a \cdot \tfrac{1}{3} + \frac{1}{a}\, \frac{2a - a^2}{3} = \tfrac{2}{3},$$

so that (18) is satisfied with the sign " $<$ " instead of " $\leqslant$ ".

Suppose therefore that (20) is not true, *i.e.* that y_0 lies in the domain G :

(21) $|\rho^k y + \rho^{-k}\bar{y}| \leqslant 1$ $(k = 0,\, 1,\, 2)$, $|y| \geqslant \beta$,   where $\beta^2 = \tfrac{1}{3}(2a - a^2)$.

From (5),

$$1 \leqslant a \leqslant \sqrt{\tfrac{3}{2}}, \quad \text{and therefore} \quad \beta \geqslant \sqrt{\{\tfrac{1}{2}(\sqrt{6} - \tfrac{3}{2})\}} = \sqrt{\left(\sqrt{(\tfrac{2}{3})} - \tfrac{1}{2}\right)} > \tfrac{1}{2}.$$

Hence G consists of six separate curvilinear triangles, each of which corresponds to and contains one of the six vertices p_h of the regular hexagon H given by

$$|\rho^k y + \rho^{-k}\bar{y}| \leqslant 1 \qquad\qquad (k = 0,\, 1,\, 2).$$

If the indices are chosen suitably, then p_1, p_3, p_5, and also p_2, p_4, p_6, form the vertices of two regular triangles of side 1 ; the complex numbers

$$\eta_h = p_{h+2} - p_h, \quad \eta_h{'} = p_{h+4} - p_h \qquad\qquad (p_{h+6} = p_h)$$

are units in and are a basis for the ring J.

Let Δ_h denote that one of the six triangles of G which lies at the vertex p_h. Then, by the translations of the y-plane

$$y \to y, \quad y \to y + \eta_h, \quad y \to y + \eta_h{'},$$

a system of 18 triangles

$$\Delta_{h0} = \Delta_h, \quad \Delta_{h1} = \Delta_h + \eta_h, \quad \Delta_{h2} = \Delta_h + \eta_h{'} \quad (h = 1,\, 2,\, ...,\, 6)$$

† See foot-note †, p. 222.

is obtained. Together, they form a domain G' consisting of six separate concave curvilinear triangles d_h with centres at the different points p_h. By the rotations

$$y \to e^{\frac{1}{3}k\pi i}y \qquad\qquad (k = 0, 1, ..., 5)$$

of the y-plane, the d_h are permuted cyclically. In particular, one of the triangles, say d_1, has its centre at the point $p_1 = \sqrt{-\frac{1}{3}}$, and its vertices are easily found to be the points

$$\{\sqrt{\tfrac{3}{4}} - \sqrt{(\beta^2 - \tfrac{1}{4})}\}i,$$

$$\tfrac{1}{2}\{-\sqrt{\tfrac{3}{4}} + \sqrt{(\beta^2 - \tfrac{1}{4})} + 1\} + \tfrac{1}{2}\sqrt{-3}\{\sqrt{\tfrac{3}{4}} - \sqrt{(\beta^2 - \tfrac{1}{4})} + 1\},$$

$$-\tfrac{1}{2}\{-\sqrt{\tfrac{3}{4}} + \sqrt{(\beta^2 - \tfrac{1}{4})} + 1\} + \tfrac{1}{2}\sqrt{-3}\{\sqrt{\tfrac{3}{4}} - \sqrt{(\beta^2 - \tfrac{1}{4})} + 1\}.$$

Obviously, therefore, for all points of G',

$$(22) \qquad y\bar{y} \leqslant \{\sqrt{\tfrac{3}{4}} - \sqrt{(\beta^2 - \tfrac{1}{4})}\}^2 = \tfrac{1}{2} + \tfrac{1}{3}(2a - a^2) - \sqrt{(2a - a^2 - \tfrac{3}{4})}.$$

⌊Suppose now that y_0 lies in Δ_h, and that therefore the points

$$y_0, \quad y_0 + \eta_h, \quad y_0 + \eta_h'$$

belong to G'. For fixed $X \equiv x_0$, the three points in the z-plane

$$Z_1 = X + \frac{b}{a}y_0, \quad Z_2 = X + \frac{b}{a}(y_0 + \eta_h), \quad Z_3 = X + \frac{b}{a}(y_0 + \eta_h')$$

form the vertices of an equilateral triangle $Q(a)$ of side

$$a = \left| \frac{b}{a} \right| = \sqrt{\left(1 - \frac{1}{a^2}\right)} \leqslant \sqrt{\tfrac{1}{3}}.$$

Therefore, by (9), their minimum distance δ_Q from the nearest lattice point satisfies the inequality

$$\delta_Q{}^2 \leqslant \delta(a)^2 = \frac{1 - a + a^2}{3} = \tfrac{1}{3}\left\{2 - \frac{1}{a^2} - \sqrt{\left(1 - \frac{1}{a^2}\right)}\right\}.$$

Hence we have proved that, if $y = y_0$ is a point in G, then there exist a number $y = y_1 \equiv y_0$ in G' [which therefore satisfies (22)] and a number $x_1 \equiv x_0$, for which

$$(23) \qquad \left(x_1 + \frac{b}{a}y_1\right)\left(\bar{x}_1 + \frac{\bar{b}}{a}\bar{y}_1\right) = \delta_Q{}^2 \leqslant \tfrac{2}{3} - \frac{1}{3a^2} - \tfrac{1}{3}\sqrt{\left(1 - \frac{1}{a^2}\right)}.$$

From (3), (22), and (23), we get

$$f(x_1, y_1) = a \left(x_1 + \frac{b}{a} y_1\right) \left(\bar{x}_1 + \frac{\bar{b}}{a} \bar{y}_1\right) + \frac{1}{a} y_1 \bar{y}_1$$

$$= a \left\{\tfrac{2}{3} - \frac{1}{3a^2} - \tfrac{1}{3} \sqrt{\left(1 - \frac{1}{a^2}\right)}\right\} + \frac{1}{a}\left(\tfrac{1}{2} + \frac{2a - a^2}{3} - \sqrt{(2a - a^2 - \tfrac{3}{4})}\right),$$

i.e.

$$f(x_1, y_1) \leqslant \tfrac{2}{3} + A, \quad \text{where} \quad A = \frac{a}{3} + \frac{1}{6a} - \tfrac{1}{3}\sqrt{(a^2 - 1)} - \frac{1}{a}\sqrt{(2a - a^2 - \tfrac{3}{4})}.$$

In this inequality, A is not positive. For put

$$a = 1 + t, \quad \text{so that} \quad 0 \leqslant t \leqslant \sqrt{\tfrac{3}{2}} - 1 < \tfrac{1}{4}.$$

Then A becomes

$$A = \frac{2t^2 + 4t + 3 - 3\sqrt{(1 - 4t^2)}}{6(1 + t)} - \tfrac{1}{3}\sqrt{(2t + t^2)}.$$

Using the inequalities for t, we get

$$\sqrt{(1 - 4t^2)} \geqslant 1 - 3t^2, \quad \text{since} \quad (1 - 3t^2)^2 = (1 - 4t^2) - t^2(2 - 9t^2) \leqslant 1 - 4t^2,$$

$$\frac{2t^2 + 4t + 3 - 3\sqrt{(1 - 4t^2)}}{6(1 + t)} \leqslant \frac{4t + 11t^2}{6(1 + t)} \leqslant t \, \frac{4 + \frac{11}{4}}{6(1 + \frac{1}{4})} = \tfrac{9}{10} t,$$

$$\tfrac{1}{3}\sqrt{(2t + t^2)} \geqslant \tfrac{1}{3}\sqrt{(2 \cdot 4t^2 + t^2)} = t,$$

and therefore

$$A \leqslant \tfrac{9}{10} t - t = -\tfrac{1}{10} t \leqslant 0.$$

This proof shows that (18) can always be satisfied with the sign " < " instead of " ≤ ", except for $a = 1$, i.e. for the form

$$f(x, y) = x\bar{x} + y\bar{y}.$$

If in this case only the sign " = " is to hold, then necessarily

$$x_1 \bar{x}_1 \geqslant \tfrac{1}{3} \text{ for } x_1 \equiv x_0 \quad \text{and} \quad y_1 \bar{y}_1 \geqslant \tfrac{1}{3} \text{ for } y_1 \equiv y_0.$$

This requires that the points x_0 and y_0 lie in the centres of triangles of the lattice L, and therefore

$$x_0 \equiv \pm y_0 \equiv \pm(\tfrac{1}{2} + \tfrac{1}{6}\sqrt{-3}).$$

8. *The principal lemma for $D = 2$.*

Theorem 4.   *Suppose that $D = 2$, and that*

$$f(x, y) = ax\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + ay\bar{y}$$

is a reduced positive definite Hermitian form of determinant 1. Then corresponding to any given complex numbers x_0 and y_0, there exist two other numbers x_1 and y_1, such that

$$(24) \qquad x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad f(x_1, y_1) \leqslant \tfrac{3}{2}.$$

The sign of equality is necessary if, and only if,

$$x_0 \equiv y_0 \equiv \tfrac{1}{2}(1 + \sqrt{-2}), \quad a = 1, \quad f(x, y) = x\bar{x} + y\bar{y}.$$

Proof.   Since x_0 and y_0 may be replaced by congruent numbers, we may suppose without loss of generality that

$$(25) \qquad |R(y_0)| \leqslant \tfrac{1}{2}, \quad |I(y_0)| \leqslant 1/\sqrt{2},$$
$$\left| R\left(x_0 + \frac{b}{a}\, y_0\right) \right| \leqslant \tfrac{1}{2}, \quad \left| I\left(x_0 + \frac{b}{a}\, y_0\right) \right| \leqslant \frac{1}{\sqrt{2}},$$

and therefore

$$\left(x_0 + \frac{b}{a}\, y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\, \bar{y}_0\right) \leqslant \tfrac{3}{4}, \quad y_0\bar{y}_0 \leqslant \tfrac{3}{4}.$$

Hence, if either of the stronger inequalities†

$$(26) \qquad y_0\bar{y}_0 < \tfrac{3}{4}(2a - a^2)$$

or‡

$$(27) \qquad \left(x_0 + \frac{b}{a}\, y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\, \bar{y}_0\right) < \tfrac{3}{4}\, \frac{2a-1}{a^2}$$

is satisfied, then

$$f(x_0, y) = a\left(x_0 + \frac{b}{a}\, y_0\right)\left(\bar{x}_0 + \frac{\bar{b}}{a}\, \bar{y}_0\right) + \frac{1}{a}\, y_0\bar{y}_0 < \begin{cases} a\tfrac{3}{4} + \dfrac{1}{a}\,\tfrac{3}{4}(2a - a^2) = \tfrac{3}{2} \;\text{ or} \\[2mm] a\tfrac{3}{4}\,\dfrac{2a-1}{a^2} + \dfrac{1}{a}\,\tfrac{3}{4} = \tfrac{3}{2}, \end{cases}$$

and so (24) holds for $x_1 = x_0$, $y_1 = y_0$ with the sign "$<$" instead of "$\leqslant$".

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† See foot-note †, p. 222.
‡ For $(2a-1)/a^2 \leqslant 1$ if $1 \leqslant a \leqslant 2$.

Suppose therefore from now onwards that neither (26) nor (27) is true, *i.e.* that y_0 lies in the domain G:

(28) $|R(y)| \leqslant \frac{1}{2}, \quad |I(y)| \leqslant 1/\sqrt{2}, \quad |y| \geqslant \beta, \quad$ where $\quad \beta = \sqrt{\{\frac{3}{4}(2a - a^2)\}},$

and that $z_0 = x_0 + \dfrac{b}{a} y_0$ satisfies the inequalities

(29) $|R(z_0)| \leqslant \frac{1}{2}, \quad |I(z_0)| \leqslant 1/\sqrt{2}, \quad |z_0| \geqslant \gamma, \quad$ where $\quad \gamma^2 = \frac{3}{4}(2a - 1)/a^2.$

We distinguish two cases, according to the value of a.

(A) $1 \leqslant a \leqslant \frac{3}{2}$. In this case,

$$\beta \geqslant \sqrt{\left\{\frac{3}{4}\left(2 \cdot \frac{3}{4} - (\frac{3}{4})^2\right)\right\}} > \sqrt{\frac{1}{2}};$$

hence the domain G consists of four separate curvilinear triangles. For given y in G, denote by ϵ_1, ϵ_2 those units ± 1, for which

$$\epsilon_1 R(y) > 0, \quad \epsilon_2 I(y) > 0,$$

and let G' be the set of all points

$$y - \epsilon_1 \eta_1 - \epsilon_2 \eta_2 \sqrt{-2} \qquad (\eta_1, \eta_2 = 0 \text{ or } 1),$$

where y assumes all possible positions in G. As in §6, G' consists of four separate parts in the form of congruent curvilinear parallelograms; it is easily verified that for all points in G',

(30) $y\bar{y} \leqslant \frac{1}{4}(4 + 6a - 3a^2) - \sqrt{(6a - 3a^2 - 2)} \quad$ for $\quad 1 \leqslant a \leqslant \frac{3}{2}.$

Suppose that y_0 belongs to the signs $\epsilon_1^0, \epsilon_2^0$, so that

$$\epsilon_1^0 R(y_0) > 0, \quad \epsilon_2^0 I(y_0) > 0,$$

and all four points

$$y_0 - \epsilon_1^0 \eta_1 - \epsilon_2^0 \eta_2 \sqrt{-2} \qquad (\eta_1, \eta_2 = 0 \text{ or } 1)$$

belong to G'. For fixed $X \equiv x_0$, the four points in the z-plane

$$Z_{\eta_1 \eta_2} = X + \frac{b}{a}(y_0 - \epsilon_1^0 \eta_1 - \epsilon_2^0 \eta_2 \sqrt{-2}) \qquad (\eta_1, \eta_2 = 0 \text{ or } 1)$$

form the vertices of a rectangle $Q(a)$ of sides

$$a = \left|\frac{b}{a}\right| = \sqrt{\left(1 - \frac{1}{a^2}\right)} \leqslant \sqrt{\frac{3}{4}}, \quad \text{and} \quad a\sqrt{2}.$$

Therefore, by (10) and (11), their minimum distance from the nearest lattice point satisfies the inequality

$$\delta_Q^2 \leqslant \delta(a)^2 \leqslant \begin{cases} \frac{3}{4} - \frac{3}{4}\sqrt{2}\,a + \frac{3}{4}a^2 = \frac{3}{2} - \frac{3}{4a^2} - \frac{3}{4}\sqrt{\left(2 - \frac{2}{a^2}\right)} & \text{for } 1 \leqslant a \leqslant \sqrt{2}, \\[2ex] \frac{3}{4}a^2 \qquad\qquad\quad = \frac{3}{4}\left(1 - \frac{1}{a^2}\right) & \text{for } \sqrt{2} \leqslant a \leqslant \frac{3}{2}. \end{cases}$$

Hence we have proved that, if $y = y_0$ *is a point in* G, *then there exist a number* $y = y_1 \equiv y_0$ *in* G' [*which therefore satisfies* (30)], *and a number* $x_1 \equiv x_0$, *for which*

$$(31) \quad \left(x_1 + \frac{b}{a}\,y_1\right)\left(\bar{x}_1 + \frac{\bar{b}}{a}\,\bar{y}_1\right) \leqslant \delta_Q^2 \leqslant \begin{cases} \frac{3}{2} - \frac{3}{4a^2} - \frac{3}{4}\sqrt{\left(2 - \frac{2}{a^2}\right)} \\[1ex] \hfill \text{for } 1 \leqslant a \leqslant \sqrt{2}, \\[2ex] \frac{3}{4} - \frac{3}{4a^2} \hfill \text{for } \sqrt{2} \leqslant a \leqslant \frac{3}{2}. \end{cases}$$

From (3), (30), and (31), we get

$$f(x_1, y_1) \leqslant \tfrac{3}{2} + A,$$

where

$$A = \begin{cases} \dfrac{3a^2 + 1 - 4\sqrt{(6a - 3a^2 - 2)}}{4a} - \frac{3}{2}\sqrt{(2a^2 - 2)} & \text{for } 1 \leqslant a \leqslant \sqrt{2}, \\[3ex] \dfrac{1}{4a}\{1 - \sqrt{(6a - 3a^2 - 2)}\} & \text{for } \sqrt{2} \leqslant a \leqslant \frac{3}{2}. \end{cases}$$

From these expressions, it is nearly trivial that A is *negative for* $\sqrt{2} \leqslant a \leqslant \frac{3}{2}$; in the other interval $1 \leqslant a \leqslant \sqrt{2}$, A is *not positive.* For put $a = 1 + t$, so that

$$0 \leqslant t \leqslant \sqrt{2} - 1, \quad A = \frac{4 + 6t + 3t^2 - 4\sqrt{(1 - 3t^2)}}{4(1 + t)} - \frac{3}{4}\sqrt{\{2(2t + t^2)\}}.$$

Then, as in § 6 and § 7,

$$\sqrt{(1 - 3t^2)} > 1 - \tfrac{9}{5}t^2, \quad \text{since} \quad (1 - \tfrac{9}{5}t^2)^2 = (1 - 3t^2) - t^2(\tfrac{3}{5} - \tfrac{81}{25}t^2) \leqslant 1 - 3t^2,$$

$$\frac{4 + 6t + 3t^2 - 4\sqrt{(1 - 3t^2)}}{4(1 + t)} \leqslant \frac{30t + 51t^2}{20(1 + t)} \leqslant t\left(\frac{3}{2} + \frac{21(\sqrt{2} - 1)}{20\sqrt{2}}\right) \leqslant 2t,$$

$$2t + t^2 \geqslant t^2\left(\frac{2}{\sqrt{2} - 1} + 1\right) = (\sqrt{2} + 1)^2 t^2, \quad \tfrac{3}{4}\sqrt{\{2(2t + t^2)\}} \geqslant \tfrac{3}{4}\sqrt{2}(\sqrt{2} + 1)t \geqslant \tfrac{5}{2}t,$$

and therefore $\qquad\qquad A \leqslant 2t - \tfrac{5}{2}t = -\tfrac{1}{2}t \leqslant 0.$

(B) $\frac{3}{2} \leqslant a \leqslant 2$. In this case, the complex number b/a satisfies by (4) the inequalities

$$(32) \qquad \left| R\left(\frac{b}{a}\right) \right| \leqslant \tfrac{1}{2}, \quad \left| I\left(\frac{b}{a}\right) \right| \leqslant \frac{1}{\sqrt{2}}, \quad \left| \frac{b}{a} \right| = \sqrt{\left(1 - \frac{1}{a^2}\right)} \geqslant \sqrt{\tfrac{5}{9}} > \sqrt{\tfrac{1}{2}},$$

and in the inequalities (29) for z_0,

$$(33) \qquad \gamma = \sqrt{\left(\tfrac{3}{4}\,\frac{2a-1}{a^2}\right)} \geqslant \tfrac{3}{4} > \sqrt{\tfrac{1}{2}}.$$

It is clear from (32) that there exist two units $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, such that

$$(34) \qquad \left| \frac{b}{a} - \frac{\epsilon_1 + \epsilon_2\sqrt{-2}}{2} \right| \leqslant \tfrac{1}{2} - \sqrt{\{(\sqrt{\tfrac{5}{9}})^2 - (\sqrt{\tfrac{1}{2}})^2\}} = \tfrac{1}{2} - \sqrt{\tfrac{1}{18}},$$

and similarly from (29) and (33) that there are two units $\eta_1 = \pm 1$, $\eta_2 = \pm 1$, such that

$$(35) \qquad |z_0 - \tfrac{1}{2}(\eta_1 + \eta_2\sqrt{-2})| \leqslant \tfrac{1}{2} - \sqrt{\{(\tfrac{3}{4})^2 - (\sqrt{\tfrac{1}{2}})^2\}} = \tfrac{1}{4}.$$

Let $E = \pm 1$ be a unit for which

$$E\,R(y_0) \geqslant 0,$$

and put

$$y_1 = y_0 - E, \qquad x_1 = x_0 + \tfrac{1}{2}(\epsilon_1 + \epsilon_2\sqrt{-2})\,E - \tfrac{1}{2}(\eta_1 + \eta_2\sqrt{-2}).$$

Then obviously

$$|R(y_1)| \leqslant 1, \quad |I(y_1)| \leqslant 1/\sqrt{2}, \quad y_1 \equiv y_0,$$

and therefore

$$(36) \qquad y_1\bar{y}_1 \leqslant \tfrac{3}{2}.$$

It is clear that $\tfrac{1}{2}(\epsilon_1 E - \eta_1)$ and $\tfrac{1}{2}(\epsilon_2 E - \eta_2)$ are integers; hence

$$x_1 \equiv x_0.$$

Finally, from (34) and (35),

$$\left| x_1 + \frac{b}{a}\,y_1 \right| = \left| \left(z_0 - \frac{\eta_1 + \eta_2\sqrt{-2}}{2}\right) - E\left(\frac{b}{a} - \frac{\epsilon_1 + \epsilon_2\sqrt{-2}}{2}\right) \right|$$

$$\leqslant \tfrac{1}{4} + (\tfrac{1}{2} - \sqrt{\tfrac{1}{18}}) = \tfrac{3}{4} - \sqrt{\tfrac{1}{18}},$$

and therefore

$$\left(x_1 + \frac{b}{a}\,y_1\right)\left(\bar{x}_1 + \frac{\bar{b}}{a}\,\bar{y}_1\right) \leqslant (\tfrac{3}{4} - \sqrt{\tfrac{1}{18}})^2 < \tfrac{4}{15}.$$

Hence, from this inequality and from (36),

$$f(x_1, y_1) = a\left(x_1 + \frac{b}{a}\,y_1\right)\left(\bar{x}_1 + \frac{\bar{b}}{a}\,\bar{y}_1\right) + \frac{1}{a}\,y_1\bar{y}_1 \leqslant \frac{4a}{15} + \frac{3}{2a}$$

$$\leqslant \tfrac{4}{14}\cdot\tfrac{3}{2} + \tfrac{3}{2}\cdot(\tfrac{3}{2})^{-1} < \tfrac{3}{2}.$$

In both cases A and B, the proof has shown that (24) can always be satisfied with the sign " $<$ " instead of " $\leqslant$ ", except for $a = 1$, *i.e.* for the form

$$f(x, y) = x\bar{x} + y\bar{y}.$$

If in this case only the sign " $=$ " is to hold, then necessarily

$$x_1\bar{x}_1 \geqslant \tfrac{3}{4} \text{ for } x_1 \equiv x_0 \quad \text{and} \quad y_1\bar{y}_1 \geqslant \tfrac{3}{4} \text{ for } y_1 \equiv y_0,$$

which requires that $x_0 \equiv y_0 \equiv \tfrac{1}{2}(1 + \sqrt{-2})$.

9. *The product of two inhomogeneous polynomials.*

By means of the preceding lemmas, we can now prove:

Theorem 5. *Suppose that $D = 1$, 2, or 3, and that a, β, γ, δ are four complex numbers of determinant $a\delta - \beta\gamma = 1$. Then to any two complex numbers x_0, y_0, there are two other complex numbers x_1, y_1, such that*

$$\text{(A)} \quad x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad |(ax_1 + \beta y_1)(\gamma x_1 + \delta y_1)| \leqslant \begin{cases} \tfrac{1}{2} & \text{for} \quad D = 1, \\ \tfrac{3}{4} & \text{for} \quad D = 2, \\ \tfrac{1}{3} & \text{for} \quad D = 3. \end{cases}$$

Here the sign of equality is necessary if

$$\text{(B)} \qquad (ax + \beta y)(\gamma x + \delta y) = (ax + by)(cx + dy)$$

identically in x, y, where a, b, c, d are integers in $K(\sqrt{-D})$ of determinant $ad - bc = 1$, and if at the same time

$$\text{(B')} \quad ax_0 + by_0 \equiv \mp(cx_0 + dy_0) \equiv \begin{cases} \tfrac{1}{2}(1 + \sqrt{-D}) & \text{for} \quad D = 1 \text{ or } 2, \\ \mp(\tfrac{1}{2} + \tfrac{1}{6}\sqrt{-3}) & \text{for} \quad D = 3. \end{cases}$$

When a/β is not an element of $K(\sqrt{-D})$, then to every $\epsilon > 0$ there is a solution x_1, y_1 of (A) *such that*

(C) $$|ax_1 + \beta y_1| < \epsilon.$$

Proof. Consider the Hermitian form

$$f_t(x, y) = t\,|\,ax + \beta y\,|^2 + \frac{1}{t}\,|\,\gamma x + \delta y\,|^2$$

and the equivalent reduced form

$$F_t(x', y') = A_t x'\bar{x}' + B_t x'y' + \bar{B}_t x'\bar{y}' + C_t y'\bar{y}'.$$

By Theorem 1, there is a value $t = t_0$, for which

$$A_{t_0} = C_{t_0};$$

if a/β is not in K, then t_0 may be assumed greater than $3/2\epsilon^2$. Let

$$x' = ax + by, \quad y' = cx + dy \qquad (ad - bc = 1)$$

be the linear integral unimodular transformation in Γ which changes $f_{t_0}(x, y)$ into $F_{t_0}(x', y')$, and put

$$x_0' = ax_0 + by_0, \quad y_0' = cx_0 + dy_0.$$

By the Theorems 2–4, there are two numbers x_1', y_1', such that

$$x_1' \equiv x_0', \quad y_1' \equiv y_0', \quad F_{t_0}(x_1', y_1') \leqslant \begin{cases} 1 & \text{for} \quad D = 1, \\ \frac{3}{2} & \text{for} \quad D = 2, \\ \frac{2}{3} & \text{for} \quad D = 3. \end{cases}$$

Denote by x_1, y_1 the two numbers satisfying

$$x_1' = ax_1 + by_1, \quad y_1' = cx_1 + dy_1.$$

Then obviously

$$x_1 \equiv x_0, \ y_1 \equiv y_0, \ f_{t_0}(x_1, y_1) = t_0\,|\,ax_1 + \beta y_1\,|^2 + \frac{1}{t_0}\,|\,\gamma x_1 + \delta y_1\,|^2 \leqslant \begin{cases} 1 & \text{for } D = 1, \\ \frac{3}{2} & \text{for } D = 2, \\ \frac{2}{3} & \text{for } D = 3, \end{cases}$$

and (A) follows immediately by the theorem of the *arithmetic and geometric means*; if a/β is not in K, then further (C) holds, since

$$|ax_1 + \beta y_1| \leqslant \sqrt{(3/2t_0)} < \epsilon.$$

The limiting cases, in which the third formula (A) is true only with the sign of equality, are derived from §§ 2 and 3 as follows:

By Theorems 2–4, necessarily $A_{t_0} = 1$ for all values $t = t_0$ for which $A_{t_0} = C_{t_0}$. Hence the semi-circle C belonging to $F_{t_0}(x', y')$ intersects the surface Σ only in points of the form $(X, Y, 1)$, where $X + iY$ is an element of J; it passes through at least one of these points, namely through $(0, 0, 1)$.

There are now two possibilities. If C degenerates into a straight line† then this is the perpendicular to the plane $Z = 0$ at the point $X = Y = 0$. Hence, by § 3,

$$\frac{\beta'}{\alpha'} = 0, \quad \frac{\delta'}{\gamma'} = \infty,$$

where $\alpha', \beta', \gamma', \delta'$ are derived from $\alpha, \beta, \gamma, \delta$ by the change of x, y into x', y'; with a suitable $\alpha' \neq 0$, we therefore get

$$F_{t_0}(x', y') = t_0 \, |\, \alpha' x' \,|^2 + \frac{1}{t_0} \left| \, \frac{1}{\alpha'} \, y' \, \right|^2,$$

so that, identically in x and y,

$$(\alpha x + \beta y)(\gamma x + \delta y) = \alpha' x' \cdot \frac{1}{\alpha'} \, y' = (ax + by)(cx + dy).$$

This proves (B); the congruences (B′) are obvious in this trivial case.

If C is a circle of finite radius, then let

$$(0, 0, 1) \quad \text{and} \quad (X, Y, 1)$$

be its two points of intersection with Σ; here $X + iY \neq 0$ is an element of J. Obviously C intersects the plane $Z = 0$ in the two points

$$\left(\frac{X}{2} \left\{ 1 + \eta \, \sqrt{ \left(1 + \frac{4}{X^2 + Y^2} \right) } \right\}, \quad \frac{Y}{2} \left\{ 1 + \eta \, \sqrt{ \left(1 + \frac{4}{X^2 + Y^2} \right) } \right\} \right)$$

$$(\eta = \pm 1).$$

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† In this case, C has the point at infinity as its second point of intersection with the plane $Z = 0$.

Hence

$$\frac{\beta'}{\alpha'} = \frac{X+iY}{2}\left\{1 - \sqrt{\left(1 + \frac{4}{X^2+Y^2}\right)}\right\},$$

$$\frac{\delta'}{\gamma'} = \frac{X+iY}{2}\left\{1 + \sqrt{\left(1 + \frac{4}{X^2+Y^2}\right)}\right\},$$

and therefore

$$F_{t_0}(x', y') = t_0\left|\alpha'\left(x' + \frac{X+iY}{2}\left\{1 - \sqrt{\left(1 + \frac{4}{X^2+Y^2}\right)}\right\}y'\right)\right|^2$$
$$+ \frac{1}{t_0}\left|\gamma'\left(x' + \frac{X+iY}{2}\left\{1 + \sqrt{\left(1 + \frac{4}{X^2+Y^2}\right)}\right\}y'\right)\right|^2,$$

where $\alpha' \neq 0$ and $\gamma' \neq 0$ are numbers satisfying

$$(X+iY)\sqrt{\left(1 + \frac{4}{X^2+Y^2}\right)}\,\alpha'\gamma' = 1.$$

Hence we have identically in x and y

$$(\alpha x + \beta y)(\gamma x + \delta y) = \frac{\left(x' + \dfrac{X+iY}{2}\,y'\right)^2 - \left\{\dfrac{X+iY}{2}\sqrt{\left(1 + \dfrac{4}{X^2+Y^2}\right)}\,y'\right\}^2}{(X+iY)\sqrt{\left(1 + \dfrac{4}{X^2+Y^2}\right)}},$$

or, more simply,

$$(37) \quad (\alpha x + \beta y)(\gamma x + \delta y) = \frac{(X-iY)x'^2 + (X^2+Y^2)x'y' - (X+iY)y'^2}{\sqrt{\{(X^2+Y^2)(X^2+Y^2+4)\}}}$$
$$= \phi(x', y') \quad \text{say.}$$

For this product, it is easily proved that the last formula (A) can always be satisfied with the sign " $<$ ". By Theorems 2–4, this is certainly true if x_0', y_0' do not satisfy the congruences

$$x_0' \equiv \pm y_0' \equiv \begin{cases} \frac{1}{2}(1 + \sqrt{-D}) & \text{for } D = 1 \text{ or } 2, \\ \pm(\frac{1}{2} + \frac{1}{6}\sqrt{-3}) & \text{for } D = 3. \end{cases}$$

On the other hand, if x_0', y_0' satisfy these congruences, then the statement follows from the formulae

$$\left|\phi\left(\frac{1+i}{2}, \frac{1+i}{2}\right)\right|^2 = \frac{1}{4} - \frac{X^2}{(X^2+Y^2)(X^2+Y^2+4)},$$

$$\left|\phi\left(\frac{1+i}{2}, \frac{1-i}{2}\right)\right|^2 = \frac{1}{4} - \frac{Y^2}{(X^2+Y^2)(X^2+Y^2+4)}$$

for $D = 1$,

$$\left|\phi\left(\frac{1+\sqrt{-2}}{2}, \frac{1+\sqrt{-2}}{2}\right)\right|^2 = \tfrac{9}{16} - \frac{9X^2}{4(X^2+Y^2)(X^2+Y^2+4)},$$

$$\left|\phi\left(\frac{1+\sqrt{-2}}{2}, \frac{1-\sqrt{-2}}{2}\right)\right|^2 = \tfrac{9}{16} - \frac{(X+2\sqrt{2}\,Y)^2}{4(X^2+Y^2)(X^2+Y^2+4)}$$

for $D = 2$, and

$$\left|\phi\left(\tfrac{1}{2}+\frac{\sqrt{-3}}{6}, \tfrac{1}{2}+\frac{\sqrt{-3}}{6}\right)\right|^2 = \tfrac{1}{9} - \frac{4X^2}{9(X^2+Y^2)(X^2+Y^2+4)},$$

$$\phi\left(\tfrac{1}{2}+\frac{\sqrt{-3}}{6}, -\tfrac{1}{2}+\frac{\sqrt{-3}}{6}\right)\right|^2 = \tfrac{1}{9} - \frac{(X+\sqrt{3}\,Y)^2}{9(X^2+Y^2)(X^2+Y^2+4)}$$

with similar expressions for $|\phi(\pm\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{-3}, \pm\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{-3})|^2$ for $D = 3$. In these formulae, at most one of the two numbers X and Y, or X and $X+2\sqrt{2}\,Y$, or X and $X+\sqrt{3}\,Y$ can vanish, so that the stated inequalities follow at once.

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