ON REDUCED POSITIVE DEFINITE TERNARY QUADRATIC FORMS

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A positive definite ternary quadratic form with real coefficients

$$f(x) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + 2b_1 x_2 x_3 + 2b_2 x_3 x_1 + 2b_3 x_1 x_2$$

is called reduced (in the sense of Seeber or Minkowski), if its coefficients satisfy the inequalities

$$(1) \quad \begin{cases} 0 < a_1 \leqslant a_2 \leqslant a_3, \quad 0 \leqslant b_1 \leqslant \tfrac{1}{2}a_2, \quad |b_2| \leqslant \tfrac{1}{2}a_1, \quad 0 \leqslant b_3 \leqslant \tfrac{1}{2}a_1, \\ \qquad b_1 - b_2 + b_3 \leqslant \tfrac{1}{2}(a_1 + a_2). \end{cases}$$

Let

$$(2) \qquad D = a_1 a_2 a_3 - (a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 - 2b_1 b_2 b_3)$$

be the determinant of $f(x)$. It was conjectured by Seeber that, for reduced forms,

$$(3) \qquad a_1 a_2 a_3 \leqslant 2D.$$

This was proved by Gauss (*Werke*, II, 188–196) in his review of Seeber's work; later proofs were given by Dirichlet (*Werke*, II, 29–48), Hermite (*Oeuvres*, I, 94–99), Korkine and Zolotareff (*Oeuvres de Zolotareff*, I, 125–129), Selling [*Journal für Math.*, 77 (1874), 143], and Minkowski (*Math. Abh.*, II, 26–27).

I show in this note‡ that (3) is an immediate consequence of (1), if a trivial property of quadratic polynomials is used. It obviously suffices to show that the function

$$(4) \qquad \lambda(b_1, b_2, b_3) = a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 - 2b_1 b_2 b_3$$

of b_1, b_2, b_3 is not greater than $\tfrac{1}{2}a_1 a_2 a_3$, if the inequalities (1) are satisfied.

LEMMA. *Let $\phi(t) = \alpha t^2 + \beta t + \gamma$ be a polynomial with real coefficients, and positive highest coefficient α. Then*

$$\phi(t) \leqslant \max\big(\phi(t_1), \phi(t_2)\big),$$

if the variable t is restricted to a finite interval $t_1 \leqslant t \leqslant t_2$.

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‡ After writing this note, I found that Zolotareff (*Oeuvres*, I, 24–25) used a similar method for another proof of (3), but his notes are very short and do not make it clear how the upper bound for λ is obtained.

Proof. $\phi(t)$ cannot attain a maximum value in an inner point of this interval, since the second derivative $\phi''(t) = 2a > 0$.

In the proof of the inequality for λ, we distinguish two cases.

(A) $b_2 \geqslant 0$. In this case, the inequality $b_1 - b_2 + b_3 \leqslant \frac{1}{2}(a_1 + a_2)$ is a consequence of the other conditions (1) and may be omitted. We remark that λ, as a function of the single variable b_1, satisfies the hypothesis of the lemma; hence

$$\lambda(b_1, b_2, b_3) \leqslant \max\Big(\lambda(0, b_2, b_3), \lambda(\tfrac{1}{2}a_2, b_2, b_3)\Big).$$

But, from (1),

$$\lambda(\tfrac{1}{2}a_2, b_2, b_3) - \lambda(0, b_2, b_3) = \tfrac{1}{4}a_1 a_2^2 - a_2 b_2 b_3 \geqslant 0,$$

and therefore $$\lambda(b_1, b_2, b_3) \leqslant \lambda(\tfrac{1}{2}a_2, b_2, b_3).$$

Similarly, we prove the two other inequalities

$$\lambda(b_1, b_2, b_3) \leqslant \lambda(b_1, \tfrac{1}{2}a_1, b_3),$$

$$\lambda(b_1, b_2, b_3) \leqslant \lambda(b_1, b_2, \tfrac{1}{2}a_1).$$

Applying each of these inequalities once, we get

$$\lambda(b_1, b_2, b_3) \leqslant \lambda(\tfrac{1}{2}a_2, b_2, b_3) \leqslant \lambda(\tfrac{1}{2}a_2, \tfrac{1}{2}a_1, b_3) \leqslant \lambda(\tfrac{1}{2}a_2, \tfrac{1}{2}a_1, \tfrac{1}{2}a_1)$$
$$= \tfrac{1}{4}(a_1 a_2^2 + a_1^2 a_3) \leqslant \tfrac{1}{2}a_1 a_2 a_3.$$

(B) $b_2^* = -b_2 \geqslant 0$. Put

$$\mu(b_1, b_2^*, b_3) = \lambda(b_1, -b_2^*, b_3) = a_1 b_1^2 + a_2 b_2^{*2} + a_3 b_3^2 + 2b_1 b_2^* b_3.$$

The conditions (1) now become

$$(5)\quad \begin{cases} 0 < a_1 \leqslant a_2 \leqslant a_3, \quad 0 \leqslant b_1 \leqslant \tfrac{1}{2}a_2, \quad 0 \leqslant b_2^* \leqslant \tfrac{1}{2}a_1, \quad 0 \leqslant b_3 \leqslant \tfrac{1}{2}a_1, \\ b_1 + b_2^* + b_3 \leqslant \tfrac{1}{2}(a_1 + a_2). \end{cases}$$

As a continuous function of b_1, b_2^*, b_3, the function μ has a maximum. This maximum can be attained only for

$$(6)\quad b_1 + b_2^* + b_3 = \tfrac{1}{2}(a_1 + a_2).$$

For otherwise it is possible to increase one of the variables b_1, b_2^*, b_3 and therefore also the value of μ, since at least one of the inequalities

$$b_1 < \tfrac{1}{2}a_2, \quad b_2^* < \tfrac{1}{2}a_1, \quad b_3 < \tfrac{1}{2}a_1$$

is satisfied.

Assume that (6) holds. We fix b_2^*, and allow b_1, and so also

$$b_3 = \tfrac{1}{2}(a_1+a_2)-b_1-b_2^*,$$

to assume all possible values; obviously b_1 is restricted to the interval

$$\tfrac{1}{2}a_2-b_2^* \leqslant b_1 \leqslant \tfrac{1}{2}a_2.$$

As a function of b_1, μ can be written as

$$\mu(b_1, b_2^*, b_3) = (a_1+a_3-2b_2^*)\, b_1^2+b_1 \,.\, \text{coefficient}+\text{coefficient};$$

here the highest coefficient $a_1+a_3-2b_2^*$ is positive. Hence by the lemma,

$$\mu(b_1, b_2^*, b_3) \leqslant \max\big(\mu(\tfrac{1}{2}a_2-b_2^*, b_2^*, \tfrac{1}{2}a_1),\ \mu(\tfrac{1}{2}a_2, b_2^*, \tfrac{1}{2}a_1-b_2^*)\big).$$

Similarly, we prove the following inequalities:

$$\mu(b_1, b_2^*, b_3) \leqslant \max\big(\mu(\tfrac{1}{2}a_2-b_3, \tfrac{1}{2}a_1, b_3),\ \mu(\tfrac{1}{2}a_2, \tfrac{1}{2}a_1-b_3, b_3)\big),$$

$$\mu(b_1, b_2^*, b_3) \leqslant \max\big(\mu(b_1, \tfrac{1}{2}a_2-b_1, \tfrac{1}{2}a_1),\ \mu(b_1, \tfrac{1}{2}a_1, \tfrac{1}{2}a_2-b_1)\big).$$

Hence, either

$$\begin{aligned}\mu(b_1, b_2^*, b_3) &\leqslant \mu(\tfrac{1}{2}a_2-b_2^*, b_2^*, \tfrac{1}{2}a_1)\\ &\leqslant \max\big\{\mu\big(\tfrac{1}{2}(a_2-a_1), \tfrac{1}{2}a_1, \tfrac{1}{2}a_1\big),\ \mu(\tfrac{1}{2}a_2, 0, \tfrac{1}{2}a_1)\big\}\\ &= \max\big(\tfrac{1}{4}(a_1 a_2^2+a_1^2 a_3),\ \tfrac{1}{4}(a_1 a_2^2+a_1^2 a_3)\big) \leqslant \tfrac{1}{2}a_1 a_2 a_3,\end{aligned}$$

or

$$\begin{aligned}\mu(b_1, b_2^*, b_3) &\leqslant \mu(\tfrac{1}{2}a_2, b_2^*, \tfrac{1}{2}a_1-b_2^*)\\ &\leqslant \max\big(\mu(\tfrac{1}{2}a_2, 0, \tfrac{1}{2}a_1),\ \mu(\tfrac{1}{2}a_2, \tfrac{1}{2}a_1, 0)\big)\\ &= \max\big(\tfrac{1}{4}(a_1 a_2^2+a_1^2 a_3),\ \tfrac{1}{4}(a_1 a_2^2+a_1^2 a_2)\big) \leqslant \tfrac{1}{2}a_1 a_2 a_3.\end{aligned}$$

It is not difficult to show by the same method that the inequality sign always holds in (3) except for those reduced forms which are equivalent to

$$x_1^2+x_2^2+x_3^2+x_1 x_2+x_1 x_3+x_2 x_3.$$

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